3. There are five vowels, a, e, i, o, u. Each vowel can be used at most once, the twenty-one consonants can be repeated, so the exponential generating function that models this is;

$$y(x) = (1+x)^5 \left(1 + x + \frac{x^2}{2!} + \cdots \right)^{21}$$

$$= (1+x)^5 e^{21x}$$

4. $S_{n,r}$ = number of ways to
a. distribute r-identical objects into n-distinct boxes with no box empty.

Each box has at least one ball, so the exponential generating function for this is,

$$\left(x + \frac{x^2}{2!} + \cdots \right)^n = \left(e^x - 1 \right)^n.$$

b. We want the coefficient of $\dfrac{x^r}{r!}$.

$$\left(e^x - 1 \right)^n = \sum_{k=0}^{n} \binom{n}{k} e^{kx} (-1)^{n-k}$$

the coefficient of $\dfrac{x^r}{r!}$

$$= \sum_{k=0}^{n} \binom{n}{k} k^r (-1)^{n-k} \qquad \left[\begin{array}{l} \text{in } e^{kx}, \\ \text{the coefficient} \\ \text{of } \dfrac{x^r}{r!} \text{ is } k^r. \end{array} \right.$$

5. You have thirteen types of cards with four cards of each type. If you deal a sequence of thirteen cards and your or only considering the values then each value can appear at most four times.

We require the exponential generating function

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)^{13}$$

and the coefficient of $\dfrac{x^{13}}{13!}$.

6. The first child gets at least two toys, and the other three can get any number. We require the exponential generating function,

$$\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)^3$$

$$= (e^x - 1 - x)\, e^{3x} .$$ We require the coefficient of $\dfrac{x^8}{8!}$.

$$e^{3x}(e^x - 1 - x) = e^{4x} - e^{3x} - x e^{3x} ,$$

the coefficient of $\dfrac{x^8}{8!}$ is,

$$4^8 - 3^8 - 8 \times 3^7$$

7

a.   For an even number of 0's, we require the exponential generating function,

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) e^{2x} \cdot$$

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) e^{2x} = \frac{1}{2}\left(e^x - e^{-x}\right) e^{2x}$$

(should be a plus)

$$= \frac{1}{2}\left(e^{3x} + e^x\right)$$

We require the coefficient of $\frac{x^r}{r!}$.

The coefficient of $\frac{x^r}{r!} = \frac{1}{2}\left(3^r + 1\right)$

b.   For an even number of 0's and an even number of 1's we require the generating function,

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right)^2 e^x \cdot$$ We want the coefficient of $\frac{x^r}{r!}$

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right)^2 e^x = \frac{1}{4}\left(e^x + e^{-x}\right)^2 e^x$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) e^x$$

$$= \frac{1}{4}\left(e^{3x} + 2e^x + e^{-x}\right).$$

The coefficient of $\frac{x^r}{r!} = \frac{1}{4}\left(3^r + 2 + (-1)^r\right)$

9. Each of the letters e, n, r, s can
a. be used at most once, the 22 other letters in the alphabet can be used more than once, so the exponential generating function required is,

$$y(x) = (1+x)^4 \left(1 + x + \frac{x^2}{2!} + \cdots \right)^{22} = (1+x)^4 e^{22x}.$$

We want the coefficient of $\frac{x^{10}}{10!}$ in $g(x)$.

$$g(x) = (1 + 4x + 6x^2 + 4x^3 + x^4) \, e^{22x}$$

Coefficient of $\frac{x^{10}}{10!}$

$$= 22^{10} + (4)22^9 \frac{10!}{9!} + (6) 22^8 \frac{10!}{8!} + (4) \frac{22^7}{7!} 10!$$
$$+ 22^6 \frac{10!}{6!}$$

b. This time, the letters e, n, r, s are used at least once, so the exponential generating function required is

$$h(x) = \left(x + \frac{x^2}{2!} + \cdots \right)^4 \left(1 + x + \frac{x^2}{2!} + \cdots \right)^{22}$$

$$= (e^x - 1)^4 e^{22x}$$

$$= (e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1) e^{22x}.$$

We want the coefficient of $\frac{x^{10}}{10!}$

$$= e^{26x} - 4e^{25x} + 6e^{24x} - 4e^{23x} + e^{22x}$$

The coefficient of $\frac{x^r}{10!}$ is

$$26^{10} - 4 \cdot 25^{10} + 6 \cdot 24^{10} - 4 \cdot 23^{10} + 22^{10}$$

10.   No digit appears exactly twice,

a. We require the exponential generating function,

$$g(x) = \left(1 + x + \frac{x^3}{3!} + \cdots\right)^3 = \left(e^x - \frac{x^2}{2!}\right)^3 .$$ We want

the coefficient of $\frac{x^r}{r!}$.

$$g(x) = e^{3x} - 3e^{2x}\frac{x^2}{2} + 3e^x\frac{x^4}{4} - \frac{x^6}{8} .$$

The coefficient of $\frac{x^r}{r!}$ is

$$= 3^r - \frac{3}{2} \cdot 2^{r-2} r(r-1) + \frac{3}{4} r(r-1)(r-2)(r-3)$$

$$= 3^r - 3r(r-1)2^{r-3} + \frac{3}{4} r(r-1)(r-2)(r-3)$$

or

$$3^r - 3 \cdot 2^{r-2} \frac{r(r-1)}{2} + 3 \cdot \frac{r(r-1)}{2}\frac{(r-2)(r-3)}{2}$$

$$= 3^r - 3\binom{r}{2}2^{r-2} + 3\binom{r}{2}\binom{r-2}{2}$$

b.   0 and 1 appear a positive even number of times, (2 times, 4 times, ...). We need the exponential generating function,

$$h(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 e^x .$$ We want the coefficient of $\frac{x^r}{r!}$

$$h(x) = \left(\frac{e^x + e^{-x}}{2} - 1\right)^2 \cdot e^x$$

$$h(x) = \left(\frac{(e^x + e^{-x})^2}{4} - 2\left(\frac{e^x + e^{-x}}{2}\right) + 1\right) e^x$$

$$h(x) = \left(\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} - e^x - e^{-x} + 1\right) e^x$$

$$= \left(\frac{e^{3x}}{4} + \frac{1}{2}e^x + \frac{e^{-x}}{4} - e^{2x} - 1 + e^x\right)$$

The coefficient of $\frac{x^r}{r!}$ is

$$= \frac{3^r}{4} + \frac{1}{2} + \frac{1}{4}(-1)^r - 2^r + 1$$

$$= \frac{1}{4}3^r + \frac{2}{2} + \frac{(-1)^r}{4} - 2^r = \frac{1}{4}\left(3^r + 6 + (-1)^r - 4\times2^r\right)$$

11. There are a couple of cases to consider. If the number of 0's and 1's are both odd, then we require the exponential generating function.

$$g_1(x) = \left(x + \frac{x^3}{3!} + \cdots\right)^2 e^{2x}$$

$$= \left(\frac{1}{2}(e^x - e^{-x})\right)^2 e^{2x}$$

$$= \frac{1}{4}(e^{2x} - 2 + e^{-2x})e^{2x} = \frac{1}{4}(e^{4x} - 2e^{2x} + 1)$$

If the number of 0's and 1's are each both even, then we require the exponential generating function.

$$g_2(x) = \left(1 + \frac{x^2}{2!} + \cdots\right)^2 e^{2x}$$

$$= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 e^{2x}$$

$$= \frac{1}{4}(e^x + e^{-x})^2 e^{2x} = \frac{1}{4}(e^{2x} + 2 + e^{-2x})e^{2x}$$

$$= \frac{1}{4}(e^{4x} + 2e^{2x} + 1)$$

The generating function we require is $g(x) = g_1(x) + g_2(x)$

$$= \frac{1}{4}(2\cdot e^{4x} + 2) = \frac{1}{2}e^{4x} + \frac{1}{2}$$

We require the coefficient of $\frac{x^r}{r!}$, and in our case, it is $\frac{1}{2}4^r$.

12. $0 \leq b_1 < b_2 \leq 4$, b_1 and b_2 represent the number of balls in the first two boxes, ie. box 1 and box 2 respectively.

The possible values of b_1 and b_2 are

(0,1)   (1,2)   (2,3)   (3,4)
(0,2)   (1,3)   (2,4)
(0,3)   (1,4)
(0,4)

The generating function required is

$$\left[1\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \right) + x\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \right) + \frac{x^2}{2!}\left(\frac{x^3}{3!} + \frac{x^4}{4!} \right) + \frac{x^3}{3!}\frac{x^4}{4!} \right]$$

14. r - distinct objects into n - different boxes with exactly m - nonempty boxes.

You need to select the m boxes that will contain the balls $\binom{n}{m}$ ways to do this. Each of these m - boxes will have atleast one ball in them. The exponential generating function required is,

$$\binom{n}{m}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^m = \binom{n}{m}\left(e^x - 1 \right)^m.$$

15

a. $\qquad a_r = \dfrac{1}{r+1}$

$$g(x) = \sum_{r=0}^{\infty} a_r \dfrac{x^r}{r!} = \sum_{r=0}^{\infty} \dfrac{1}{r+1} \dfrac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \dfrac{x^r}{(r+1)!} = \dfrac{1}{x} \sum_{r=0}^{\infty} \dfrac{x^{r+1}}{(r+1)!}$$

$$= \dfrac{1}{x}(e^x - 1)$$

b. $\quad g(x) = \sum_{r=0}^{\infty} a_r \dfrac{x^r}{r!} = \sum_{r=0}^{\infty} r! \dfrac{x^r}{r!}$

$$= \sum_{r=0}^{\infty} x^r = \dfrac{1}{1-x} \quad .$$

17. $\qquad f(x) = \sum_{r=0}^{\infty} \dfrac{a_r}{r!} x^r \qquad g(x) = \sum_{r=0}^{\infty} \dfrac{b_r}{r!} x^r$

$$\text{let } a_r' = \dfrac{a_r}{r!} \quad \text{and} \quad b_r' = \dfrac{b_r}{r!}$$

$$f(x) = \sum_{r=0}^{\infty} a_r' x^r \qquad g(x) = \sum_{r=0}^{\infty} b_r' x^r$$

The Cauchy Product of $f(x)g(x)$

$$= \sum_{r=0}^{\infty} c_r' x^r \quad \text{where} \quad c_r' = \sum_{k=0}^{r} a_k' b_{r-k}'$$

$$= \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} \dfrac{a_k}{k!} \dfrac{b_{r-k}}{(r-k)!} r! \right) \dfrac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} \binom{r}{k} a_k b_{r-k} \right) \dfrac{x^r}{r!}$$

19. In example 6, we used generating functions to determine the number of quaternary sequences with an even number of 0's and an odd number of 1's.
We will now do this using a combinatorial argument.

Consider all $r-1$ digit quaternary sequences. An $r-1$ digit quaternary sequence comes in four types.

type I; even number of 0's, odd number of 1's.
type II; odd number of 0's, even number of 1's.
type III; even number of 0's, even number of 1's.
type IV; odd number of 0's, odd number of 1's.

There are equal numbers of sequences of type I and type II (i.e. there is a one to one correspondence between them), We'll say that there are x of them each. Suppose that there are y - sequences of type III and z - sequences of type IV.
There are in total 4^{r-1} quaternary sequences.
Hence $x + x + y + z = 4^{r-1}$.
Lets consider an r-digit quaternary sequence. We can get an r-digit quaternary sequence with an even number of 0's and an odd number of 1's, from an $r-1$ digit quaternary sequence from the four types available.
If we have an $r-1$ digit quaternary sequence of type I, then we get an r-digit quaternary sequence with an even number of zeros and odd number of 1's by adding a 2 or a 3, for the rth digit. There are $2x$ sequences of this type and we want these.

If we have an $r-1$ digit quaternary sequence of type II, then we can't get an r-digit quaternary sequence with an even number of 0's and odd number of 1's from th_ no matter what digit is selected for the rth digit.

If we have an $r-1$ digit quaternary sequence of type III, then we can get an r-digit quaternary sequence with an even number of 0's and an odd number of 1's, by selecting a 1 for the last digit. There are y sequences of this type.

If we have an $r-1$ digit quaternary sequence of type IV, then we can get an r-digit quaternary sequence with an even number of 0's and an odd number of 1's, by selecting a 0 for the rth digit. There are z sequences of this type.

These are all the possible ways to obtain an r-digit quaternary sequence with an even number of 0's and an odd number of 1's.

Adding all these cases together we get that the number of r-digit quaternary sequences with an even number of 0's and an odd number of 1's

$$= 2x + y + z = 4^{r-1}.$$

Section 6.5

1

c. $\qquad u_r = 3r^2$

Start off with $\dfrac{1}{1-x} = 1+x+x^2+...$

$$= \sum_{r=0}^{\infty} x^r$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \cdot \sum_{r=0}^{\infty} r x^{r-1} = \sum_{r=0}^{\infty} r x^r$$

$$x \cdot \frac{1}{(1-x)^2} = \sum_{r=0}^{\infty} r x^r$$

$$x \cdot \left(\frac{d}{dx} \frac{x}{(1-x)^2}\right) = x \cdot \sum_{r=0}^{\infty} r^2 x^{r-1} = \sum_{r=0}^{\infty} r^2 x^r$$

$$\frac{x(1+x)}{(1-x)^3} = \sum_{r=0}^{\infty} r^2 x^r$$

$$\frac{3x(1+x)}{(1-x)^3} = \sum_{r=0}^{\infty} 3r^2 x^r$$

d. Find the generating function whose coefficient is $3r$ and add it to the generating function whose coefficient is 7.

Start off with $\dfrac{1}{1-x} = \sum_{r=0}^{\infty} x^r$

$$X \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) = X \cdot \sum_{r=0}^{\infty} r x^{r-1} = \sum_{r=0}^{\infty} r x^r$$

$$\frac{X}{(1-x)^2} = \sum_{r=0}^{\infty} r x^r$$

$$\frac{3x}{(1-x)^2} = \sum_{r=0}^{\infty} 3r x^r$$

$$\frac{7}{1-x} = \sum_{r=0}^{\infty} 7 x^r \qquad . \qquad \text{Hence our}$$

desired generating function is,

$$\frac{3x}{(1-x)^2} + \frac{7}{1-x} = \frac{3x + 7(1-x)}{(1-x)^2} = \frac{7 - 4x}{(1-x)^2}$$

e. Look back at example 2.

We start with $4! \, (1-x)^{-5}$, the coefficient of x^r which is a_r, is

$$4! \binom{r+5-1}{r} = 4! \binom{r+4}{r} = 4! \frac{(r+4)!}{r! \, 4!}$$
$$= (r+4)(r+3)(r+2)(r+1)$$

Note that the coefficient of x^{r-4} in
   $4! \, (1-x)^{-5} = r(r-1)(r-2)(r-3)$.

Our series requires this as the coefficient for x^r. We multiply the above series by x^4, and the series $4! \, x^4 (1-x)^{-5}$ has $r(r-1)(r-2)(r-3)$ as the coefficient of x^r.

2

a. Obtain the generating function whose coefficient is $a_r = r$.

Start with

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$$

$$x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \cdot \sum_{r=0}^{\infty} r x^{r-1}$$

$$= \sum_{r=0}^{\infty} r x^r$$

$$\frac{x}{(1-x)^2} = \sum_{r=0}^{\infty} r x^r$$

To obtain the sum $0 + 1 + 2 + \cdots + n$, use theorem 1, we must the coefficient of x^n in $h(x) = \dfrac{x}{(1-x)^3}$.

$$h(x) = x \cdot \left(1 + \binom{1+3-1}{1} x + \cdots + \binom{r+3-1}{r} x^r + \cdots \right)$$

coefficient of x^n

$$= \binom{n-1+3-1}{n-1} = \binom{n+1}{n-1}$$

$$= \binom{n+1}{2} = \frac{(n+1)\,n}{2}$$

b. Obtain the generating function whose coefficient $a_r = 13$.

$$\frac{13}{(1-x)} = \sum_{r=0}^{\infty} 13 x^r$$

Using theorem 1, $\qquad$ $13+13+\ldots+13$

is the coefficient of x^n in $h(x) = \dfrac{13}{(1-x)^2}$.

$$\frac{13}{(1-x)^2} = 13 \cdot \left(1 + \binom{2+1-1}{1}x + \ldots + \binom{r+2-1}{r}x^r + \ldots\right)$$

The coefficient of x^n $=$ $13\binom{n+2-1}{n}$

$$= 13\binom{n+1}{n} = 13(n+1)$$

c. From 1c, we have
the generating function whose coefficient

$$a_r = 3r^2.$$

From theorem 1,
We want the coefficient of x^n in

$$h(x) = \frac{\dfrac{3x(1+x)}{(1-x)^3}}{(1-x)} = \frac{3x(1+x)}{(1-x)^4}$$

$$h(x) = (3x + 3x^2)\left(1 + \binom{1+4-1}{1}x + \ldots + \binom{r+4-1}{r}x^r + \ldots\right),$$

The coefficient of x^n ,

$$= 3\binom{n-1+4-1}{n-1} + 3\cdot\binom{n-2+4-1}{n-2}$$

$$= 3\binom{n+2}{n-1} + 3\cdot\binom{n+1}{n-2}$$

$$= 3\binom{n+2}{3} + 3\binom{n+1}{3}$$

$$= \frac{(n+2)(n+1)n}{2} + \frac{(n+1)n(n-1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}$$

d.   From 1d,   we have the generating function   whose coefficient is
$$a_r = 3r + 7.$$

From theorem 1,   we want the coefficient of x^n in $h(x) = \dfrac{3x}{(1-x)^3} + \dfrac{7}{(1-x)^2}$.

$$h(x) = 3x \left(1 + \binom{1+3-1}{1} x + \cdots + \binom{r+3-1}{r} x^r + \cdots \right)$$

$$+ 7 \left(1 + \binom{1+2-1}{1} x + \cdots + \binom{r+2-1}{r} x^r + \cdots \right)$$

The coefficient of x^n

$$= 3 \cdot \binom{n-1+3-1}{n-1} + 7 \cdot \binom{n+2-1}{n}$$

$$= 3 \binom{n+1}{n-1} + 7 \cdot \binom{n+1}{n}$$

$$= 3 \binom{n+1}{2} + 7(n+1)$$

e.   From 1e,   we have the generating function whose coefficient is

$$a_r = r(r-1)(r-2)(r-3).$$

From theorem 1, we want the coefficient of x^n in $h(x) = \dfrac{4!\, x^4}{(1-x)^6}$.

$$h(x) = 4!\, x^4 \left(1 + \binom{1+6-1}{1} x + \cdots + \binom{r+6-1}{r} x^r + \cdots \right)$$

The coefficient of x^n

$$= 4! \binom{n-4+6-1}{n-4} = 4! \binom{n+1}{n-4}$$

$$= 4! \binom{n+1}{5} .$$

3. $a_r = r(r+2)$.

Find the generating function whose coefficient is $(r+2)(r+1)$ and subtract from it, the generating function whose coefficient is $r-2$. .

The function,

$$\frac{2!}{(1-x)^3}$$ has the coefficient $a_r = 2! \binom{r+3-1}{r}$

$$= 2! \binom{r+2}{r} = (r+2)(r+1)$$

i.e. $$\frac{2!}{(1-x)^3} = \sum_{r=0}^{\infty} (r+2)(r+1) x^r$$

$$\frac{x}{(1-x)^2} = \sum_{r=0}^{\infty} r x^r$$

$$\frac{2}{1-x} = \sum_{r=0}^{\infty} 2x^r .$$

The desired generating function is

$$\frac{2}{(1-x)^3} - \frac{x}{(1-x)^2} - \frac{2}{1-x}$$

$$= \frac{2-x+x^2-2x^2+4x-2}{(1-x)^3} = \frac{3x-x^2}{(1-x)^3}$$

4.
a. $P(r,2) = \frac{r!}{(r-2)!} = r(r-1) = r^2-r$

$P(r,1) = \frac{r!}{(r-1)!} = r$

$r^2 = P(r,2) + P(r,1)$

$$P(r,3) + P(r,2) + P(r,1)$$

$$= r(r-1)(r-2) + r(r-1) + r$$
$$= r^3 - 2r^2 + 2r$$

$$= r^3 - 2(P(r,2) + P(r,1)) + 2P(r,1)$$
$$= r^3 - 2P(r,2)$$

$$\Rightarrow r^3 = P(r,3) + 3P(r,2) + P(r,1)$$

b. Find a generating function for $3r^3 - 5r^2 + 4r$.

From part a,

$$3r^3 - 5r^2 + 4r = 3(P(r,3) + 3P(r,2) + P(r,1))$$
$$- 5(P(r,2) + P(r,1)) + 4P(r,1)$$

$$= 3P(r,3) + 4P(r,2) + 2P(r,1)$$

$P(r,3) = r(r-1)(r-2)$. Find generating function with $a_r = r(r-1)(r-2)$

Start with $3!\,(1-x)^{-4}$, this has coefficient

$$a_r = 3!\binom{r+4-1}{r} = (r+3)(r+2)(r+1).$$

$3!\,x^3(1-x)^{-4}$ has $r(r-1)(r-2)$ as its coefficient a_r
$P(r,2) = r(r-1)$. Find generating function with $a_r = r(r-1)$.

Start with $2!\,(1-x)^{-3}$, this has coefficient
$$a_r = 2!\binom{r+3-1}{r} = 2!\binom{r+2}{r} = (r+2)(r+1).$$

$x^2\,2!\,(1-x)^{-3}$ has $r(r-1)$ as its coefficient a_r.

generating function with coefficient

$a_r = r$ is, $\dfrac{x}{(1-x)^2}$

We now conclude that the generating function with coefficient $a_r = 3r^3 - 5r^2 + 4r$,

is $3\left[\dfrac{3! \; x^3}{(1-x)^4}\right] + 4\left[\dfrac{2! \; x^2}{(1-x)^3}\right] + \left[\dfrac{x}{(1-x)^2}\right] 2$

$= \dfrac{18 \, x^3}{(1-x)^4} + \dfrac{8 x^2}{(1-x)^3} + \dfrac{2x}{(1-x)^2}$.

5

a. $a_r = (r-1)^2$.

See example 1 on page 273.

For $a_r = r^2$, generating function is $\dfrac{x(1+x)}{(1-x)^3}$

For $a_r = -2r$, generating function is $\dfrac{-2x}{(1-x)^2}$

For $a_r = 1$, generating function is $\dfrac{1}{1-x}$

Add up these generating functions,

$\dfrac{x(1+x)}{(1-x)^3} - \dfrac{2x}{(1-x)^2} + \dfrac{1}{1-x}$

$= \dfrac{x(1+x) - 2x(1-x) + (1-x)^2}{(1-x)^3}$

$= \dfrac{4x^2 - 3x + 1}{(1-x)^3}$

b.
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$-\ln(1-x) = \int \frac{1}{1-x} dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} x^r .$$

$$a_r = \frac{1}{r},$$

required generating function is $-\ln(1-x)$.

6.
$$h(x) = \sum_{r=0}^{\infty} a_r x^r$$

$$h(x)(1-x) = \sum_{r=0}^{\infty} a_r x^r - \sum_{r=0}^{\infty} a_r x^{r+1}$$

Coefficient of x^r in $h(x)(1-x)$
$= $ coefficient of x^r in $h(x) - $ coefficient of x^r in $x h(x)$,

$$= a_r - a_{r-1} .$$

7. Cauchy product of $h(x) \cdot \frac{1}{1-x}$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + \cdots)$$

$$= \sum_{r=0}^{\infty} c_r x^r \quad \text{where } c_r = \sum_{k=0}^{r} a_k$$

Section 6.5 #8

$$h(x) = \sum_{r=0}^{\infty} a_r x^r \qquad\qquad S_r = \sum_{k=r+1}^{\infty} a_k$$

generating function for S_r,

$$\sum_{r=0}^{\infty} S_r x^r = \sum_{r=0}^{\infty} \sum_{k=r+1}^{\infty} a_k$$

$$= \quad a_1 + a_2 + a_3 + \cdots + a_j + \cdots$$
$$+ \quad a_2 x + a_3 x + \cdots + a_j x + \cdots$$
$$+ \quad a_3 x^2 + \cdots + a_j x^2 + \cdots$$
$$\vdots$$
$$+ \quad a_j x^{j-1} + \cdots$$

reverse the order of Summation.

$$\sum_{j=1}^{\infty} \sum_{k=0}^{j-1} a_j x^k = \sum_{j=1}^{\infty} a_j \sum_{k=0}^{j-1} x^k$$

$$= \sum_{j=1}^{\infty} \left(a_j \frac{1-x^j}{1-x} \right) = \frac{1}{1-x} \sum_{j=1}^{\infty} a_j (1-x^j)$$

$$= \frac{1}{1-x} \left(\sum_{j=1}^{\infty} a_j - \sum_{j=1}^{\infty} a_j x^j \right)$$

$$= \frac{1}{1-x} \left(\alpha - x \cdot \sum_{j=1}^{\infty} a_j x^{j-1} \right)$$

$$= \frac{1}{1-x} \left(\alpha - x h(x) \right)$$

$$= \frac{x h(x) - \alpha}{x-1}$$

since α is a constant   generating function $\dfrac{x h(x)}{x-1}$