Challenge set 4

part II

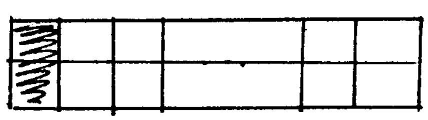
Exercises 7.1

5. Three types of pennies.
Two types of nickels.
One type of dime.
One type of quarters.

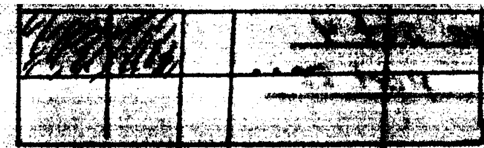
Let a_n = number of ways to distribute a total of n¢ among the three types of pennies, two types of nickels, one type of dime and one type of quarter.

If we distribute 1¢ of money, then there are $\binom{1+3-1}{1}$ ways of picking the penny, and $3a_{n-1}$ ways to distribute the remaining $n-1$ ¢. If we distribute 5¢ of money, then there are $\binom{1+2-1}{1}$ ways of picking the nickel, and $2a_{n-5}$ ways to distribute the remaining $n-5$¢. If we distribute 25¢ of money, then there a_{n-25} ways to distribute the remaining $n-25$¢. If we distribute 10¢ of money, then there are a_{n-10} ways to distribute the remaining $n-10$ cents. In the last two cases we distribute out 25¢ and 10¢ of money via a quarter and nickel respectively.

9. a_n = the number of ways to fill a $2 \times n$ checkerboard, with dominoes.



If we place the first domino such that it fills one of the columns, then we have a $2 \times (n-1)$ checkerboard left which can be filled in a_{n-1} ways.



(3) If we place the first domino such ~~that it fills~~ two spaces in the first row, we are left with a $2 \times (n-2)$ checkerboard and a horizontal 1×2 block. The $2 \times (n-2)$ checkerboard can be filled in a_{n-2} ways, the horizontal 1×2 block can only be filled one way.

Hence $a_n = a_{n-1} + a_{n-2}$.

15. $a_n =$ amount of money in the savings account after n years.

$$a_n = \underbrace{a_{n-1}}_{\text{money from last year}} + \underbrace{0.06\, a_{n-1}}_{\text{interest earned on last years money}} + \underbrace{50(1.06)}_{\text{new deposit earns 1 year interest}}$$

$$a_n = 1.06\,(a_{n-1}) + 50(1.06)$$

21. Let $a_n =$ number of n-digit ternary sequence with "012" occuring for the first time at the end.

If the first digit is 1 $\underbrace{\_\ \_\ \ldots\ \_}_{n-1 \text{ digit sequence left}}$

a_{n-1} $n-1$ digit sequences with "012" occuring for the first time at the end.

If the first digit is 2 $\underbrace{\_\ \_\ \ldots\ \_}_{n-1 \text{ digit sequence left}}$

a_{n-1} $n-1$ digit sequences with "012" occuring for the first time at the end.

If the first digit is 0 $\_\ \_\ \ldots\ \_$, then there a_{n-1} $(n-1)$ digit ternary sequences that have 012 occuring for the first time at the end.

There are a_{n-3} $(n-1)$ digit ternary sequences that start with 12, and have "012" occuring for the first time at the end

$$a_n = a_{n-1} + a_{n-1} + \overbrace{a_{n-1} - a_{n-3}}^{a_{n-1} - a_{n-3}}$$
$$= 3a_{n-1} - a_{n-3}.$$

27. Look at it from the perspective of the objects of type I.

If we take one object of type I, then there are $a_{n-1,k-1}$ ways of selecting $n-1$ objects from $k-1$ types with atmost 3 objects of each type.

If we take two objects of type II, then there are $a_{n-2,k-1}$ ways of selecting $n-2$ objects from $k-1$ types with at most 3 objects of each type.

If we take three objects of type I, then there are $a_{n-3,k-1}$ ways of selecting $n-3$ objects from $k-1$ types with atmost 3 objects of each type.

If we take no objects of type I, then there are $a_{n,k-1}$ ways of selecting n objects from $k-1$ types with at most 3 objects of each type. $a_{n,k} = a_{n,k-1} + a_{n-1,k-1} + a_{n-2,k-1} + a_{n-3,k}$

31. Follow example 11b carefully.

a_n = number of n-digit binary sequences with an even number of zero's and an even number of ones.

b_n = number of n-digit binary sequences with an even number of zero's and an odd number of ones.
c_n = number of n-digit binary sequences with an odd number of zero's and an even number of one's.

$2^n - a_n - b_n - c_n$ = number of n-digit binary sequences with an odd number of zero's and odd number of one's.

an n-digit binary sequence is obtained by having a 1 for the first digit followed by an $n-1$ digit sequence with an even number of zero's and odd number of one's, or by having 0 for the first digit followed by an $n-1$ digit sequence with an odd number of zero's and an even number of one's.

$$a_n = b_{n-1} + c_{n-1}$$

b_n - # of n digit binary sequence with an even number of zero's and an odd number of ones.

If the first digit is 1, then we need an $n-1$ digit sequence with an even number of zero's and even number of one's. a_{n-1}.

If the first digit is 0, then we need an $n-1$ digit Sequence with an odd number of number of one's and an odd number of zero's.

$$2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$$

$$b_n = a_{n-1} + 2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$$
$$= 2^{n-1} - b_{n-1} - c_{n-1}$$

c_n - # of n digit binary sequences with an odd number of zero's and an even number of ones.

If the first digit is a 1, then we need an $n-1$ digit sequence with an odd number of ones and odd number of zero's. $2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$.

If the first digit is a 0, then we need an $n-1$ digit sequence with an even number of zero's and an even number of ones. a_{n-1}.

$$c_n = a_{n-1} + 2^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$$
$$= 2^{n-1} - b_{n-1} - c_{n-1}$$

Summary of recurrence relations:

$$a_n = b_{n-1} + c_{n-1}$$
$$b_n = c_n = 2^{n-1} - b_{n-1} - c_{n-1}$$

41 a. $f(n,k)$ = # of k-subsets of the integers 1 through n with no pairs of consecutive integers

First, lets look at $f(5,2)$. We are working with the integers $\{1,2,3,4,5\}$. $k=2$

$\{1,3\}$ $\{1,4\}$ $\{1,5\}$
$\{2,4\}$ $\{2,5\}$ $\{3,5\}$

Six 2-subsets of 5 with no pair of consecutive integers.

if 1 belongs to the subset there are $f(3,1)=3$ 2-subsets of 5 that contain 1.

If 1 doesn't belong to the subset, there are $f(4,2)=3$ 2-subsets of 5 that doesn't contain 1.

In general if 1 belongs to the subset, then there are $f(n-2,k-1)$ $k-1$ subsets of $n-2$ numbers with no pair of consecutive numbers.

If 1 doesn't belong then there are $f(n-1,k)$ k-subsets of $n-1$ numbers with no pair of numbers consecutive. $\therefore f(n,k) = f(n-2,k-1) + f(n-1,k)$

47. a. $$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

induction:

$n=1$: LHS $= F_0 + F_1 = 2$ RHS $= F_3 - 1 = F_1 + F_2 - 1 = F_1 + F_1 + F_0 - 1 = 2$

Suppose $$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

show: $$\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$$

$$\sum_{i=0}^{n+1} F_i = \sum_{i=0}^{n} F_i + F_{n+1} = \underbrace{F_{n+2} - 1}_{I.H} + F_{n+1} = F_{n+3} - 1$$

b. $$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

$n=0$: LHS $= F_0^2 = 1^2 = 1$ RHS $= F_0 F_1 = 1 \cdot 1 = 1$

Suppose $$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

show: $$\sum_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$$

$$\sum_{i=0}^{n+1} F_i^2 = \sum_{i=0}^{n} F_i^2 + F_{n+1}^2 = \underbrace{F_n F_{n+1}}_{I.H.} + F_{n+1}^2$$

$$= F_{n+1} (F_n + F_{n+1}) = F_{n+1} F_{n+2}$$

c. $$\sum_{k=0}^{n} F_{2k} = F_{2n+1}$$

$n=0$: LHS $= F_0 = 1$ RHS $= F_{0+1} = F_1 = 1$

Suppose $$\sum_{k=0}^{n} F_{2k} = F_{2n+1}$$

Show: $$\sum_{k=0}^{n+1} F_{2k} = F_{2n+3}$$

$$\sum_{k=0}^{n+1} F_{2k} = \sum_{k=0}^{n} F_{2k} + F_{2n+2} = F_{2n+1} + F_{2n+2} = F_{2n+3}$$

e. $\sum_{i=1}^{2n} (-1)^{i+1} F_i = -F_{2n-1}$

$n=1$ LHS RHS

$= F_1 - F_2 \qquad = -F_1 = -1$

$= 1-2$

$= -1$

Suppose: $\sum_{i=1}^{2n} (-1)^{i+1} F_i = -F_{2n-1}$ show $\sum_{i=1}^{2n+2} (-1)^{i+1} F_i = -F_{2n+1}$

$$\sum_{i=1}^{2n+2} (-1)^{i+1} F_i = \sum_{i=1}^{2n} (-1)^{i+1} F_i + (-1)^{2n+2} F_{2n+1} + (-1)^{2n+3} F_{2n+2}$$

$$= -F_{2n-1} + F_{2n+1} - F_{2n+2}$$

$$= -F_{2n-1} + F_{2n} + F_{2n-1} - F_{2n+2}$$

$$= F_{2n} - (F_{2n+1} + F_{2n})$$

$$= -F_{2n+1}$$

d. $F_{n-1} F_{n+1} = F_n^2 + (-1)^{n-1} \qquad n \geq 2$

$n=1$

$F_0 F_2 = F_1^2 + (-1)^0$

$= 1 \cdot 2 \qquad = 1^2 + 1 = 2$

$= 2$

Suppose $F_{n-1} F_{n+1} = F_n^2 + (-1)^{n-1}$

show: $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$

$$F_{n+1}^2 + (-1)^n = F_{n+1}(F_n + F_{n-1}) + (-1)^n$$

$$= F_n F_{n+1} + F_{n-1} F_{n+1} + (-1)^n$$

$$= F_n F_{n+1} + \underbrace{F_n^2 + (-1)^{n-1}}_{I.H.} + (-1)^n$$

$$= F_n (F_{n+1} + F_n) + (-1)^{n-1} + (-1)^n$$

$$= F_n F_{n+2} + (-1)^{n-1}(1 + (-1)) = F_n F_{n+2}.$$

17. Let a_n = the number of n-digit binary sequences with atleast one instance of consecutive 0's.

If the first digit is a 1,

1 _ _ ... _

then we are left with an $n-1$ digit binary sequence and this sequence must have atleast one instance of consecutive 0's.
There a_{n-1} of these.

If the first digit is a 0, then there are two subcases we need to consider.

Subcase 1.

Second digit is a 1.

$0\,1\ \underbrace{\_\ \_\ \cdots\ \_}_{n-2 \text{ digits}}$

$n-2$ digit binary sequence left and there are a_{n-2} of these with atleast one occurence of consecutive 0's.

Subcase 2

Second digit is a 0.

$0\,0\ \underbrace{\_\ \_\ \cdots\ \_}_{n-2 \text{ digits}}$

$n-2$ digit binary sequence left and since we already have an instance of consecutive 0's, we can fill in the $n-2$ positions with a 0 or a 1.
There are 2^{n-2} $n-2$ digit binary sequences.

Recurrence relation for this is

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2}$$

20 Let $a_n =$ the number of n-digit ternary sequences in which no 1 appears to the right of any 2.

If the first digit is a 1, then we are left with an $n-1$ digit sequence in which no 1 appears to the right of any 2. There are a_{n-1} of these.

If the first digit is a 0, then we are left with an $n-1$ digit sequence in which no 1 appears to the right of any 2. There are a_{n-1} of these.

If the first digit is a 2, then we are left with an $n-1$ digit sequence in which no 1 appears to the right of any 2. Since we already have a 2, the remaining $n-1$ digits can only be filled in with a 0 or a 2, i.e. 2^{n-1} ways.

Recurrence relation for this is,

$$a_n = a_{n-1} + a_{n-1} + 2^{n-1} = 2a_{n-1} + 2^{n-1}.$$

24. For 2 people, there are $\binom{2}{2}$ ways to pair them off for a tennis match.
$a_1 = 1$.
For 4 people, there are $\dfrac{\binom{4}{2}\binom{2}{2}}{2!}$ ways to pair them off for tennis matches.

$$a_2 = \frac{\binom{4}{2} a_1}{2!}$$

For 6 people, there are $\dfrac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}$

$$= \frac{\binom{6}{2}}{3} \frac{\binom{4}{2}\binom{2}{2}}{2!} = \frac{\binom{6}{2}}{3} a_2$$

$$a_3 = \frac{\binom{6}{2}}{3} a_2$$

Conjecture: $a_n = \dfrac{\binom{2n}{2}}{n} a_{n-1} = (2n-1)a_{n-1}.$

proof by induction:

$$a_1 = \frac{\binom{2}{2}}{1} a_0 = 1$$

There is only one way to pair off 2 people for a tennis match.

Suppose $a_k = (2k-1)\,a_{k-1}$.

We must prove: $a_{k+1} = (2k+1)\,a_k$.

From $2k+2$ people pair off 2 people in $\binom{2k+2}{2}$ ways. From the remaining

$2k$ people there are $(2k-1)\,a_{k-1}$ ways to pair them off. Order doesn't matter with these pair off's, so we could insert our first pair into $k+1$ positions before, in between, and after the other k pairs formed from the remaining $2k$ people.

Hence $a_{k+1} = \dfrac{\binom{2k+2}{2}}{k+1} (2k-1)\,a_{k-1}$,

$$a_{k+1} = \frac{(2k+2)(2k+1)}{2(k+1)} (2k-1)\,a_{k-1}$$

$$= (2k+1)(2k-1)\,a_{k-1} = (2k+1)\,a_k.$$

28. Let $a_{n,k}$ = the number of ways to distribute n distinct objects into k indistinguishable boxes with no box empty.

The boxes are only distinguishable by the number of balls they have in them.

Suppose a box has only one ball. There are $\binom{n}{1}$ ways to select a ball for this box. There are $n-1$ balls left and $k-1$ boxes left to distribute these balls to. There are $a_{n-1,k-1}$ ways to do this.
We can generalize this argument. Suppose a box has j balls in it. There are $\binom{n}{j}$ ways to select the balls for this box. There are $n-j$ balls left, and $k-1$ boxes left to distribute these balls to. There are $a_{n-j,k-1}$ ways to do this.
This will end when we have k balls left. $a_{k,k}=1$ because we must one ball in each box and since the boxes are indistinguishable, it doesn't matter what order we do this in.

Our recurrence relation is,

$$a_{n,k} = \sum_{j=1}^{n-k} a_{n-j,n-1} .$$

29. Please refer back to question 11a, on page 264.
Let $a_{n,k}$ = the number of the partitions of an integer n into k parts.

$$a_{n,k} = a_{n-1,k-1} + a_{n-k,k} .$$

32.

a. Let a_n = the number of n-digit quaternary sequences with an even number of 0's. We break this down into cases involving the first digit.

i. First digit is a 3.

$\underbrace{3 \; \_ \; \_ \; \ldots \; \_}_{n-1 \text{ digit sequence}}$. We are left with an n-1 digit quaternary sequence with an even number of 0's. **There are a_{n-1}** of these.

ii. First digit is a 2.

$\underbrace{2 \; \_ \; \_ \; \ldots \; \_}_{n-1 \text{ digit sequence}}$. Same situation as case i.

We get a_{n-1} n-1 digit quaternary sequences with an even number of 0's.

iii. First digit is a 1.

$\underbrace{1 \; \_ \; \_ \; \ldots \; \_}_{n-1 \text{ digit sequence}}$. Same situation as in case i and case ii. We get a_{n-1} from this.

iv. First digit is 0.

$\underbrace{0 \; \_ \; \_ \; \ldots \; \_}_{n-1 \text{ digit sequence}}$

We are left with an n-1 digit quaternary sequence and we require this to have an odd number of 0's.

Let b_n = the number of quaternary sequences with an odd number of 0's. For this case there are

b_{n-1} $n-1$ digit quaternary sequences with an odd number of 0's.

$$a_n = 3a_{n-1} + b_{n-1} \quad .$$

Using an argument similiar to the one used for a_n, we will get
$b_n = 3b_{n-1} + a_{n-1}$.

If the first digit is a 1, 2, or 3, then we are left with an $n-1$ digit quaternary sequence requiring an odd number of 0's. In each case we get b_{n-1}. This accounts for $3b_{n-1}$ in the formula for b_n.
If the first digit is 0, then we are left with an $n-1$ digit quaternary sequence requiring an even number of 0's. There are a_{n-1} of these.

b. Let a_n = the number of n-digit quaternary sequences with an even total number of 0's and 1's. We break this down into cases involving the first digit.

If the first digit is a 2 or 3, then we are left with an $n-1$ digit quaternary sequence requiring an even total of 0's and 1's. In each case we get a_{n-1}, for a total of $2a_{n-1}$.

If the first digit is a 0 or 1, then we are left an $n-1$ digit quaternary sequence requiring an odd **total number** of 0's and 1's.
Let b_n = the number of n-digit quaternary sequences with an odd **total number** of 0's and 1's.
In each case we get b_{n-1} for a total of $2b_{n-1}$. Hence $a_n = 2a_{n-1} + 2b_{n-1}$.