Nonlinear Schrödinger Evolutions from Low Regularity Initial Data

J. Colliander

University of Toronto

PIMS August 2009

1. Cubic NLS Initial Value Problem on $\mathbb{R}^2$

1. Cubic NLS Initial Value Problem on $\mathbb{R}^2$

We consider the initial value problems:

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^{\pm}(\mathbb{R}^2))$$

The $+$ case is called defocusing; $-$ is focusing. $NLS_3^{\pm}$ is ubiquitous in physics. The solution has a dilation symmetry

$$u^{\lambda}(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

which is invariant in $L^2(\mathbb{R}^2)$. This problem is L^2-critical.

(This talk mostly addresses the defocusing case.)

Time Invariant Quantities

$$\boxed{\text{Mass}} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\boxed{\text{Momentum}} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t)\nabla u(t) dx.$$

$$\boxed{\text{Energy}} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx \pm \frac{1}{2}|u(t)|^4 dx.$$

Hamiltonian

Kinetic

Potential

- Mass is $\boxed{L^2}$; Momentum scales $\boxed{H^{1/2}}$; Energy involves $\boxed{H^1}$.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$.
  Monotone or Almost Conserved Localizations?

Linear Schrödinger Propagator and Estimates

The solution of the linear Schrödinger initial value problem

$$\begin{cases} (i\partial_t + \Delta)u = 0 \\ u(0,x) = u_0(x). \end{cases} \qquad (LS(\mathbb{R}^d))$$

is denoted $u(t,x) = e^{it\Delta}u_0$. The solution can be given explicitly

- Fourier Multiplier Representation:

$$e^{it\Delta}u_0(x) = c_\pi \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2} \widehat{u_0}(\xi) d\xi.$$

- Convolution Representation:

$$e^{it\Delta}u_0(x) = c_\pi^1 \frac{1}{(it)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy.$$

LINEAR SCHRÖDINGER PROPAGATOR AND ESTIMATES

The solution of the linear Schrödinger initial value problem

$$\begin{cases} (i\partial_t + \Delta)u = 0 \\ u(0,x) = u_0(x). \end{cases} \qquad (LS(\mathbb{R}^d))$$

is denoted $u(t,x) = e^{it\Delta}u_0$. The solution can be given explicitly

- Fourier Multiplier Representation:

$$e^{it\Delta}u_0(x) = c_\pi \int_{\mathbb{R}^d} e^{ix\cdot\xi}\underline{e^{-it|\xi|^2}}\,\widehat{u_0}(\xi)d\xi.$$

Modulus 1 Multiplier

- Convolution Representation:

$$e^{it\Delta}u_0(x) = c_\pi^1 \frac{1}{(it)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy.$$

Linear Schrödinger Propagator and Estimates

The solution of the linear Schrödinger initial value problem

$$\begin{cases} (i\partial_t + \Delta)u = 0 \\ u(0,x) = u_0(x). \end{cases} \qquad (LS(\mathbb{R}^d))$$

is denoted $u(t,x) = e^{it\Delta}u_0$. The solution can be given explicitly

- Fourier Multiplier Representation:

$$e^{it\Delta}u_0(x) = c_\pi \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2} \widehat{u_0}(\xi) d\xi.$$

- $\boxed{\text{Convolution Representation}}$:

$$e^{it\Delta}u_0(x) = c_\pi^1 \underbrace{\frac{1}{(it)^{d/2}}}_{\boxed{\text{decay}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy.$$

Estimates for Linear Schrödinger Propagator

- Fourier Multiplier Representation $\implies$ Unitary in H^s:

$$\|D_x^s e^{it\Delta} u_0\|_{L_x^2} = \|D_x^s u_0\|_{L_x^2}.$$

- Convolution Representation $\implies$ Dispersive estimate:

$$\|e^{it\Delta} u_0\|_{L_x^2} \leq \frac{C}{t^{d/2}} \|u_0\|_{L_x^1}.$$

- Spacetime estimates? Strichartz estimates hold, for example,

$$\|e^{it\Delta} u_0\|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|u_0\|_{L^2(\mathbb{R}_x^2)}.$$

(Strichartz estimates linked to Fourier restriction phenomena.)

Local-in-time theory for $NLS_3^{\pm}(\mathbb{R}^2)$

- $\forall\ u_0 \in L^2(\mathbb{R}^2)\ \exists\ T_{lwp}(u_0)$ determined by
$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$
$\exists$ unique $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$ solving $NLS_3^{+}(\mathbb{R}^2)$.
- $\forall\ u_0 \in H^s(\mathbb{R}^2), s > 0,\ T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists: $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.
- Define the maximal forward existence time $T^*(u_0)$ by
$$\|u\|_{L^4_{tx}([0,T^*-\delta]\times\mathbb{R}^2)} < \infty$$
for all $\delta > 0$ but diverges to ∞ as $\delta \searrow 0$.
- $\exists$ small data scattering threshold $\mu_0 > 0$
$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

Global-in-time theory?

What is the ultimate fate of the local-in-time solutions?

L^2-critical Scattering Conjecture:
$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Moreover, $\exists\ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \pm\infty} \|e^{\pm it\Delta} u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Same statement for focusing $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known for large radial data [Killip-Tao-Visan 07].

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{1}{3}$	resonant cut & I-Morawetz	[C-Roy08]
$s > \frac{1}{4}$	resonant cut & I-Morawetz	[Dodson09]
$s > 0$?		

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{1}{3}$	resonant cut & I-Morawetz	[C-Roy08]
$s > \frac{1}{4}$	resonant cut & I-Morawetz	[Dodson09]
$s > 0$?		

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

2. Bourgain's High-Low Fourier Truncation

2. Bourgain's High-Low Fourier Truncation

IMRN International Mathematics Research Notices
1998, No. 5

Refinements of Strichartz' Inequality and Applications to 2D-NLS with Critical Nonlinearity

J. Bourgain

Summary

Consider the 2D IVP

$$\begin{cases} iu_t + \Delta u + \lambda |u|^2 u = 0 \\ u(0) = \varphi \in L^2(\mathbb{R}^2). \end{cases} \tag{†}$$

The theory on the Cauchy problem asserts a unique maximal solution

$$u \in \mathcal{C}(]-T_*, T^*[; L^2(\mathbb{R}^2)) \cap L^4(]-T_*, T^*[; L^4(\mathbb{R}^2))$$

2. Bourgain's High-Low Fourier Truncation

Consider the Cauchy problem for defocusing cubic NLS on $\mathbb{R}^2$:

$$\begin{cases} (i\partial_t + \Delta)u = +|u|^2 u \\ u(0,x) = \phi_0(x). \end{cases} \qquad (NLS_3^+(\mathbb{R}^2))$$

We describe the first result to give global well-posedness below H^1.

- $NLS_3^+(\mathbb{R}^2)$ is GWP in H^s for $s > \frac{2}{3}$ [Bourgain 98].
- First use of Bilinear Strichartz estimate was in this proof.
- Proof cuts solution into low and high frequency parts.
- For $u_0 \in H^s, \ s > \frac{2}{3}$, Proof gives (and $\boxed{\text{crucially exploits}}$),

$$u(t) - e^{it\Delta}\phi_0 \in H^1(\mathbb{R}^2_x).$$

Setting up; Decomposing Data

- Fix a large target time T.
- Let $N = N(T)$ be large to be determined.
- Decompose the initial data:

$$\phi_0 = \phi_{low} + \phi_{high}$$

where

$$\phi_{low}(x) = \int_{|\xi|<N} e^{ix\cdot\xi}\widehat{\phi_0}(\xi)d\xi.$$

- Our plan is to evolve:

$$\phi_0 = \phi_{low} + \phi_{high}$$

$$u(t) = u_{low}(t) + u_{high}(t).$$

Setting up; Decomposing Data

- Fix a large target time T.
- Let $N = N(T)$ be large to be determined.
- Decompose the initial data:

$$\phi_0 = \phi_{low} + \phi_{high}$$

where

$$\phi_{low}(x) = \int_{|\xi|<N} e^{ix\cdot\xi}\widehat{\phi_0}(\xi)d\xi.$$

- Our plan is to evolve:

$$\begin{tikzcd} \phi_0 \arrow[d, "\boxed{\text{NLS}}"] & = & \phi_{low} \arrow[d, "\boxed{\text{NLS}}"] & + & \phi_{high} \arrow[d, "\boxed{\text{DE}}"] \\ u(t) & = & u_{low}(t) & + & u_{high}(t). \end{tikzcd}$$

Setting up; Decomposing Data

Low Frequency Data Size:

- Kinetic Energy:

$$\begin{aligned} \|\nabla \phi_{low}\|_{L^2}^2 &= \int_{|\xi|<N} |\xi|^2 |\widehat{\phi_0}(\xi)|^2 dx \\ &= \int_{|\xi|<N} |\xi|^{2(1-s)} |\xi|^{2s} |\widehat{\phi_0}(\xi)|^2 dx \\ &\leq N^{2(1-s)} \|\phi_0\|_{H^s}^2 \leq C_0 N^{2(1-s)}. \end{aligned}$$

- Potential Energy: $\|\phi_{low}\|_{L_x^4} \leq \|\phi_{low}\|_{L^2}^{1/2} \|\nabla \phi_{low}\|_{L^2}^{1/2}$

$$\implies H[\phi_{low}] \leq C N^{2(1-s)}.$$

High Frequency Data Size:

$$\|\phi_{high}\|_{L^2} \leq C_0 N^{-s}, \ \|\phi_{high}\|_{H^s} \leq C_0.$$

LWP of Low Frequency Evolution along NLS

The NLS Cauchy Problem for the low frequency data

$$\begin{cases} (i\partial_t + \Delta)u_{low} = +|u_{low}|^2 u_{low} \\ u_{low}(0,x) = \phi_{low}(x) \end{cases} \qquad (NLS)$$

is well-posed on $[0, T_{lwp}]$ with $T_{lwp} \sim \|\phi_{low}\|_{H^1}^{-2} \sim N^{-2(1-s)}$.

We obtain, as a consequence of the local theory, that

$$\|u_{low}\|_{L^4_{[0,T_{lwp}],x}} \leq \frac{1}{100}.$$

LWP of High Frequency Evolution along DE

The NLS Cauchy Problem for the high frequency data

$$\begin{cases} (i\partial_t + \Delta)u_{high} = +2|u_{low}|^2 u_{high} + \text{similar} + |u_{high}|^2 u_{high} \\ u_{high}(0,x) = \phi_{high}(x) \end{cases} \qquad (DE)$$

is also well-posed on $[0, T_{lwp}]$.

Remark: The LWP lifetime of *NLS* evolution of u_{low} AND the LWP lifetime of the *DE* evolution of u_{high} are controlled by $\|u_{low}(0)\|_{H^1}$.

Extra Smoothing of Nonlinear Duhamel Term

The high frequency evolution may be written

$$u_{high}(t) = e^{it\Delta} u_{high} + w.$$

The local theory gives $\|w(t)\|_{L^2} \lesssim N^{-s}$. Moreover, due to smoothing (obtained via bilinear Strichartz), we have that

$$w \in H^1, \; \|w(t)\|_{H^1} \lesssim N^{1-2s+}. \qquad \text{(SMOOTH!)}$$

Let's postpone the proof of (SMOOTH!).

Nonlinear High Frequency Term Hiding Step!

- $\forall\ t \in [0, T_{lwp}]$, we have

$$u(t) = u_{low}(t) + e^{it\Delta}\phi_{high} + w(t).$$

- At time T_{lwp}, we define data for the progressive sheme:

$$u(T_{lwp}) = u_{low}(T_{lwp}) + w(T_{lwp}) + e^{iT_{lwp}\Delta}\phi_{high}.$$

$$u(t) = u^{(2)}_{low}(t) + u^{(2)}_{high}(t)$$

for $t > T_{lwp}$.

Nonlinear High Frequency Term Hiding Step!

- $\forall\ t \in [0, T_{lwp}]$, we have

$$u(t) = u_{low}(t) + e^{it\Delta}\phi_{high} + w(t).$$

- At time T_{lwp}, we define data for the progressive sheme:

$$u(T_{lwp}) = u_{low}(T_{lwp}) + w(T_{lwp}) + e^{iT_{lwp}\Delta}\phi_{high}.$$

$$u(t) = u^{(2)}_{low}(t) + u^{(2)}_{high}(t)$$

for $t > T_{lwp}$.

HAMILTONIAN INCREMENT: $\phi_{low}(0) \longmapsto u^{(2)}_{low}(T_{lwp})$

The Hamiltonian increment due to $w(T_{lwp})$ being added to low frequency evolution can be calcluated. Indeed, by Taylor expansion, using the bound (SMOOTH!) and energy conservation of u_{low} evolution, we have using

$$H[u_l^{(2)}(T_{lwp})] = H[u_l(0)] + (H[u_l(T_{lwp}) + w(T_{lwp})] - H[u_l(T_{lwp})])$$
$$\sim N^{2(1-s)} + N^{2-3s+} \sim N^{2(1-s)}.$$

Moreover, we can accumulate N^s increments of size N^{2-3s+} before we double the size $N^{2(1-s)}$ of the Hamiltonian. During the iteration, Hamiltonian of "low frequency" pieces remains of size $\lesssim N^{2(1-s)}$ so the LWP steps are of uniform size $N^{-2(1-s)}$. We advance the solution on a time interval of size:

$$N^s N^{-2(1-s)} = N^{-2+3s}.$$

For $s > \frac{2}{3}$, we can choose N to go past target time T. ■

How do we prove (SMOOTH!)?

Bourgain's Bilinear Strichartz Estimate: For (dyadic) $N \leq L$

$$\|e^{it\Delta} f_L e^{it\Delta} g_N\|_{L^2_{t,x}} \leq \frac{N^{\frac{2-1}{2}}}{L^{\frac{1}{2}}} \|f_L\|_{L^2_x} \|g_N\|_{L^2_x}.$$

Corollary

For $s \geq \frac{1}{2}$

$$\begin{aligned} \|D^s_x(u_1 u_2)\|_{L^2_{[0,\delta],x}} \leq & C(\|u_1\|_{X^{s,1/2+}_{[0,\delta]}} \|u_2\|_{X^{0,1/2+}_{[0,\delta]}} \\ & + \|u_1\|_{X^{1/2,1/2+}_{[0,\delta]}} \|u_2\|_{X^{s-1/2,1/2+}_{[0,\delta]}}). \end{aligned}$$

Thus, the Bilinear Estimate allows us move half a derivative off the high frequency part and instead onto of the low frequency part.

Treatment of a typical term in w

- Using the controls we have on u_{low}, u_{high} from the local theory on $[0, T_{lwp}]$, we want to prove for

$$w = \int_0^t e^{i(t-t')\Delta} |u_{low}|^2 u_{high}(t') dt'$$

  that $\sup_{t \in [0, T_{lwp}]} \|\nabla w\|_{L^2} < N^{1-2S+}$.
- By Sobolev embedding, we have

$$\|w\|_{L^\infty_{[0,T_{lwp}]} H^1} \leq \|w\|_{X^{1,1/2+}_{[0,T_{lwp}]}}.$$

- The mapping $f \longmapsto \int_0^t e^{i(t-t')\Delta}$ is formally $f \longmapsto (i\partial_t + \Delta)^{-1} f$ which, due to time localization, is essentially $\widehat{f} \longmapsto \langle \tau + |\xi|^2 \rangle \widehat{f}$. It suffices to control $\|D_x |u_{low}|^2 u_{high}\|_{X^{0,-1/2+}}$. Proceed by duality....

Treatment of a typical term in w

$$\|w\|_{L^\infty_{[0,T_{lwp}]}H^1} \leq \sup_{\|g\|_{X^{0,1/2-}}\leq 1} \langle g, D_x(|u_{low}|^2 u_{high})\rangle.$$

$$\lesssim \sup_g \langle g D_x u_{low}, u_{low} u_{high}\rangle + \sup_g \langle g u_{low}, D_x(u_{low} u_{high})\rangle$$

$$= \text{easier} + \sup_g \langle D_x^{1/2}(g u_{low}), D_x^{1/2}(u_{low} u_{high}\rangle.$$

The corollary and the available bounds then give (SMOOTH!).

3. Bilinear Strichartz Estimate

3. Bilinear Strichartz Estimate

- Recall the Strichartz estimate for $(i\partial_t + \Delta)$ on $\mathbb{R}^2$:

$$\|e^{it\Delta}u_0\|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C\|u_0\|_{L^2(\mathbb{R}_x^2)}.$$

- We can view this trivially as a bilinear estimate by writing

$$\|e^{it\Delta}u_0 \; e^{it\Delta}v_0\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C\|u_0\|_{L^2(\mathbb{R}_x^2)}\|v_0\|_{L^2(\mathbb{R}_x^2)}.$$

- Bourgain refined this trivial bilinear estimate for functions having certain Fourier support properties.

Bilinear Strichartz Estimate

Theorem

For (dyadic) $N \leq L$ and for $x \in \mathbb{R}^2$,

$$\|e^{it\Delta} f_L e^{it\Delta} g_N\|_{L^2_{t,x}} \leq \frac{N^{\frac{1}{2}}}{L^{\frac{1}{2}}} \|f_L\|_{L^2_x} \|g_N\|_{L^2_x}.$$

- Here spt $(\widehat{f_L}) \subset \{|\xi| \sim L\},\ g_N$ similar.
- Observe that $\sqrt{\frac{N}{L}} \ll 1$ when $N \ll L$.

3. Bourgain's Proof

Bourgain: IMRN98

Proof. Since the standard Strichartz inequality yields (112) without the

$$\left(\frac{M_1}{M_2}\right)^{\frac{1}{2}}\text{-factor,}$$

we may assume $M_2 \gg M_1$.

Writing

$$(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2) = \int \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\xi_2)e^{i[(\xi_1+\xi_2).x+(|\xi_1|^2+|\xi_2|^2)t]}\,d\xi_1 d\xi_2,$$

it follows from Parseval's identity and Cauchy-Schwarz that

$$\begin{aligned}
\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_2^2 &= \int d\xi d\lambda \left|\int \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\xi-\xi_1)\delta_0(|\xi_1|^2+|\xi-\xi_1|^2-\lambda)\,d\xi_1\right|^2 \\
&\leq \|\psi_1\|_2^2\|\psi_2\|_2^2\Bigg[\sup_{\lambda,|\xi|\sim M_2} \operatorname{mes}_{(1)}[\xi_1 \,|\, |\xi_1|\sim M_1 \\
&\qquad\qquad \text{and } |\xi_1|^2+|\xi-\xi_1|^2=\lambda]\Bigg] \\
&< C\frac{M_1}{M_2}.
\end{aligned}$$

Proof Based on Change of Variables

Ideas from (Kenig-Ponce-Vega); see [C-Delort-Kenig-Staffilani].

Recall the Fourier multiplier representation of the propagator:

$$\begin{aligned} e^{it\Delta} f(x) =& c_\pi \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{-it|\xi|^2} \widehat{f}(\xi) d\xi \\ =& c_\pi \int_{\mathbb{R}^{1+2}} e^{i(x\cdot\xi + t\tau)} \delta_0(\tau + |\xi|^2) \widehat{f}(\xi) d\tau d\xi. \end{aligned}$$

spacetime inverse
Fourier transform

With $f = f_L$ and $g = g_N$, we wish to estimate

$$\| e^{it\Delta} f \;\, e^{it\Delta} g \|_{L^2_{t,x}} = \| \mathcal{F}[e^{it\Delta} f \;\, e^{it\Delta} g] \|_{L^2_{\tau,\xi}}.$$

Using Fourier tranform property, $\mathcal{F}(ab) = \widehat{a} * \widehat{b}$, we find....

Fourier Manipulations; Dirac Evaluations

We wish to estimate (in $L^2_{\tau,\xi}$) the expression

$$\int_{\substack{\tau = \tau_1 + \tau_2 \\ \xi = \xi_1 + \xi_2}} \delta_0(\tau_1 + |\xi_1|^2)\widehat{f}(\xi_1)\delta_0(\tau_2 + |\xi_2|^2)\widehat{g}(\xi_2).$$

Evaluating the δ functions, we find $\tau_j = -|\xi_j|^2$, so

$$\int_{\substack{\tau = -|\xi_1|^2 - |\xi_2|^2 \\ \xi = \xi_1 + \xi_2}} \widehat{f}(\xi_1)\widehat{g}(\xi_2)$$

We proceed by duality. Let's test this against $d(\tau, \xi)$....

Duality Reduces Matters to Certain Integral

$$\|e^{it\Delta}f\; e^{it\Delta}g\|_{L^2_{t,x}} = \sup_{\|d\|_{L^2_{\tau,\xi}}\leq 1} \left\langle d(\tau,\xi)\ ,\ \int_{\substack{\tau=-|\xi_1|^2-|\xi_2|^2\\ \xi=\xi_1+\xi_2}} \widehat{f}(\xi_1)\widehat{g}(\xi_2) \right\rangle .$$

$$= \sup_{d} \int d(-|\xi_1|^2-|\xi_2|^2, \xi_1+\xi_2)\ \widehat{f}(\xi_1)\widehat{g}(\xi_2)d\xi_1 d\xi_2.$$

Fourier manipulations reduce matters to bounding an integral.

Our task: Show the integral above is bounded by

$$\lesssim \sqrt{\frac{N}{L}}\|f\|_{L^2}\|g\|_{L^2}\|d\|_{L^2}.$$

Setting Up the Change of Variables

Let's define a change of variables motivated by the arguments of d:

$$u = -|\xi_1|^2 - |\xi_2|^2,\ v = \xi_1 + \xi_2.$$

- Note that $u \in \mathbb{R}$ and $v \in \mathbb{R}^2$. Thus, $dudv$ is a measure in 3d while $d\xi_1 d\xi_2$ is a measure in 4d.
- Note also that ξ_2 is the argument of $g = g_N$ so it is localized to the smaller dyadic shell $|\xi_2| \sim N \ll L$.
- Let's denote the components of $\xi_j \in \mathbb{R}^2$ with superscripts:

$$\xi_j = (\xi_j^1, \xi_j^2).$$

- The full change of variables is the defined via

$$dudv\ d\xi_2^1 = |J|\ d\xi_1^1 d\xi_1^2 d\xi_2^2\ d\xi_2^1.$$

We have an extra variable outside the changed integral.

The Jacobian

The Jacobian matrix J is calculated as

$$J = \begin{bmatrix} \frac{\partial u}{\partial \xi_1^1} & \frac{\partial v^1}{\partial \xi_1^1} & \frac{\partial v^2}{\partial \xi_1^1} \\ \frac{\partial u}{\partial \xi_2^1} & \frac{\partial v^1}{\partial \xi_2^1} & \frac{\partial v^2}{\partial \xi_2^1} \\ \frac{\partial u}{\partial \xi_2^2} & \frac{\partial v^1}{\partial \xi_2^2} & \frac{\partial v^2}{\partial \xi_2^2} \end{bmatrix}.$$

The explicit forms for u, v permit calculating

$$|J| = 2|\xi_1^2 - \xi_2^2|.$$

Since $|\xi_1| \sim L$, we may assume by rotation that $|J| \sim L$.

Changing Variables

Our task: Estimate, for $|\xi_1| \sim L$, $|\xi_2| \sim N$, the integral

$$\int_{|\xi_2^1| \lesssim N} \int_{\xi_1, \xi_2^2} d(-|\xi_1|^2 - |\xi_2|^2, \xi_1 + \xi_2)\ \widehat{f}(\xi_1)\widehat{g}(\xi_2) d\xi_1^1 d\xi_1^2 d\xi_2^2\ d\xi_2^1.$$

We insert the Jacobian and reexpress inner integration as

$$\int_{\xi_1, \xi_2^2} d(-|\xi_1|^2 - |\xi_2|^2, \xi_1 + \xi_2)\ \frac{\widehat{f}(\xi_1)\widehat{g}(\xi_2)}{|J|} |J| d\xi_1^1 d\xi_1^2 d\xi_2^2.$$

Changing variables, we observe this equals

$$\int_{u,v} d(u,v) H(u,v;\xi_2^1)|J| du dv$$

where

$$H(u,v;\xi_2^1) = \frac{\widehat{f}(\xi_1)\widehat{g}(\xi_2)}{|J|}.$$

Cauchy-Schwarz; Jacobian Remnant

We apply Cauchy-Schwarz in u, v to bound by

$$\|d\|_{L^2}\left(\int_{u,v}|H(u,v;\xi_2^1)|^2 dudv\right)^{1/2}.$$

We drop $\|d\|_{L^2} \leq 1$ by duality and change variables back. We get

$$\left(\int_{\xi_1,\xi_2^2}\left|\frac{\widehat{f}(\xi_1)\widehat{g}(\xi_2)}{|J|}\right|^2 |J| d\xi_1^1 d\xi_1^2 d\xi_2^2\right)^{1/2}.$$

One factor of the Jacobian denominator remains! We gain $L^{-1/2}$.

We still have the extra outside integration....

Trivial Cauchy-Schwarz on Extra Integral

Recalling what we must control, using what we have obtained....

$$\frac{1}{L^{1/2}} \int\limits_{|\xi_2^1| \lesssim N} \left(\int\limits_{\xi_1, \xi_2^2} \left| \widehat{f}(\xi_1) \widehat{g}(\xi_2) \right|^2 d\xi_1^1 d\xi_1^2 d\xi_2^2 \right)^{1/2} d\xi_2^1.$$

Apply Cauchy-Schwarz in ξ_2^1 and <u>pay the penalty</u> of $N^{1/2}$.

We gain over the trivial bilinear estimate by the factor

$$\sqrt{\frac{(\text{measure of extra support})}{|J|}} = \sqrt{\frac{N}{L}}.$$

4. The I-Method of Almost Conservation

4. The I-Method of Almost Conservation

Let $H^s \ni u_0 \longmapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A smoothing operator $I = I_N : H^s \longmapsto H^1$. The NLS evolution $u_0 \longmapsto u$ induces a smooth reference evolution $H^1 \ni Iu_0 \longmapsto Iu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.
- A modified energy $\widetilde{E}[Iu]$ built using the reference evolution.

First Version of the I-method: $\widetilde{E} = H[Iu]$

For $s < 1, N \gg 1$ define smooth monotone $m : \mathbb{R}^2_\xi \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N. \end{cases}$$

The associated Fourier multiplier operator, $\widehat{(Iu)}(\xi) = m(\xi)\widehat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Set $\widetilde{E}[Iu(t)] = H[Iu(t)]$. Other choices of $\widetilde{E}$ are mentioned later.

AC Law Decay and Sobolev GWP index

1. **Modified LWP.** Initial v_0 s.t. $\|\nabla I v_0\|_{L^2} \sim 1$ has $T_{lwp} \sim 1$.
2. **Goal.** $\forall\ u_0 \in H^s$, $\forall\ T > 0$, construct $u : [0, T] \times \mathbb{R}^2 \to \mathbb{C}$.
3. $\iff$ **Dilated Goal.** Construct $u^\lambda : [0, \lambda^2 T] \times \mathbb{R}^2 \to \mathbb{C}$.
4. **Rescale Data.** $\|I \nabla u_0^\lambda\|_{L^2} \lesssim N^{1-s} \lambda^{-s} \|u_0\|_{H^s} \sim 1$ provided we choose $\lambda = \lambda(N) \sim N^{\frac{1-s}{s}} \iff N^{1-s}\lambda^{-s} \sim 1$.
5. **Almost Conservation Law.** $\|I \nabla u(t)\|_{L^2} \lesssim H[Iu(t)]$ and
$$\sup_{t \in [0, T_{lwp}]} H[Iu(t)] \leq H[Iu(0)] + N^{-\alpha}.$$
6. **Delay of Data Doubling.** Iterate modified LWP N^α steps with $T_{lwp} \sim 1$. We obtain rescaled solution for $t \in [0, N^\alpha]$.
$$\lambda^2(N) T < N^\alpha \iff T < N^{\alpha + \frac{2(s-1)}{s}} \text{ so } s > \frac{2}{2+\alpha} \text{ suffices.}$$

FIRST VERSION OF THE I-METHOD: $\widetilde{E} = H[Iu]$

A Fourier analysis established the almost conservation property of $\widetilde{E} = H[Iu]$ with $\alpha = \frac{3}{2}$ which led to...

THEOREM (CKSTT 02)

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$.

- The smoothing property $u(t) - e^{it\Delta}u_0 \in H^1$ is not obtained.
- Same result for $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Here Q is the ground state (unique positive solution of $-Q + \Delta Q = -Q^3$).
- Fourier analysis leading to $\alpha = \frac{3}{2}$ in fact gives $\alpha = 2$ for most frequency interactions.

Almost Conservation Law for $H[Iu]$

Proposition

Given $s > \frac{4}{7}, N \gg 1$, and initial data $\phi_0 \in C_0^\infty(\mathbb{R}^2)$ with $E(I_N u_0) \leq 1$, then there exists a $T_{lwp} \sim 1$ so that the solution

$$u(t,x) \in C([0, T_{lwp}], H^s(\mathbb{R}^2))$$

of $NLS_3^+(\mathbb{R}^2)$ satisfies

$$E(I_N u)(t) = E(I_N u)(0) + O(N^{-\frac{3}{2}+}),$$

for all $t \in [0, T_{lwp}]$.

Ideas in the Proof of Almost Conservation

- Standard Energy Conservation Calculation:

$$\partial_t H(u) = \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u) dx$$
$$= \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u - i u_t) dx = 0.$$

cancellation

- For the smoothed reference evolution, we imitate....

$$\partial_t H(Iu) = \Re \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - \Delta Iu - iIu_t) dx$$
$$= \Re \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - I(|u|^2 u)) dx \neq 0.$$

commutator!

- The increment in modified energy involves a commutator,

$$H(Iu)(t) - H(Iu)(0) = \Re \int_0^t \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - I(|u|^2 u)) dx dt.$$

- Littlewood-Paley, Case-by-Case, (Bi)linear Strichartz, $X_{s,b}$....

Remarks

- The almost conservation property

$$\sup_{t\in[0,T_{lwp}]} \widetilde{E}[Iu(t)] \leq \widetilde{E}[Iu_0] + N^{-\alpha}$$

  leads to GWP for

$$s > s_\alpha = \frac{2}{2+\alpha}.$$

- The I-method is a *subcritical method*. To prove the Scattering Conjecture at $s = 0$ via the I-method would require $\alpha = +\infty$.
- The I-method *localizes the conserved density in frequency*. Similar ideas appear in recent critical scattering results.
- There is a *multilinear corrections algorithm* for defining other choices of $\widetilde{E}$ which yield a better AC property.