Mat 344
Solution Sketch for Challenge Set 1

1.

a. The positive integers less than or equal to 100 are the integers 1,2,3,…,100. There are 50 even integers and 50 odd integers.

b. Let S={1,2,3,…,100}. Let A = {x: x is an even positive integer}, B={x: x is a positive integer divisible by 5}, and C={x: x is a positive integer divisible by 11}. Let N(A or B)= {x: x is an integer divisible by 2 or 5}. The number of integers not divisible by 2 or 5 = 100-N(A or B). We need to determine N(A or B). There is a counting formula for obtaining N(A or B). It is N (A or B)=N(A)+N(B)-N(A and B). N(A and B)={x: x is a positive integer divisible by 2 and 5} = {x: x is a positive integer divisible by 10}. N(A) = 50, N(B)=20, and N(A and B)=10. N(A or B)= 50+20-10=60. Hence the number of integers not divisible by 2 or 5 = 100-60=40.

c. N(A or B or C) = {x: x is a positive integer divisible by 2, 5, 11}. The number of integers not divisible by 2, 5, 11 = 100-N(A or B or C). There is a counting formula for determining N(A or B or C). It is N(A or B or C) = N(A)+N(B)+N[C] -N(A and B)-N(A and C)-N(B and C)+N(A and B and C). N(A and C)={x: x is a positive integer divisible by 2 and 11}={x; x is divisible by 22}. N(B and C)={x: x is a positive integer divisible by 5 and 11}={x: x is divisible by 55}. N(A and B and C)= {x: x is a positive integer divisible by 2,5,11}={x: x is divisible by 110}. N(A)=50, N(B)=20, N[C]=9, N(A and B)=10, N(A and C)=4, N(B and C)=1, N(A and B and C)=0 since no integer is S is divisible by 110. Hence N(A or B or C) = 50+20+9-10-4-1=64. The number of integers not divisible by 2,5, or 11 = 100-64=36.

2.

a. Each pen has a choice of going to person A or person B. By the fundamental principle of counting, there are 2x2x2x2 =16 ways of distributing 4 different color pens to 2 different people.

b. Out of the sixteen cases obtained in part a, there will be two cases where one person gets all the pens, i.e. person A gets all four pens, or person B gets all four pens. Therefore, there will be 16-2=14 cases where each person will receive at least one pen.

3. There are 10x10=100 potential addresses that can be constructed using two digits, however since 00 is not an address, there are 100-1=99 different addresses on Memory Lane.

4.

a. The first part is a simple application of the pigeon hole principle. This principle states that if you try to put n+1 pigeons into n holes, then at least one hole will have more than one pigeon. If we let the 12 months of the year be the holes, and the 13 people be the pigeons, then by applying the pigeon hole principle, one of months (holes) will have two people who have a birthday in it.

b. Since the numbers {1,2,…,10} are either even or odd, it is necessary to break this down into cases.

case 1. 5 even integers, and 1 odd integer. Since we picked all the even integers, 2 divides 4, 6,8,10. We have a number that divides another exactly.

case 2. 4 even integers, and 2 odd integers. If **2** and **1** are selected then we are done. If we don't have **2** , then the four even integers left are 4,6,8,10, and since 4 divides 8 exactly, we have a number which divides another exactly.

case 3. 2 even integers and 4 odd integers. If **2** and **1** are selected then we are done (i.e. we have an integer which divides another one exactly). If we don't have **1**, then the four odd integers left are 3,5,7,9, and since 3 divides 9 exactly, we have a number which divides another exactly.

case 4. 1 even integer and 5 odd integers. Since all the odd integers have been picked, the number **1** has been picked and **1** divides every integer exactly.

case 5. 3 even integers and 3 odd integers. This is the most difficult case. If **2** and **1** are selected then we are done. We are now working with the four even integers, 4,6,8,10 and the four odd integers, 3,5,7,9. There are four ways to select three integers from a set of four integers. We could examine the 4x4=16 cases that arise, and discover that in each case, there will be one integer that divides another exactly, however with some selectivity, we can shorten this process. Since 4 divides 8 and 3 divides 9, it follows that we don't need to have 4 and 8, or 3 and 9 together when we select three even and three odd integers. If they are together then there is an integer which divides another one exactly. The following are the remaining cases:

i. {4,6,10}, {3,5,7}- 3 divides 6 exactly.
ii. {4,6,10}, {5,7,9}- 5 divides 10 exactly.
iii. {6,8,10}, {3,5,9}- 3 divides 6 exactly.
iv. {6,8,10},{5,7,9}- 5 divides 10 exactly.

5. Let teams A,B,C, and D play in a round robin tournament. The following chart gives an example of what could happen.

	A	B	C	D	Points
A	--	W	W	W	3
B	L	--	L	W	1
C	L	W	--	W	2
D	L	L	L	--	0
				Total	6

With four teams in a round robin tournament there will be 6 games played. As a result of this there will be 6 wins and 6 losses arising. The six wins will account for a total of 6 points. Hence the sum of the scores doesn't depend upon the outcome of the games.

6. For any arrangement of men on the dance floor, there are 5x4x3x2x1=5!=120 ways the women can be matched with the men. Another approach to this is to pick the 5 couples first, i.e. 5x5x4x4x3x3x2x2x1x1=5!x5! ways, and then realize that it doesn't matter what order they are put on the dance floor. Either way, we get $\frac{5!x5!}{5!} = 5! = 120$.

7. Any time person j shakes hands with person k , the number two will be added to the total number of handshakes. Therefore the total number of handshakes must be even. However, the total number of a handshakes = the sum of all the handshakes made by each person. Let x(j)= the number of handshakes made by person j. $\sum_{j=1}^{6} x(j) =$ total number of handshakes.

Let S = {j: x(j) shakes hands an odd number of times}.
<u>Claim:</u> Card(S) is even.

<u>Proof:</u> If card (S) is odd, then the number of people who shake hands an odd number of times is odd. These people will contribute an odd number of handshakes to the total number of handshakes because the sum of an odd number of odd numbers is odd (try and prove this). The remaining people shake hands an even number of times and will contribute an even number to the total number of handshakes. Thus, the total number of handshakes = even + odd = odd number, contradicting that the total number of handshakes is even.

8. With one cut, there will be two slices.
You want the second cut to intersect the first cut so you will get 4 slices. If not then you will get 3 slices.

You will want the third cut not to pass through the intersection of the first two cuts in order to get the maximum of seven slices. If the third cut passes through the first two cuts , then you will get six slices.

According to this scheme, the maximum number of slices is obtained when no three lines are concurrent(i.e. three lines do not intersect in a point). Note that when the third line cuts the other two lines, three new regions (pizza slices) are created if it doesn't cross through the intersection. The fourth cut should cross the other three in a way such that no three lines are concurrent. This will create 4 new regions (pizza slices) and yield a maximum of 11 slices.

Finally, the fifth cut should cross the other four cuts in a way such that no three lines are concurrent. This will create 5 new regions (pizza slices) and yield a maximum of 16 slices.

In general, let a(n) = maximum number of slices that can be obtained from a pizza when n cuts are made. Our analysis has shown that, a(1)=2, a(2)=a(1)+2=4, a(3)=a(2)+3=7, a(4)=a(3)+4=11, a(5)=a(4)+5=16,…, a(n)=a(n-1)+n. This recursion can be solved, and you will learn how to do this later in the course. For now, I will give you the answer, it is

$a(n) = 1 + \dfrac{n^2+n}{2}$. Try and prove this by induction.

9.

$A_n = \{n, n+1, \ldots, 2n+1\}$. Find $\bigcup_{n=0}^{\infty} A_n$

claim: $\bigcup_{n=0}^{\infty} A_n = \{0,1,2,\ldots,\} =$ Whole numbers

Proof:

$\Rightarrow \bigcup_{n=0}^{\infty} A_n \subseteq \{0,1,2,\ldots\}$

Let $x \in \bigcup_{n=0}^{\infty} A_n$. $x \in A_{n_0}$ for some $n = n_0$. This means that

$x \in \{n_0, n_0+1, \ldots, 2n_0+1\}$, x is a positive integer and $x \in \{0,1,2,\ldots\}$.

$\Leftarrow \{0,1,2,\ldots\} \subseteq \bigcup_{n=0}^{\infty} A_n$. Let $n \in \{0,1,2,\ldots\}$. By the definition of A_n,

$A_n = \{n, n+1, \ldots, 2n+1\}$, $n \in A_n$, and hence $n \in \bigcup_{n=0}^{\infty} A_n$.

Claim: $\bigcap_{n=0}^{\infty} A_n = \emptyset$.

Proof:

If not, suppose $m \in \bigcap_{n=0}^{\infty} A_n$. This means that $m \in A_n$ for all n. However,

$m \notin A_{m+1} = \{m+1, m+2, \ldots, 2(m+1)+1\}$. This is a contradiction.

10.

a. X and Y are finite sets. Suppose $f: X \to Y$ is one to one. It follows from this that card (f(X)) = card(X). Since $f(X) \subseteq Y$, card (f(X)) $\leq$ card (Y). Putting these facts together, card(X) = card(f(X)) $\leq$ card(Y).

b. Suppose $f: X \to Y$ is onto, i.e. f(X) = Y.
claim: card (f(X)) $\leq$ card (X).
proof:
Let $X = \{x_1, x_2, \ldots, x_n\}$. $f(X) = \{f(x_1), f(x_2), \ldots, f(x_n)\}$. Even if all of the $f(x_i)$ are distinct, card(f(X)) is at most card (X). Hence card (f(X)) $\leq$ card(X).
With these facts, it now follows that card(Y) = card(f(X)) $\leq$ card(X).

c. Suppose $f: X \to Y$ is one to one and onto. From part(a), card (X) $\leq$ card(Y) and from part b, card (Y) $\leq$ card (X). Putting these together, card (X) = card (Y).

d. $\Rightarrow$ Suppose card (X) = card (Y) and $f: X \to Y$ is one to one. We want to prove that f is onto, i.e. f(X) = Y. Since $f: X \to Y$ is one to one, card (X) = card(f(X)).
By hypothesis, card (X) = card(Y), and since $f(X) \subseteq Y$, with card (f(X)) = card (Y), it follows that f(X) can't be a proper subset, and we must conclude f(X) = Y.
$\Leftarrow$ Suppose $f: X \to Y$ is onto, i.e. f(X) = Y and card(X) = card(Y). We want to prove that $f: X \to Y$ is one to one. Card (f(X)) = card(Y) = card(X). If card(f(X)) = card(X), then the elements of $\{f(x_1), f(x_2), \ldots, f(x_n)\}$ must be distinct. If not then for some i, and j, $f(x_i) = f(x_j)$, and card(f(X)) < n = card (X). It now follows that $f: X \to Y$ is one to one.

11.
It is not hard to see that n must be at least 10, in order to for $n^3 < 2^n$, because $n^3 > 2^n$ for $n = 1, 2, \ldots, 9$. From the graph of $y = \log_2(x)$, $\log_2(x) > 1$ for $x > 2$, so $\log_2(n) > 1$ if $n > 10$. $\mathrm{Log}_2(n) < n$ because $n < 2^n$ (you can prove this by induction).
$n < n\log_2(n)$ because $\log_2(n) > 1$ for $n \geq 10$. $n\log_2(n) < n x n = n^2 < n^3$. We still need to show that $n^3 < 2^n$ for $n \geq 10$. It's not easy to do this by induction, so we will use a different method. Since $\log_2(x)$ is an increasing function, it suffices to prove that $\log_2(n^3) = 3\log_2(n) < \log_2(2^n) = n$. Use calculus. Define $f(x) = x - 3\log_2(x)$, show that f(x) is increasing for $x \geq 10$. $f'(x) = 1 - \frac{3}{x}\frac{1}{\ln(2)}$. For $x \geq 10$, $f'(x) \geq 0$.