Low Regularity Aspects of NLS Blowup

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PIMS, May 2009

1. Blowup Solutions Exist

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We consider the Cauchy problem for L^2 critical focusing NLS:

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^-(\mathbb{R}^2))$$

The solution has an L^2-invariant dilation symmetry

$$u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

Time invariant conserved quantities:

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx - \frac{1}{2}|u(t)|^4 dx.$$

$NLS_3^-(\mathbb{R}^2)$ H^1-GWP Theory

- Weinstein's H^1-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u,\, T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- Q is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

Pseudoconformal Symmetry

- Pseudoconformal transformation:

$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$

- $\mathcal{PC}$ is L^2-critical *NLS* solution symmetry:
  Suppose $0 < t_1 < t_2 < \infty$. If

$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$

  then

$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$

  solves

$$i\partial_\tau v + \Delta_y v = \pm|v|^{4/d} v.$$

- $\mathcal{PC}$ is an L^2-Strichartz isometry:
  If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then

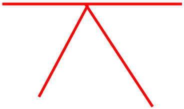
$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}},$$

  is an explicit blowup solution.



- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in S is conically concentrated into a point.

- Minimal mass H^1 blowup solution characterization:
  $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2dx < \infty$ blows up.
- How do these solutions blow up?

Lemma (Subcritical Scaling Lower Bound):

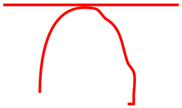
If $H^s \ni u_0 \longmapsto u(t)$ with $s > 0$ solving $NLS_3^-(\mathbb{R}^2)$ for all t near T^* in the maximal finite (forward) interval of existence $[0, T^*)$ then

$$\frac{c}{(T^* - t)^{s/2}} \leq \|D^s u(t)\|_{L^2_x}.$$

Scaling invariance and LWP theory:

- $v(\tau, y) := \frac{1}{\lambda} u(t + \frac{\tau}{\lambda^2}, \frac{y}{\lambda}) \implies \|D^s v(0)\|_{L^2} = \frac{1}{\lambda^s} \|D^s u(t)\|_{L^2}$.
- Choose λ so that $\|D^s v(0)\|_{L^2} = 1 \implies \lambda = \|D^s u(t)\|_{L^2}^{\frac{1}{s}}$.
- LWP $\implies v(0) \longmapsto v(t)$ for $\tau \in [0, 1] \iff t + \frac{1}{\lambda^2} < T^*$.
- $\lambda^2 > \frac{1}{T^* - t} \implies$ claim.

Mass Concentration Property: H^1 theory



H^1 Theory of Mass Concentration

- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ implies
$$\liminf_{t \nearrow T^*} \int_{|x|<(T^*-t)^{1/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$
[Merle-Tsutsumi]
- H^1 blowups parabolically concentrate at least the ground state mass. Explicit blowups S concentrate mass much faster.
- Fantastic recent progress on the H^1 blowup theory. [Merle-Raphaël]

Mass Concentration Property: L^2 Theory

L^2 **Theory of Mass Concentration**

- $L^2 \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\limsup_{t \nearrow T^*} \sup_{\text{cubes } I, side(I) \leq (T^* - t)^{1/2}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

[Bourgain]

L^2 blowups parabolically concentrate some mass.

- For large L^2 data, do there exist tiny concentrations?
- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
- Upgrading $\limsup$ into $\liminf$ appears challenging.

$NLS_3^-(\mathbb{R}^2)$: Conjectures/Questions

- **Scattering Below the Ground State Mass?**

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \overset{???}{\implies} u_0 \longmapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$

- **Minimal Mass Blowup Characterization?**

$$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \longmapsto u, T^* < \infty \overset{???}{\implies} u = S,$$

  modulo symmetries. Intermediate step: Characterize in H^s?

- **Concentrated mass amounts are quantized?**
  Ground and excited state profiles are only asymptotic profiles?
- **Are there any general upper bounds?** lim sup **vs.** lim inf **?**
- **What are the possible "singular sets" for NLS blowups?**

L^2 Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

$\boxed{\text{Wider Window}}$

H^s-blowup solutions concentrate ground state mass. [C-Raynor-C.Sulem-Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1$, $T^* < \infty \implies \exists\ t_n \nearrow T^*$ s.t. $u(t_n) \to Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence). For H^s blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an H^s analog of an H^1 result of [Weinstein] which preceded the minimal H^1 blowup solution characterization.

L^2 Critical Case: Partial Results

- Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

$$\limsup_{t\nearrow T^*} \sup_{cubes\ \ I, side(I)\le(T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}} \int_I |u(t,x)|^2 dx \ge \|u_0\|_{L^2}^{-M}.$$

[C-Roudenko]

2. Ground State Mass Concentration for H^s

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Theorem (C-Raynor-Sulem-Wright 05)

For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

- $\{radial\}$ removed by concentration compactness. [Tzirakis] $NLS_5^-(\mathbb{R})$
- Higher dimension generalization $NLS^-_{1+\frac{4}{d}}(\mathbb{R}^d)$. [Visan-Zhang]

Ground State Mass Concentration for H^1

Recall [Merle-Tsutsumi]. $H^1 \cap \{radial\} \ni u_0 \longmapsto u$ with $T^* < \infty$.

- **Rescalings (weakly) converge to asymptotic profile.**

  Consider $\{u(t_n, \cdot)\}_{n\in\mathbb{N}} = \{u_n(\cdot)\}_{n\in\mathbb{N}}$ along $t_n \nearrow T^*$. Form

  Rescaled Snapshots

  $$v_n(\cdot) = \lambda_n^{-1} u_n(\lambda_n^{-1}(\cdot))$$

  with $\lambda_n = \|\nabla u_n\|_{L^2} \gtrsim (T^* - t_n)^{-1/2}$ so that $\|\nabla v_n\|_{L^2} = 1$.
  Thus, $\exists\ v \in H^1$ such that $v_n \rightharpoonup v$ in H^1 along subsequence.

- **Compactness and energy of rescaled asymptotic object.**

  Radial & Rellich Compactness $\implies$ $v_n \to v$ strongly in L^4.
  $|E[v_n]| = \lambda_n^{-2}|E[u(t_n)]| \to 0 \implies E[v] \leq 0$.

- $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$; **undo scaling.**

Ground State Mass Concentration for H^1

Recall [Merle-Tsutsumi]. $H^1 \cap \{radial\} \ni u_0 \longmapsto u$ with $T^* < \infty$.

- **Rescalings (weakly) converge to asymptotic profile.**

  Consider $\{u(t_n, \cdot)\}_{n\in\mathbb{N}} = \{u_n(\cdot)\}_{n\in\mathbb{N}}$ along $t_n \nearrow T^*$. Form

  $$v_n(\cdot) = \lambda_n^{-1} u_n(\lambda_n^{-1}(\cdot))$$

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GROUND STATE MASS CONCENTRATION FOR H^s

We imitate the [Merle-Tsutsumi] result using modified energy.

- **Blowup Parameter:**

$$\lambda(t) = \|u(t)\|_{H^s};\ \Lambda(t) = \sup_{\tau \in [0,t]} \lambda(\tau).$$

- **Modified Blowup Parameter:**

$$\sigma(t) = \|I\langle\nabla\rangle u(t)\|_{L^2};\ \Sigma(t) = \sup_{\tau \in [0,t]} \sigma(\tau).$$

Recall,

$$\|f\|_{H^s} \leq \|I\langle\nabla\rangle f\|_{L^2} \leq N^{1-s}\|f\|_{H^s}.$$

Thus, $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$.

Ground State Mass Concentration for H^s

Lemma (Modified Kinetic $\gg$ Modified Total Energy)

$\forall\ s > 0.86$ if $H^s \ni u_0 \longmapsto u$ on maximal $[0, T^*)$ then $\forall T < T^*\ \exists\ N = N(T)$ such that

$$|E[I_{N(T)} u(T)]| \leq C_0 \Lambda(T)^{p(s)}$$

with $p(s) < 2$ and $C_0 = C_0(s, T^*, \|u_0\|_{H^s})$.

- Modified Kinetic Energy $\gg$ Modified Total Energy.
- $N(T) = C\Lambda(T)^{\frac{p(s)}{2(1-s)}}$.
- Proof based on almost conservation; multilinear analysis.

Ground State Mass Concentration for H^s

1. **Rescale by modified kinetic energy.**
   Choose any *maximizing sequence* $t_n \nearrow T^*$ satisfying $\|u(t_n)\|_{H^s} = \Lambda(t_n)$. Define $v_n(y) = \sigma_n^{-1} I_{N(t_n)} u(t_n, \sigma_n^{-1} y)$ where $N(t_n)$ is as in the Lemma.
2. **Weak convergence and L^4 compactness.**
   Rescaling $\implies \|\nabla v_n\|_{H^1} \to 1$ so $\exists\ v \in H^1$ s.t. $v_n \rightharpoonup v$ along subsequence. Radial & Rellich $\implies v_n \to v$ strongly L^4.
3. **Energy of asymptotic object.**
   $|E[v_n]| = \sigma_n^{-2} |E[I_N u_n]| \leq \sigma_n^{-2} \Lambda^{p(s)}(t_n) \leq (\Lambda(t_n))^{p(s)-2} \to 0.$
4. **Undo the rescaling.**
   Unravelling scaling using lower bound $\sigma_n \gtrsim (T^* - t_n)^{-s/2}$ completes proof.

3. Concentration & Strichartz Explosion

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Idea!

- Ground state soliton $u(t,x) = e^{it}Q(x)$ satisfies

$$\|u\|_{L^4([j,j+1]_t \times \mathbb{R}^2_x)} = \eta = O(1), \ \forall\, j \in \mathbb{N}.$$

- L^4-isometry & explicit $S = \mathcal{PC}[e^{it}Q] \sim |\tau|^{-1} Q(\tau^{-1} y) e^{i\cdots}$,

$$\|S\|_{L^4([-\frac{1}{j}, -\frac{1}{j+1}]_\tau \times \mathbb{R}^2_y)} = \eta, \ \forall\, j \in \mathbb{N}.$$

- Thus, $\|S\|_{L^4([-1,t]\times\mathbb{R}^2)} \sim \frac{1}{|t|}$; Mass concentrated in $|y| \lesssim |t|$.
- Contrast with [Merle-Tsutsumi], [Bourgain] Concentration: $\|u\|_{L^4([-1,t]\times\mathbb{R}^2)} \nearrow \infty \implies$ Mass concentrated in $|y| \lesssim |t|^{1/2}$.
- Observation?
  Strichartz explosion rate $= f$(concentration window size).

Heuristic: Window size & L^4 Explosion

- When $\|u\|_{L^4([t_n,t_{n+1}]\times\mathbb{R}^2)} \sim \eta$ [Bourgain] shows parabolic concentration: $\exists\ t_n^* \in [t_n, t_{n+1}]$ and $x_0 \in \mathbb{R}^2$ where

$$\int_{|x-x_0|\lesssim|t_{n+1}-t_n|^{1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}.$$

- In [C-Roudenko], we observe (overstated!):

$$\|u\|_{L^4_{[0,T^*-t]\times\mathbb{R}^2}} := f(T^*-t) \nearrow \infty \text{ as } t \nearrow T^*$$

$$\Updownarrow$$

$$\sup_{x_0\in\mathbb{R}^2} \int_{|x-x_0|\lesssim[-\partial_t f(T^*-t)]^{-1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}$$

- Why? By first order Taylor approximation, we have $\eta \sim f(T^*-t_{n+1}) - f(T^*-t_n) \sim [-\partial_t f(T^*-t_n)](t_{n+1}-t_n)$.

Ideas in Bourgain's Proof

- Decompose $[0, T^*)$ into $\bigcup [t_n, t_{n+1})$ on which

$$\|u\|_{L^4([t_n,t_{n+1}]\times\mathbb{R}^2)} = \eta \sim \frac{1}{100}.$$

- For $t \in [t_n, t_{n+1})$, we have $u \sim e^{i(t-t_n)\Delta} u(t_n)$.
- Strichartz Refinements and the conditions

$$\|f\|_{L^2} < \|u_0\|_{L^2};\ \|e^{it\Delta} f\|_{L^4} > \eta$$

  spawn a spacetime tube decomposition of $e^{it\Delta} f$.
- $\exists$ **concentration time** $t_n^* \in [t_n, t_{n+1})\ \forall\ n$.
  Thus, proof is more refined than the $\limsup$ claim.
- Taylor expansion connects $(t_{n+1} - t_n)$ with $T^* - t_n$.

Squares Lemma

Lemma (Bourgain)

$\forall\ \epsilon > 0$ and $\forall\ f \in L^2(\mathbb{R}^2)\ \exists\ \{\widehat{f}_r\}_{1 \le r \le R(\epsilon)}$ such that

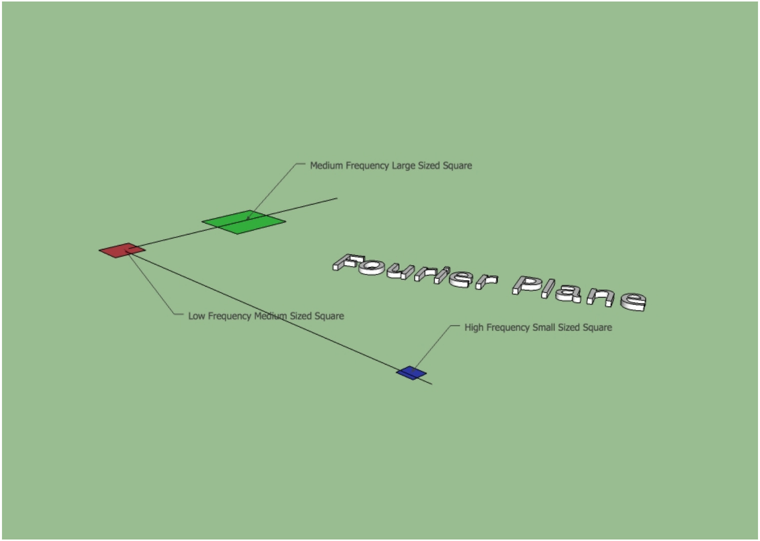
- *spt $\widehat{f}_r \subset \tau_r \subset \mathbb{R}^2$ with τ_r a square of side A_r centered at ξ_r*
- $|\widehat{f}_r| \le \frac{1}{A_r}$
- $\|\widehat{f}_r\|_{L^2} \ge \delta(\epsilon) > 0$

and

$$\|e^{it\Delta} f - \sum_{r=1}^{R(\epsilon)} e^{it\Delta} f_r\|_{L^4_{t,x}} \le \epsilon.$$

The linear Schrödinger evolution of any L^2 function is approximated by the evolution of a function with Fourier support on a system of squares and bounded Fourier transform.

Squares Lemma



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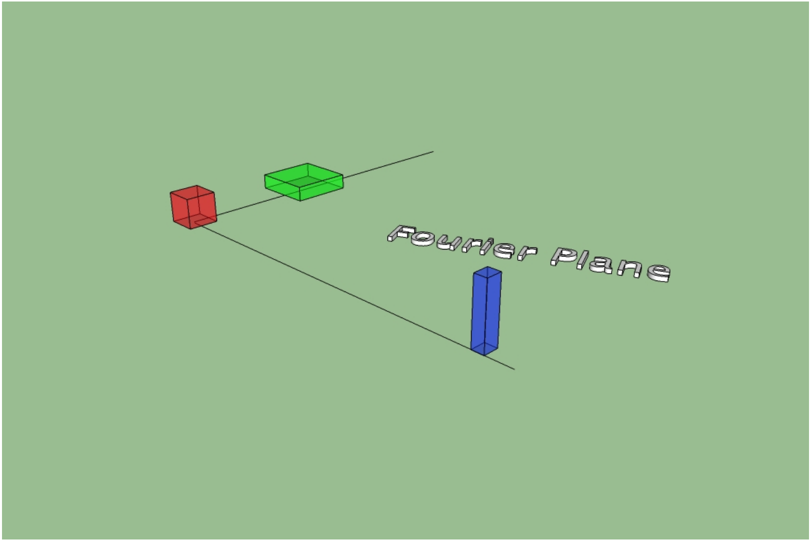


Tubes Lemma

Lemma (Bourgain)

Consider a function g satisfying: (Think of g as one of the f_r.)

- *spt* $\widehat{g} \subset \tau \subset \mathbb{R}^2$ *with* τ *a square of side* A *centered at* ξ_0
- $|\widehat{g}| \leq \frac{1}{A}$.

$\forall\ \epsilon > 0\ \exists$ *spacetime tubes* $\{Q_s\}_{1 \leq s \leq S(\epsilon)}$ *of form*

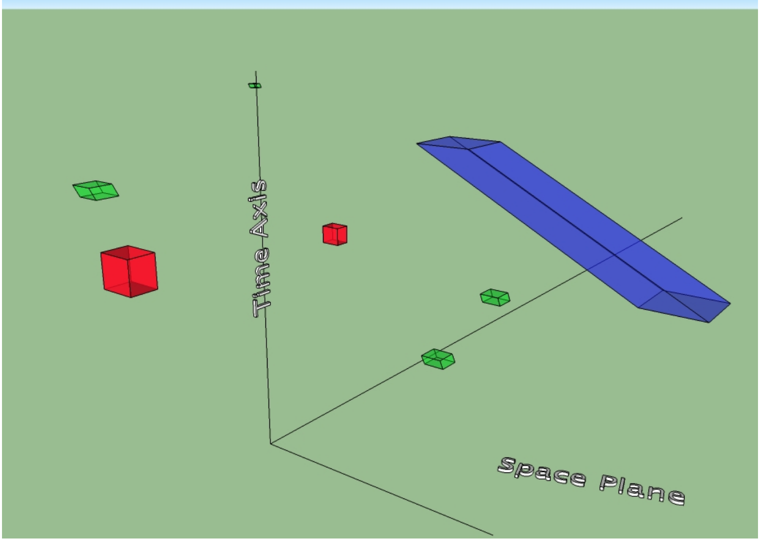
- $Q_s = \{(t, x) \in \mathbb{R}^3 : x - 2t\xi_0 \in \tau_s, t \in J_s\}$
- τ_s *is a (dual sized to* τ*) cube of side* $\frac{1}{A}, |J_s| = \frac{1}{A^2}$

and

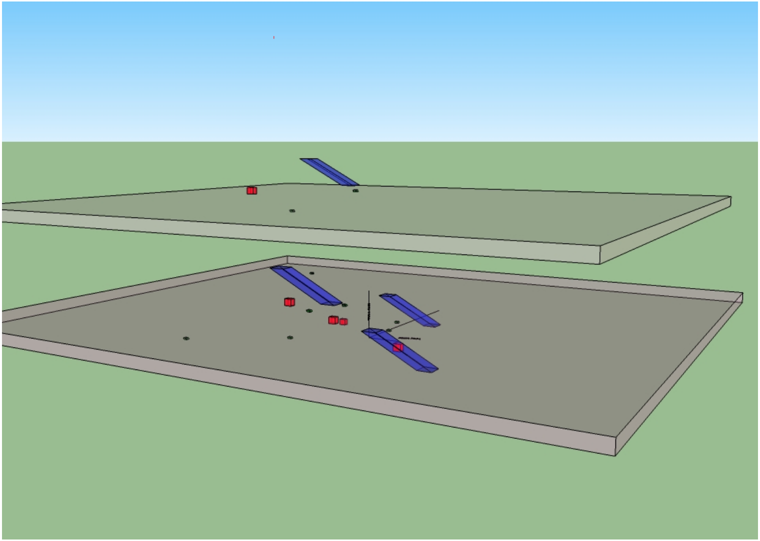
$$\left(\int_{\mathbb{R}^3 \setminus \cup_s Q_s} |e^{it\Delta} g|^4 dxdt \right)^{1/4} < \epsilon.$$

There is just dust outside the tubes!

Tubes Lemma



Tubes Lemma with Time Slices



Strichartz Explosion $\implies$ Tight Window

Theorem (C-Roudenko)

Suppose $T^* < \infty$ *and* $\|u\|_{L^{\frac{2(d+2)}{d}}([0,t]\times\mathbb{R}^d)} \gtrsim (T^* - t)^{-\beta}$. *Then*

$$\limsup_{t \nearrow T^*} \sup_{\substack{\text{cubes } J \in \mathbb{R}^d : \\ l(J) < (T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |u(t,x)|^2 \, dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

Furthermore, $\forall\ t \in (0, T^*)$ $\exists$ *a cube* $\tau(t) \subseteq \mathbb{R}^d_\xi$ *of size* $l(\tau(t)) \gtrsim (T^* - t)^{-(\frac{1}{2}+\frac{\beta}{2})}$ *such that*

$$\limsup_{t \nearrow T^*} \sup_{\substack{\text{cubes } J \in \mathbb{R}^d : \\ l(J) < (T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |P_{\tau(t)} u(t,x)|^2 \, dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

Thickened Time Interval of Concentration

Lemma (Frequency localized waves persist)

Let $f \in L^2_x(\mathbb{R}^d)$ and spt $\hat{f} \subset [0,1]^d$ and suppose

$$\int_{[0,1]^d} |f(x)|^2 \, dx \geq c_1 > 0.$$

Then for $|t| < c(c_1, \|f\|_{L^2})$ concentration persists

$$\int_{[0,1]^d} |e^{it\Delta} f(x)|^2 \, dx \geq \frac{c_1}{2}.$$

- Frequency localization in conclusion shows concentration persists for t in an interval containing t_n^* of size $(T^* - t)^{1+\beta}$.
- Thickened concentration interval may not cover $[t_n, t_{n+1}]$.

Tight Window $\implies$ Strichartz Explosion

Let $F(t) = \|u\|^4_{L^4([0,t]\times\mathbb{R}^2)}$ and $P_{L(t)} = P_{\{|\xi|\leq L(t)\}}$.

Lemma (Pointwise Derivative Lower Bound)

Suppose $\exists\ \alpha \geq \frac{1}{2}, \epsilon > 0$ *such that*

$$\limsup_{t\nearrow T^*} \sup_{\substack{\text{cubes } J\subset\mathbb{R}^d :\\ l(J) < (T^*-t)^\alpha}} \int_J |P_{L(t)}u(t,x)|^2\, dx \geq \epsilon.$$

Then $\exists\ t_n \nearrow T^*$ *such that*

$$F'(t_n) \gtrsim (T^* - t_n)^{-2\alpha}.$$

On thickened concentration time intervals, we integrate the derivative lower bound to get a Strichartz lower bound.

Cautious Remark Concerning lim inf

- Consider $NLS_3^-(\mathbb{R}^2)$ posed at time $t = -\epsilon$ with data

$$\phi_\epsilon(x) = e^{i\epsilon^{-1}|x|^2} e^{i\epsilon^{-1}} Q(x).$$

- Dilated explicit solution which blows up at $t = 0 = T^*$!
- The parabolic scale related to distance to blowup time is $\sqrt{\epsilon}$. For τ a cube of side $\sqrt{\epsilon}$, observe that ϕ_ϵ is non-concentrated

$$\int_\tau |\phi_\epsilon|^2 dx \lesssim \epsilon.$$

- Consider data $(1-\delta)\phi_\epsilon$....

Phase oscillations violently influence L^2 blowup behavior.

4. Rough Blowup Solutions of $NLS_3^-(\mathbb{R}^2)$

Known Maximal-in-Time Solution Scenarios

1. Soliton solutions exist: $u(t,x) = e^{it}R(x)$
   - $Q(x)$ ground state; also excited states.
   - non-scattering; Strichartz S^0 norms diverge global-in-time.
   - a priori H^1 control if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [Weinstein]
2. $\{\text{radial}\} \cap L^2 \ni u_0 \longmapsto u$ scatters if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [KTV]
3. $\mathcal{PC}$ transformation + solitons $\implies$ explicit (fast) $\frac{1}{t}$-blowups.
   - $\mathcal{PC}$ is a Stricharz S^0 isometry.
   - $\exists$ other $\frac{1}{t}$-blowups [Bourgain-Wang; Krieger-Schlag].
   - Stability?
4. Virial Blowup Solutions
   - Obstructive argument
   - Qualitative properties?

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

What Happens?

NLS Blowup Dynamic?

Question: What are the dynamical properties of $NLS_3^-(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation

$\log\log$ BLOWUP REGIME

- Numerical/Persuasive arguments [LPSS] led to:
  - Prediction of blowups with $\log\log$ speed:

$$\|u(t)\|_{H^1} \sim \sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}.$$

  - Prediction that such blowups are generic/stable/observed.
  - Identification of certain mechanisms forecasting $\log\log$.
- $NLS_5^-(\mathbb{R}^1)$ has $\log\log$ blowup solutions. [Perelman]
- Detailed Description of $\log\log$ regime in series by [MR].

Qualitative Aspects of $\log\log$ regime

- Robust, open set in H^1.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomona in critical space (L^2 instability?).
- Conjectured quantization properties?
- Boundary of $\log\log$ regime in phase space?

Phase Space Basin

Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- Small Excess Mass: $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.
- Negative Total Energy: $H[u_0] < 0$.

The associated solution $u_0 \longmapsto u$ explodes with $T^* < \infty$ and

- $\exists\ (\lambda(t), x(t), \gamma(t) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.
- Sharp log log speed law holds:

$$\lambda(t)\sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$

- $u^* \notin H^s$ for $s > 0$; $u^* \notin L^p$ for $p > 2$. (Rough residual)

Theorem (Raphaël): H^1 Stability of $\log\log$

- Fact: $\mathcal{PC}$ + $\log\log$ for $E < 0 \implies \exists$ $\log\log$ with $E > 0$.
- H^1-Stability Theorem: The set of data with $u_0 \in H^1$ with small excess mass blowing up in $\log\log$ regime is open in H^1.
- Develops **bootstrap** approach to *constructing* $\log\log$.
- Further applications of Raphaël's bootstrap/stability:
  - Domains: [Planchon-R:Ω]
  - Singular $S^1 \subset \mathbb{R}^2$: [R:Ring]
  - Singular $S^{d-1} \subset \mathbb{R}^d$: [R-Szeftel:Codimension One Spheres]
  - Singular $\mathbb{T}^{d-2} \subset \mathbb{R}^3$: [Zwiers: Codimension Two Tori]
  - Higher Codimensional Singular Sets?
  - Rough Blowups

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Theorem (C-Raphaël): H^s Stability of $\log\log$

- Let $u_0 \in H^1$ evolve into the $\log\log$ regime.
- $\forall\ s > 0\ \exists\ \epsilon = \epsilon(s, u_0) > 0$ such that $\forall\ v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^-(\mathbb{R}^2)$ solution $v_0 \longmapsto v$ blows up in $\log\log$ regime.

Thus, the H^1 $\log\log$ blowup solutions constructed by [MR] are contained in an open superset of $\log\log$ blowups in H^s, $\forall\ s > 0$.

REMARKS ABOUT THE H^s STABILITY OF $\log\log$

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of H^s $\log\log$ blowups.
- The condition $s > 0$ is expected to be optimal. Instability?
  Small L^2 (but huge H^s) perturbation destroys rough residual mass ($u^* \notin H^s, \ \forall \ s > 0$) leading to fast $\frac{1}{t}$-blowup?
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via I-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled I-method?

Aspects of the [MR] Analysis

- Geometrical description of $\log \log$ blowup solutions.
  - Various profiles $Q, Q_b, \widetilde{Q}_b, \widetilde{Q}_{b(t)} + \zeta_{b(t)}$. (Obscure Notation)
  - Modulation parameters related to solution symmetries.
  - Three zones: blowup core, radiation, distant/decoupled.
- Virial/Coercivity constraints; Orthogonality conditions.
- A key role played by Energy conservation.

Geometrical Description

- Near T^*, $\log\log$ blowups satisfy **geometrical ansatz**

$$u(t,x) = \frac{1}{\lambda(t)}(Q_{b(t)} + \epsilon)\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}.$$

- Parameters $(\lambda(t), x(t), \gamma(t), b(t))$ solve ODEs forced by $F(\epsilon)$.
- ODEs emerge from geometrical ansatz, taking inner products with equation, imposing orthogonality conditions.
  (These choices change across the [MR] works.)

Energy Conservation in [MR] analysis

- Control of ϵ:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- Energy conservation and $\lambda \searrow 0 \implies$

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|. \longrightarrow \boxed{0}$$

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)

5. Singular Ring Solutions of Cubic NLS on $\mathbb{R}^3$

This section describes work of **I. Zwiers** (Toronto Ph.D Student).

5. Singular Ring Solutions of Cubic NLS on $\mathbb{R}^3$

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Consider the cubic focusing NLS initial value problem on $\mathbb{R}^3$:

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0, \mathbf{x}) = u_0(\mathbf{x}). \end{cases} \qquad (NLS_3^-(\mathbb{R}^3))$$

Inspired by work of P. Raphaël, consider cylindrical coordinates

$$\mathbf{x} = (r, \theta, z) \in \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}$$

and seek a cylindrically symmetric solution (independent of θ). A solution like this is a function of $(r, z) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying

$$(i\partial_t + \partial_r^2 + \partial_z^2)u = -|u|^2 u + \text{error}.$$

This equation resembles $NLS_3^-(\mathbb{R}^2_{z,x})$ with stable $\log\log$ blowups.

Numerical Collapse to Circle [Gavish-Fibich]

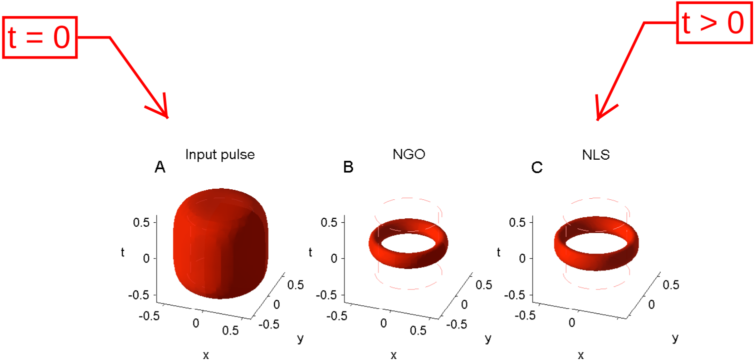

Figure 1: Iso-amplitude plot of the amplitude of the solution of the three-dimensional cubic NLS with initial condition $\psi_0 = 20e^{-(x^2+y^2)^2-z^2}$. A: Initial condition. B: NGO prediction at $t = 0.02$. C: NLS solution at $t = 0.0099$.

Theorem(I. Zwiers) Singular Ring for $NLS_3^-(\mathbb{R}^3)$

$\exists$ cylindrically symmetric initial data $u_0 \longmapsto u(t)$ along $NLS_3^-(\mathbb{R}^3)$ for $t \in [0, T^*)$ (forward maximal, finite) and, as $t \nearrow T^*$:

- $\exists$ $(\lambda(t), \rho(t), \zeta(t), \gamma(t)) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ such that

$$u(t,x) - \frac{1}{\lambda(t)} Q\left(\frac{[r,z]-[\rho(t),\zeta(t)]}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2(\mathbb{R}^3)$$

- Sharp $\log\log$ speed law holds:

$$\lambda(t)\sqrt{\frac{\log|\log(T^*-t)|}{T^*-t}} \to \sqrt{2\pi}$$

- Singularity point converges $[\rho(t), \zeta(t)] \to [r_0, z_0] \sim (1,0)$
- Regularity persists outside singularity: $\forall\ R > 0$,

$$u^* \in H^1(|[[\rho(t), \zeta(t)] - [r_0, z_0]| > R).$$

Remarks on Zwiers' Theorem

- Exploits $L^2(\mathbb{R}^2)$-critical $\log\log$ machinery of [Merle-Raphaël].
- Inspired by singular circle solution of $NLS_5^-(\mathbb{R})$ of [Raphaël].
- Solutions of $NLS_5(\mathbb{R}^N)$ singular on $\mathbb{S}^{N-1}$ were constructed by [Raphaël-Szeftel].
  Regularity persistence result of [Z] built on ideas from [RS].
- Zwiers singular ring solution provides another example of "Type II" singularity in the energy supercritical regime.
- Scaling Heuristics (based on mass concentration) suggest these solutions saturate dimension upper bounds on possible singular sets:

$$\dim_H(\{\mathbf{x} : (T^*, \mathbf{x}) \text{is singular}) \leq 2s_c = 2(\frac{d}{2} - \frac{2}{p-1})?$$

Connect this with partial regularity results of Scheffer, Cafarelli-Kohn-Nirenberg on Navier-Stokes?