Almost Sure Well-Posedness of Cubic NLS on the Torus Below L^2

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joint work with Tadahiro Oh (U. Toronto)

1. Introduction: Background, Motivation, New Results

Cubic Nonlinear Schrödinger Equation

Consider the following Cauchy problem:

$$\begin{cases} iu_t - u_{xx} \pm u|u|^2 = 0 \\ u|_{t=0} = u_0, \ x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \end{cases} \qquad (NLS_3^\pm(\mathbb{T}))$$

with random initial data below $L^2(\mathbb{T})$.

Main Goals:

- Establish almost sure LWP with initial data u_0 of the form

$$u_0(x) = u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle |n|^\alpha \rangle} e^{inx}, \quad \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$$

  $\{g_n\}_{n\in\mathbb{Z}}$ = standard $\mathbb{C}$-valued Gaussians on $(\Omega, \mathcal{F}, \mathbb{P})$.

- Extend local-in-time solutions to global-in-time solutions **without** an available invariant measure.

Well-Posedness vs. Ill-Posedness Threshold Heuristics

- Dilation Symmetry and Scaling Invariant Sobolev Norm

$NLS_3(\mathbb{R})$: Any solution u spawns a family of solutions

$$u_\lambda(t,x) = \frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}).$$

$$\|D_x^s u_\lambda(t)\|_{L^2} = \lambda^{s+\frac{1}{2}} \| = \|D_x^s u(t)\|_{L^2}.$$

The dilation invariant Sobolev index $s_c = -\frac{1}{2}$.
We expect $NLS_3(\mathbb{T})$ is *ill-posed* for $s < -\frac{1}{2}$.

- Galilean Invariance and Galilean Invariant Sobolev Norm

The galilean symmetry leaves the L^2 norm invariant.
We expect that $NLS_3(\mathbb{T})$ is *ill-posed* for $s < 0$.

Well-Posedness Speculations on $NLS_3^{\pm}(\mathbb{T})$ below L^2

Known Results:

- $NLS_3(\mathbb{T})$ is globally well-posed in $L^2(\mathbb{T})$. [B93]
- Data-solution map $H^s \ni u_0 \longmapsto u(t) \in H^s$ not uniformly continuous for $s < 0$. [BGT02], [CCT03], [M08].
- Data-solution map unbounded on H^s for $s < -\frac{1}{2}$. [CCT03] "Norm Inflation"
- A priori (local-in-time) bound on $\|u(t)\|_{H^s(\mathbb{T})}$ AND weak solutions without uniqueness for $s \geq -\frac{1}{6}$ (CHT09). (Similar prior work on $NLS_3(\mathbb{R})$ [CCT06], [KT06].)

Speculations?

1. No norm inflation for $NLS_3(\mathbb{R})$ and $NLS_3(\mathbb{T})$ in H^s for $s > -\frac{1}{2}$?
2. LWP (merely continuous dependence on data) in H^s for $s > -\frac{1}{2}$?

(Finite Dimensional) Invariant Gibbs Measures

A function $H : \mathbb{R}^n_p \times \mathbb{R}^n_q \to \mathbb{R}$ induces Hamiltonian flow on $\mathbb{R}^{2n}$:

$$\begin{cases} \dot{p} = \nabla_q H \\ \dot{q} = -\nabla_p H \end{cases} \qquad \text{(Hamilton's Equation)}$$

- The vector field $X = (\nabla_q H, -\nabla_p H)$ is divergence free:

  $$\mathrm{div}_{\mathbb{R}^{2n}} X = (\nabla_p, \nabla_q) \cdot (\nabla_q H, -\nabla_p H) = 0.$$

  Thus, Lebesgue measure $\prod_{j=1}^n dp_j dq_j$ is invariant under the Hamiltonian flow. (Liouville's Theorem) Poincaré Recurrence

- $H(p(t), q(t))$ is invariant under the flow:

  $$\frac{d}{dt} H(p(t), q(t)) = \nabla_p H \cdot \dot{p} + \nabla_q H \cdot \dot{q} = 0.$$

  (Hamiltonian Conservation)

(Finite Dimensional) Invariant Gibbs Measures

We can combine Hamiltonian conservation and Lebesgue measure invariance to build other flow-invariant measures:

$$d\mu_f(p, q) = f(H(p, q)) \prod_{j=1}^{n} dp_j dq_j.$$

The *Gibbs measure* arises when we choose f to be a Gaussian and normalize it to have total measure 1:

$$d\mu = Z^{-1} e^{-H(p,q)} \prod_{j=1}^{n} dp_j dq_j.$$

The Gibbs measure can be shown to be well-defined in infinite dimensions even though the Lebesgue measure can't be.

Invariant Gibbs Measures for $NLS_3(\mathbb{T})$

- Time Invariant Quantities for $NLS_3^{\pm}$ flow:

$$\text{Mass} = \int_{\mathbb{T}} |u(t,x)|^2 dx.$$

$$\text{Energy} = H[u(t)] = \int_T \frac{1}{2}|\nabla u(t)|^2 dx \pm \frac{1}{4}|u(t)|^4 dx.$$

- The Gibbs measure associated to $NLS_3^{\pm}(\mathbb{T})$,

$$d\mu = Z^{-1} e^{-\frac{1}{2}\int |u|^2 dx - \frac{1}{2}\int |u_x|^2 dx \mp \frac{1}{4}\int |u|^4 dx} \prod_{x\in\mathbb{T}} du(x),$$

(with an appropriate L^2 cutoff in the focusing case) is normalizable and invariant under $NLS_3(\mathbb{T})$ flow. [B94]

- Gibbs measure is absolutely cts. w.r.t. *Wiener Measure*

$$d\rho_1 = Z_1^{-1} e^{-\frac{1}{2}\int |\partial_x^1 u|^2 dx} \prod_{x\in\mathbb{T}} du(x).$$

Invariant Gibbs Measures for $NLS_3^+(\mathbb{T}^2)$

- LWP for $NLS_3^+(\mathbb{T}^2)$ is known for $H^{0+}(\mathbb{T}^2)$ (not L^2).[*B*93]
- For $\mathbb{T}^2$, Wiener measure is supported on $H^{0-}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$.
- Nevertheless, the Gibbs measure for the defocusing *Wick ordered* cubic NLS on $\mathbb{T}^2$

$$iu_t - \Delta u + (u|u|^2 - 2u \fint |u|^2 dx) = 0 \qquad (WNLS(\mathbb{T}^2))$$

  was normalized and proved to be flow-invariant. [B96]
- $WNLS(\mathbb{T}^2)$ is GWP on support of Gibbs measure! [B96]

Questions: Is $NLS_3^+(\mathbb{T}^2)$ *ill-posed* on L^2?
Or on any space between the Gibbs measure support and H^{0+}?

Invariant Gibbs Measures for $NLS_3^+(\mathbb{T}^2)$

B96

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Communications in Mathematical Physics

Invariant Measures for the 2*D*-Defocusing Nonlinear Schrödinger Equation

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Random Data Cauchy Theory

Consider the cubic NLW on $\mathbb{T}^3$:

$$\begin{cases} \Box u + u^3 = 0 \\ (u(0), \partial_t u(0)) = (f_0, f_1) \in H^s \times H^{s-1}. \end{cases} \tag{NLW}$$

- This problem is $\dot{H}^{\frac{1}{2}}$-critical. Ill-posedness known for $s < \frac{1}{2}$. Well-posedness is known for $s \geq \frac{1}{2}$.
- Consider the Gaussian randomization map:

$$\Omega \times H^s \ni (\omega, f) \longmapsto f^\omega(x) = \sum_{n \in \mathbb{Z}^3} \widehat{f}(n) g_n(\omega) e^{in \cdot x}.$$

- NLW is almost surely well-posed for randomized supercritical data $(f_0^\omega, f_1^\omega) \in H^s \times H^{s-1}, s \geq \frac{1}{4}$. [BTz]

BTz

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Inventiones mathematicae

Random data Cauchy theory for supercritical wave equations I: local theory

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New Results (joint work with Tadahiro Oh)

Theorem (Almost Sure Local Well-Posedness for $WNLS(\mathbb{T})$)

$WNLS(\mathbb{T})$ is almost surely locally well-posed (with respect to a canonical Gaussian measure) on $H^s(\mathbb{T})$ for $s > -\frac{1}{3}$.

Theorem (Almost Sure Global Well-Posedness for $WNLS(\mathbb{T})$)

$WNLS(\mathbb{T})$ is almost surely globally well-posed (with respect to a canonical Gaussian measure) on $H^s(\mathbb{T})$ for $s > -\frac{1}{8}$.

More Precise Statements Later

2. Canonical Gaussian Measures

Canonical Gaussian Measures on Sobolev spaces

We can regard

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle |n|^\alpha \rangle} e^{inx}$$

as a typical element in the support of the Gaussian measure

$$d\rho_\alpha = Z_\alpha^{-1} e^{-\frac{1}{2}\int |u|^2 dx - \frac{1}{2}\int |D^\alpha u|^2 dx} \prod_{x\in\mathbb{T}} du(x),$$

where $D = \sqrt{-\partial_x^2}$.

We will make sense of $d\rho_\alpha$ using the Gaussian weight even though the Lebesgue measure does not make sense.

Construction of ρ_α

Define the Gaussian measure $\rho_{\alpha,N}$ on $\mathbb{C}^{2N+1}$ by

$$d\rho_{\alpha,N} = Z_N^{-1} e^{-\frac{1}{2}\sum_{|n|\le N}(1+|n|^{2\alpha})|\widehat{u}_n|^2} \prod_{|n|\le N} d\widehat{u}_n.$$

We may view $\rho_{\alpha,N}$ as induced probability measure on $\mathbb{C}^{2N+1}$ under the map

$$\omega \to \left\{ \frac{g_n(\omega)}{(1+|n|^{2\alpha})^{\frac{1}{2}}} \right\}_{|n|\le N}$$

Question: As $N \to \infty$, where does the limit $\rho_\alpha = \lim_{N\to\infty} \rho_{\alpha,N}$ make sense as a *countably additive* probability measure?

Gaussian Measures on Hilbert Spaces

H, *real* separable Hilbert space

$B : H \to H$, linear, positive, self-adjoint operator

$\{e_n\}_{n=1}^{\infty}$, eigenvectors of B forming an O.N. basis of H

$\{\lambda_n\}_{n=1}^{\infty}$, corresponding eigenvalues

- Define

$$\rho_N(M) = (2\pi)^{-\frac{N}{2}} \Big(\prod_{n=1}^{N} \lambda_n^{-\frac{1}{2}} \Big) \int_F e^{-\frac{1}{2}\sum_{n=1}^{N} \lambda_n^{-1} x_n^2} \prod_{n=1}^{N} dx_n$$

where $\begin{cases} M = \{x \in H : (x_1, \cdots, x_N) \in F\}, \;\; F, \;\; \text{Borel set in } \mathbb{R}^N \\ x_n = \langle x, e_n \rangle = j\text{th coordinate of } x. \end{cases}$

Thus, ρ_N is a Gaussian measure on $E_N = \text{span}\{e_1, \cdots, e_N\}$.

- Now, define ρ on H by $\rho\big|_{E_N} = \rho_N$.

Gaussian Measures on Hilbert Spaces

Fact

(1) ρ is countably additive if and only if B is of trace class, i.e. $\sum \lambda_n < \infty$

(2) If (1) holds, then $\rho_N \rightharpoonup \rho$ as $N \to \infty$

Now, let $B = \mathrm{diag}(1 + |n|^{2\sigma} : n \neq 0)$. Then, we have

$$-\frac{1}{2} \sum_{|n| \leq N} (1 + |n|^{2\alpha}) |\widehat{u}_n|^2 \sim -\frac{1}{2} \sum_{|n| \leq N} (1 + |n|^{-2\sigma})(1 + |n|^{2(\sigma+\alpha)}) |\widehat{u}_n|^2$$

$$= -\frac{1}{2} \langle B^{-1} \widehat{u}_n, \widehat{u}_n \rangle_{H^{\sigma+\alpha}}.$$

B is of trace class iff $\sum_{n \in \mathbb{Z}} \langle n \rangle^{2\sigma} < \infty$ iff $\sigma < -\frac{1}{2}$.

Thus $\rho_\alpha = \lim_{N \to \infty} \rho_{\alpha,N}$ defines countably additive measure on

$$H^{\alpha - \frac{1}{2} -} = \cap_{s < \alpha - \frac{1}{2}} H^s \setminus H^{\alpha - \frac{1}{2}}.$$

Typical Element as Gaussian Random ~~Fourier~~ Series

Typical elements in the support of ρ_α may be represented

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e_n(x), \quad \text{where } e_n(x) = \frac{1}{\langle |n|^\alpha \rangle} e^{inx}$$

- Given another O.N. basis $\{\widetilde{e}_n\}$ of $H^\alpha(\mathbb{T})$, we have

  $$u_0^\omega = \sum_n \widetilde{g}_n(\omega)\widetilde{e}_n,$$

  where $\{\widetilde{g}_n\}$ is another family of i.i.d. standard Gaussians.
- u_0 can be regarded as Gaussian randomization of the function ϕ with Fourier coefficient $\widehat{\phi}_n = \frac{1}{\langle |n|^\alpha \rangle}$.
  - $\phi \in H^s$, $s < \alpha - \frac{1}{2}$ but not in $H^{\alpha - \frac{1}{2}}$.
  - Gaussian randomization of the Fourier coefficients does *not* give *any* smoothing a.s. (in the Sobolev scale.) Hence,

    $$\operatorname{supp}(\rho_\alpha) \subset \cap_{s < \alpha - \frac{1}{2}} H^s \setminus H^{\alpha - \frac{1}{2}}.$$

3. Wick Ordered Cubic NLS

WNLS instead of *NLS*

- *WNLS* is equivalent to $NLS_3(\mathbb{T})$ when $u_0 \in L^2(\mathbb{T})$.
  - Note that $\mu = \fint |u|^2 dx = \frac{1}{2\pi} \int |u|^2 dx$ is conserved.
  - Now set $v \to e^{\pm 2i\mu t} u$ to see that $NLS_3^{\pm}$ is equivalent to the *Wick ordered cubic NLS*:

$$\begin{cases} iv_t - v_{xx} \pm \left(v|v|^2 - 2v \fint |v|^2 dx\right) = 0 \\ v|_{t=0} = u_0 = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{|n|^\alpha} e^{inx} \in H^{\alpha-\frac{1}{2}-}. \end{cases} \tag{WNLS}$$

- *WNLS* is inequivalent to NLS_3 for $u_0 \in H^s$ with $s < 0$. Indeed, we can't define the phase μ in this case.

We *choose* to consider *WNLS* for $\alpha < \frac{1}{2}$ (below $L^2(\mathbb{T})$) *instead* of NLS_3.

Ill-Posedness of *WNLS* below $L^2(\mathbb{T})$

- The [BGTz] example $u_{N,a}(x,t) = ae^{i(Nx+N^2t\mp|a|^2t)}$, with $a \in \mathbb{C}$ and $N \in \mathbb{N}$, still solves *WNLS*. Thus, uniform continuity of the flow map for *WNLS* fails in H^s, $s < 0$.
- Molinet's ill-posedness result does not apply to *WNLS*.
- Local-in-time solutions to (WNLS) in $\mathcal{F}L^p \supset L^2(\mathbb{T})$, $2 < p < \infty$, by a power series method. [Christ07] Uniqueness is unknown.

4. Almost Sure Local Well-Posedness

Nonlinear smoothing under randomization

Duhamel formulation of *WNLS*:

$$u(t) = \Gamma u(t) := S(t)u_0 \pm i \int_0^t S(t-t')\mathcal{N}(u)(t')dt'$$

where $S(t) = e^{-i\partial_x^2 t}$ and $\mathcal{N}(u) = u|u|^2 - 2u \fint |u|^2$.

- $S(t)u_0$ has the same regularity as u_0 for each fixed $t \in \mathbb{R}$. i.e. $S(t)u_0^\omega \in H^{\alpha-\frac{1}{2}-} \setminus H^{\alpha-\frac{1}{2}}$ a.s., below L^2 for $\alpha \le \frac{1}{2}$.
- Main Observation: By nonlinear smoothing, Duhamel term

$$\int_0^t S(t-t')\mathcal{N}(u)(t')dt' \in L^2(\mathbb{T})$$

  even when $\alpha \le \frac{1}{2}$.
- This permits us to run a contraction argument in $X^{s,b}$.

Contraction around random linear solution

- Let Γ denote the right hand side of Duhamel's formula.
- For each small $\delta > 0$, we construct $\Omega_\delta \subset \Omega$ with $\rho_\alpha(\Omega_\delta^C) < e^{-\frac{1}{\delta^c}}$ so that Γ defines a contraction on $S(t)u_0^\omega + B$ on $[-\delta, \delta]$ for all $\omega \in \Omega_\delta$, where B denotes the ball of radius 1 in the Bourgain space $X^{0,\frac{1}{2}+,\delta}$.
- Recall the Bourgain space $X^{s,b}(\mathbb{T} \times \mathbb{R})$ defined by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{L^2_n L^2_\tau(\mathbb{Z}\times\mathbb{R})}$$

  and its local-in-time version $X^{s,b,\delta}$ on the time interval $[\delta, \delta]$.
- For $\omega \in \Omega_\delta$ with $\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^c}}$, it suffices to prove

$$\|\mathcal{N}(u)\|_{X^{0,-\frac{1}{2}+,\delta}} \lesssim \delta^\theta, \quad \theta > 0$$

Theorem (Almost Sure LWP for *WNLS*)

Let $\alpha > \frac{1}{6}$. Then, *WNLS* is LWP almost surely in $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$.
More precisely, there exist $c > 0$ such that for each $\delta \ll 1$, there exists a set $\Omega_T \in \mathcal{F}$ with the following properties:

(i) $\mathbb{P}(\Omega_T^c) = \rho_\alpha \circ u_0(\Omega_T^c) < e^{-\frac{1}{\delta^c}}$, where $u_0 : \Omega \to H^{\alpha-\frac{1}{2}-}(\mathbb{T})$.

(ii) $\forall\, \omega \in \Omega_T \ \exists$ a unique solution u of *WNLS* in

$$S(t)u_0^\omega + C([-\delta,\delta]; L^2(\mathbb{T})) \subset C([-\delta,\delta]; H^{\alpha-\frac{1}{2}-}(\mathbb{T})).$$

In particular, we have almost sure LWP with respect to the Gaussian measure ρ_α supported on $H^s(\mathbb{T})$ for each $s > -\frac{1}{3}$.

Invariance of White Noise under *WNLS* flow?

- When $\alpha = 0$, ρ_0 corresponds to the white noise

$$d\rho_0 = Z_0^{-1} e^{-\frac{1}{2}\int |u|^2 dx} \prod_{x \in \mathbb{T}} du(x)$$

  which is supported on $H^{-\frac{1}{2}-}$ and is formally invariant.
- Our results are partial results in this direction.
- White noise invariance for KdV has been established. $[QV], [OQV]$

Decompose Nonlinearity: Wick Order Cancellation

We write the nonlinearity $\mathcal{N}(u)$ as

$$\mathcal{N}(u) = u|u|^2 - 2u \fint |u|^2 = \mathcal{N}_1(u,u,u) - \mathcal{N}_2(u,u,u)$$

where

$$\begin{cases} \mathcal{N}_1(u_1,u_2,u_3)(x) = \sum_{n_2 \neq n_1, n_3} \widehat{u}_1(n_1)\overline{\widehat{u}_2(n_2)}\widehat{u}_3(n_3)e^{i(n_1-n_2+n_3)x} \\ \mathcal{N}_2(u_1,u_2,u_3)(x) = \sum_n \widehat{u}_1(n)\overline{\widehat{u}_2(n)}\widehat{u}_3(n)e^{inx}. \end{cases}$$

Main Goal: Prove the trilinear estimates ($j = 1, 2$):

$$\|\mathcal{N}_j(u_1,u_2,u_3)\|_{X^{0,-\frac{1}{2}+,\delta}} \lesssim \delta^\theta, \quad \theta > 0.$$

Random-Linear vs. Deterministic-Smooth

We analyze the $\mathcal{N}_1, \mathcal{N}_2$ contributions leading to

$$\|\mathcal{N}_j(u_1, u_2, u_3)\|_{X^{0,-\frac{1}{2}+,\delta}} \lesssim \delta^{\theta}, \;\; \theta > 0,$$

by assuming u_j is one of the following forms:

- Type I (random, linear, rough):

$$u_j(x,t) = \sum_n \frac{g_n(\omega)}{|n|^{\alpha}} e^{i(nx+n^2t)} \in H^{\alpha-\frac{1}{2}-}$$

- Type II (deterministic, smooth):

$$u_j \;\text{ with } \|u_j\|_{X^{0,\frac{1}{2}+,\delta}} \leq 1.$$

This is done using a case-by-case analysis. (similar to [B96])
Trilinear terms: I, I, I; I, I, II; I, II, II; etc.

Inputs Used in Trilinear Analysis

- The $X^{s,b}$ machinery, dyadic decomposition, standard stuff.
- Algebraic identity, divisor estimate: For $n = n_1 - n_2 + n_3$,

$$n^2 - (n_1^2 - n_2^2 + n_3^2) = 2(n_2 - n_1)(n_2 - n_3).$$

- Strichartz controls on deterministed Type II terms.
  The periodic L^4-Strichartz

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{3}{8}}}.$$

  and interpolations with the trivial estimate

$$\|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}}.$$

- Probabilistic estimates on Type I terms.

  Randomizing Fourier coefficients does not lead to more smoothness but does lead to improved L^p properties.

Large Deviation Estimate

Lemma

Let $f^\omega(x,t) = \sum c_n g_n(\omega) e^{i(nx+n^2t)}$, where $\{g_n\}$ is a family of complex valued standard i.i.d. Gaussian random variables. Then, for $p \geq 2$, there exists δ, $T_0 > 0$ such that

$$\mathbb{P}(\|f^\omega\|_{L^p(\mathbb{T}\times[-T,T])} > C\|c_n\|_{l^2_n}) < e^{-\frac{c}{T^\delta}}$$

for $T \leq T_0$.

This gives good L^p control on the Type I terms.

5. Almost Sure Global Well-Posedness

We establish almost sure global well-posedness for *WNLS* by adapting Bourgain's high/low Fourier truncation method.

Overview of Discussion:

1. Precise statement of almost sure GWP result.
2. Describe high/low Fourier truncation method for GWP.
3. Explain adaptation to prove almost sure GWP.

Theorem (Almost Sure GWP for *WNLS*)

Let $\alpha \in (\frac{3}{8}, \frac{1}{2}]$. Then, *WNLS* is LWP almost surely in $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$. More precisely, for almost every $\omega \in \Omega$ $\exists!$ solution

$$u \in e^{-it\partial_x^2} u_0^\omega + C(\mathbb{R}; H^{\alpha-\frac{1}{2}}(\mathbb{T}))$$

of *WNLS* with initial data given by the random Fourier series

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{1+|n|^\alpha} e^{inx}.$$

In particular, we have almost sure global well-posedness with respect to the Gaussian measure supported on $H^s(\mathbb{T}), s > -\frac{1}{8}$.

Bourgain's High-Low Fourier Truncation

B98

IMRN International Mathematics Research Notices
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Refinements of Strichartz' Inequality and Applications to 2D-NLS with Critical Nonlinearity

J. Bourgain

Summary

Consider the 2D IVP

$$\begin{cases} iu_t + \Delta u + \lambda |u|^2 u = 0 \\ u(0) = \varphi \in L^2(\mathbb{R}^2). \end{cases} \qquad (\dagger)$$

The theory on the Cauchy problem asserts a unique maximal solution

$$u \in \mathcal{C}(]-T_*, T^*[; L^2(\mathbb{R}^2)) \cap L^4(]-T_*, T^*[; L^4(\mathbb{R}^2))$$

Bourgain's High-Low Fourier Truncation

Consider the Cauchy problem for defocusing cubic NLS on $\mathbb{R}^2$:

$$\begin{cases} (i\partial_t + \Delta)u = +|u|^2 u \\ u(0,x) = \phi_0(x). \end{cases} \qquad (NLS_3^+(\mathbb{R}^2))$$

We describe the first result to give GWP below H^1.

- $NLS_3^+(\mathbb{R}^2)$ is GWP in H^s for $s > \frac{2}{3}$ [Bourgain 98].
- Proof cuts solution into low and high frequency parts.
- For $u_0 \in H^s,\ s > \frac{2}{3}$, Proof gives (and crucially exploits),

$$u(t) - e^{it\Delta}\phi_0 \in H^1(\mathbb{R}^2_x).$$

Setting up; Decomposing Data

- Fix a large target time T.
- Let $N = N(T)$ be large to be determined.
- Decompose the initial data:

$$\phi_0 = \phi_{low} + \phi_{high}$$

where

$$\phi_{low}(x) = \int_{|\xi|<N} e^{ix\cdot\xi}\widehat{\phi_0}(\xi)d\xi.$$

- Our plan is to evolve:

$$\begin{array}{ccccc} \phi_0 & = & \phi_{low} & + & \phi_{high} \\ \Big\downarrow \boxed{\text{NLS}} & & \Big\downarrow \boxed{\text{NLS}} & & \Big\downarrow \boxed{\text{DE}} \\ u(t) & = & u_{low}(t) & + & u_{high}(t). \end{array}$$

Setting up; Decomposing Data

Low Frequency Data Size:

- Kinetic Energy:

$$\begin{aligned} \|\nabla \phi_{low}\|_{L^2}^2 &= \int_{|\xi|<N} |\xi|^2 |\widehat{\phi_0}(\xi)|^2 dx \\ &= \int_{|\xi|<N} |\xi|^{2(1-s)} |\xi|^{2s} |\widehat{\phi_0}(\xi)|^2 dx \\ &= N^{2(1-s)} \|\phi_0\|_{H^s}^2 \leq C_0 N^{2(1-s)}. \end{aligned}$$

- Potential Energy: $\|\phi_{low}\|_{L_x^4} \leq \|\phi_{low}\|_{L^2}^{1/2} \|\nabla \phi_{low}\|_{L^2}^{1/2}$

$$\implies H[\phi_{low}] \leq C N^{2(1-s)}.$$

High Frequency Data Size:

$$\|\phi_{high}\|_{L^2} \leq C_0 N^{-s}, \ \|\phi_{high}\|_{H^s} \leq C_0.$$

LWP of Low Frequency Evolution along NLS

The NLS Cauchy Problem for the low frequency data

$$\begin{cases} (i\partial_t + \Delta)u_{low} = +|u_{low}|^2 u_{low} \\ u_{low}(0,x) = \phi_{low}(x) \end{cases}$$

is well-posed on $[0, T_{lwp}]$ with $T_{lwp} \sim \|\phi_{low}\|_{H^1}^{-2} \sim N^{-2(1-s)}$.

We obtain, as a consequence of the local theory, that

$$\|u_{low}\|_{L^4_{[0,T_{lwp}],x}} \leq \frac{1}{100}.$$

LWP of High Frequency Evolution along DE

The NLS Cauchy Problem for the low frequency data

$$\begin{cases} (i\partial_t + \Delta) u_{high} = +2|u_{low}|^2 u_{high} + \text{similar} + |u_{high}|^2 u_{high} \\ u_{high}(0, x) = \phi_{high}(x) \end{cases}$$

is also well-posed on $[0, T_{lwp}]$.

Remark: The LWP lifetime of *NLS* evolution of u_{low} AND the LWP lifetime of the *DE* evolution of u_{high} are controlled by $\|u_{low}(0)\|_{H^1}$.

Extra Smoothing of Nonlinear Duhamel Term

The high frequency evolution may be written

$$u_{high}(t) = e^{it\Delta} u_{high} + w.$$

The local theory gives $\|w(t)\|_{L^2} \lesssim N^{-s}$. Moreover, due to smoothing (obtained via bilinear Strichartz), we have that

$$w \in H^1, \ \|w(t)\|_{H^1} \lesssim N^{1-2s+}. \qquad \text{(SMOOTH!)}$$

Let's assume (SMOOTH!).

Nonlinear High Frequency Term Hiding Step!

- $\forall\, t \in [0, T_{lwp}]$, we have

$$u(t) = u_{low}(t) + e^{it\Delta}\phi_{high} + w(t).$$

- At time T_{lwp}, we define data for the progressive sheme:

$$u(T_{lwp}) = u_{low}(T_{lwp}) + w(T_{lwp}) + e^{iT_{lwp}\Delta}\phi_{high}.$$

$$u(t) = u^{(2)}_{low}(t) + u^{(2)}_{high}(t)$$

for $t > T_{lwp}$.

Hamiltonian Increment: $\phi_{low}(0) \longmapsto u^{(2)}_{low}(T_{lwp})$

The Hamiltonian increment due to $w(T_{lwp})$ being added to low frequency evolution can be calcluated. Indeed, by Taylor expansion, using the bound (SMOOTH!) and energy conservation of u_{low} evolution, we have using

$$
\begin{aligned}
H[u^{(2)}_{low}(T_{lwp})] &= H[u_{low}(0)] + (H[u_{low}(T_{lwp}) + w(T_{lwp})] - H[u_{low}(T_{lwp})]) \\
&\sim N^{2(1-s)} + N^{2-3s+} \sim N^{2(1-s)}.
\end{aligned}
$$

Moreover, we can accumulate N^s increments of size N^{2-3s+} before we double the size $N^{2(1-s)}$ of the Hamiltonian. During the iteration, Hamiltonian of "low frequency" pieces remains of size $\lesssim N^{2(1-s)}$ so the LWP steps are of uniform size $N^{-2(1-s)}$. We advance the solution on a time interval of size:

$$N^s N^{-2(1-s)} = N^{-2+3s}.$$

For $s > \frac{2}{3}$, we can choose N to go past target time T.

Why did the scheme progress?

Along the the time steps $T_{lwp}, 2T_{lwp}, \ldots, \lfloor N^s \rfloor T_{lwp}$,
the low and high frequency data have uniform properties:

- High frequency Duhamel term small in H^1.
- Low frequency data: Hamiltonian Conservation!
- High frequency data: Linear!

For almost sure GWP result, similar scheme progresses:

- High frequency Duhamel term small in L^2.
- Low frequency data: Mass Conservation!
- High frequency data: Linear!
  $\Longrightarrow$ uniform Gaussian probability bounds.

Adaptation for Almost Sure GWP Proof

Adaptation for Almost Sure GWP Proof

We are studying the Cauchy problem for *WNLS*

$$\begin{cases} iv_t - v_{xx} \pm (v|v|^2 - 2v \fint |v|^2 dx) = 0 \\ v|_{t=0} = u_0 = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{|n|^\alpha} e^{inx} \in H^{\alpha - \frac{1}{2} -}. \end{cases} \tag{WNLS}$$

Let $s = \alpha - \frac{1}{2} < 0$. By a large deviation estimate,

$$\mathbb{P}(\|u_0^\omega\|_{H^s} > K) \le e^{-cK^2}.$$

Restrict to $u_0^\omega \in \Omega_K = \{\omega : \|u_0^\omega\|_{H^s} < K\}$. Eventually, $K \nearrow \infty$.

Let $\phi_0 = \mathbb{P}_{|\xi|\le N} u_0^\omega$ and $\phi_0 + \psi_0 = u_0$. Low frequency part has

$$\|\phi_0\|_{L^2} \le N^{-s} K.$$

Low Frequency *WNLS* Evolution

Consider the flow of the low frequency part:

$$\begin{cases} i\partial_t u^1 - \partial_x^2 u^1 \pm \mathcal{N}(u^1) = 0 \\ u^1|_{t=0} = \phi_0. \end{cases}$$

- This problem is GWP with $\|u^1(t)\|_{L^2} = \|\phi_0\|_{L^2} \lesssim N^{-s}K$.
- Standard local theory $\implies$ spacetime control:

$$\|u^1\|_{X^{0,\frac{1}{2}+}_{[0,\delta]}} \lesssim \|\phi_0\|_{L^2} \lesssim N^{-s}K,$$

where δ is the time of local existence, i.e.
$\delta = \delta(N^{-s}K) \lesssim \delta(\|\phi_0\|_{L^2})$.

High Frequency *DE* evolution

Consider the difference equation for the high-frequency part:

$$\begin{cases} i\partial_t v^1 - \partial_x^2 v^1 \pm \left(\mathcal{N}(u^1 + v^1) - \mathcal{N}(u^1)\right) = 0 \\ v^1|_{t=0} = \psi_0 = \sum_{|n|>N} \frac{g_n(\omega)}{1+|n|^\alpha} e^{inx}. \end{cases} \qquad (DE)$$

Then, $u(t) = u^1(t) + v^1(t)$ solves *WNLS* as long as v^1 solves *DE*.

- Probabilistic local theory applies to *DE*.
- u^1 is *large* in the $X^{0,\frac{1}{2}+,\delta}$ norm; only quadratic in *DE*.

High/Low Time Step Iteration

- By the probabilistic local theory, we can show that DE is LWP on $[0, \delta]$ except on a set of measure $e^{-\frac{1}{\delta^c}}$. We then obtain

$$v^1(t) = S(t)\psi_0 + w^1(t),$$

where the nonlinear Duhamel part $w^1(t) \in L^2(\mathbb{T})$.

with estimate

- At time $t = \delta$, we hide $w^1(\delta)$ inside ϕ^1:

$$\phi_1 := u^1(\delta) + w^1(\delta) \in L^2$$

$$\psi_1 := S(\delta)\psi_0 = \sum_{|n| \geq N} \frac{g_n e^{in^2\delta}}{|n|^\alpha} e^{inx}.$$

  - Low frequency data ϕ_1 has (essentially) same L^2 size.
  - High frequency data ψ_1 has same bounds as ψ_0.

Measure Zero Issue?

Measure Zero Issue?

?

- Almost sure LWP involves set with nontrivial complement.
- Almost sure GWP claimed almost everywhere w.r.t Ω.
- Measure theory clarifies this apparent contradiction.

Proposition (GWP off small set)

Let $\alpha > \frac{1}{4}$. Given $T > 0, \varepsilon > 0$ (unlinked!), $\exists\, \Omega_{T,\varepsilon} \in \mathcal{F}$ such that:

(i) $\mathbb{P}(\Omega^c_{T,\varepsilon}) = \rho_\alpha \circ u_0(\Omega^c_{T,\varepsilon}) < \varepsilon$, where $u_0 : \Omega \to H^{\alpha-\frac{1}{2}-}(\mathbb{T})$.

(ii) $\forall\, \omega \in \Omega_{T,\varepsilon} \;\exists$ unique solution u of *WNLS* in

$$S(t)u_0 + C([-T,T]; L^2(\mathbb{T})) \subset C([-T,T]; H^{\alpha-\frac{1}{2}-}(\mathbb{T}))$$

Measure Zero Issue?

- For fixed $\gamma > 0$:
  Apply Proposition with $T_j = 2^j$ and $\varepsilon_j = 2^{-j}\gamma$ to get $\Omega_{T_j,\varepsilon_j}$.
- Let $\Omega_\gamma = \bigcap_{j=1}^{\infty} \Omega_{T_j,\varepsilon_j}$.
  WNLS is globally well-posed on Ω_γ with $\mathbb{P}(\Omega_\gamma^c) < \gamma$.
- Now, let $\widetilde{\Omega} = \bigcup_{\gamma>0} \Omega_\gamma$.
  WNLS is GWP on $\widetilde{\Omega}$ and $\mathbb{P}(\widetilde{\Omega}^c) = 0$.