A Frequency Cascading Solution of the 2D periodic Schrödinger equation

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1 Introduction

2 Overview of Proof

3 Concatenated Sliders for Toy Model

4 Construction of Resonant Set $\Lambda$
1. Introduction
The NLS Initial Value Problem

[Joint work with Keel, Staffilani, Takaoka and Tao]

We consider the defocusing initial value problem:

\[
\begin{cases}
(-i\partial_t + \Delta) u = |u|^2 u \\
u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2.
\end{cases}
\]

\( (NLS(\mathbb{T}^2)) \)

Smooth solution \( u(x, t) \) exists globally and

\[
\text{Mass} = M(u) = \|u(t)\|^2 = M(0)
\]

\[
\text{Energy} = E(u) = \int \left( \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(x, t)|^4 \right) dx = E(0)
\]

We want to understand the shape of \( |\hat{u}(t, \xi)| \). The conservation laws impose \( L^2 \)-moment constraints on this object.
Notion of Frequency Cascade

**Definition**

A frequency cascade is the phenomenon of global-in-time solutions shifting their mass toward increasingly high frequencies.

This shift is also called a *forward cascade*.

- A way to measure the cascade is to study

  \[ \| u(t) \|_{\dot{H}^s}^2 = \int |\hat{u}(t, \xi)|^2 |\xi|^{2s} d\xi \]

  and prove that it grows for large times \( t \).

- The cascade is incompatible with *scattering* and *integrability*. 

Incompatible with Scattering & Integrability

- **Scattering**: \( \forall \) global solution \( u(t, x) \in H^s \ \exists \ u_0^+ \in H^s \) such that,

\[
\lim_{t \to +\infty} \| u(t, x) - e^{it\Delta} u_0^+ (x) \|_{H^s} = 0.
\]

Note: \( \| e^{it\Delta} u_0^+ \|_{H^s} = \| u_0^+ \|_{H^s} \ \Rightarrow \ \| u(t) \|_{H^s} \) is bounded.

- **Complete Integrability**: The 1d equation

\[
(i\partial_t + \Delta)u = -|u|^2 u
\]

has infinitely many conservation laws. Combining them in the right way one gets that \( \| u(t) \|_{H^s} \leq C_s \) for all times.
Past Results

- **Bourgain**: (late 90’s)
  For the periodic IVP $\textit{NLS}(\mathbb{T}^2)$ one can prove
  \[ \| u(t) \|_{H^s}^2 \leq C_s |t|^{4s}. \]

  The idea is to improve the local estimate for $t \in [-1, 1]$
  \[ \| u(t) \|_{H^s} \leq C_s \| u(0) \|_{H^s}, \quad \text{for } C_s \gg 1 \]
  
  (\implies \| u(t) \|_{H^s} \lesssim C |t| \text{ upper bounds}) to obtain
  \[ \| u(t) \|_{H^s} \leq 1 \| u(0) \|_{H^s} + C_s \| u(0) \|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1, \]
  
  for some $\delta > 0$. This iterates to give
  \[ \| u(t) \|_{H^s} \leq C_s |t|^{1/\delta}. \]

- **Improvements**: Staffilani, Colliander-Delort-Kenig-Staffilani.
**Past Results**

- **Bourgain:** (late 90’s)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x, t)$ to the modified wave equation
  \[
  (\partial_{tt} - \tilde{\Delta}) u = u^p
  \]
such that $\|u(t)\|_{H^s} \sim |t|^m$.

- **Physics:** Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

- **Kuksin** has studied a small dispersion NLS
  \[
i\partial_t w + \delta \Delta w = |w|^2 w
  \]
  with odd, periodic boundary conditions and with $0 < \delta \ll 1$.
  Smooth norms of relatively generic data, with unit sized $L^2$ norm, are shown to grow larger than a negative power of $\delta$.
  These results correspond to large data solutions of the $\delta = 1$ problem $NLS^+_3(\mathbb{T}^2)$.
Exploding Sobolev Norms Conjecture?

- Solutions to dispersive equations on $\mathbb{R}^d$ have bounded high Sobolev norms.
- There are solutions to nonlinear dispersive equations on $\mathbb{T}^d$ with exploding Sobolev norms. In particular for $\text{NLS}(\mathbb{T}^2)$ there exists $u(t, x)$ such that
  \[\|u(t)\|_{H^s}^2 \to \infty \text{ as } t \to \infty.\]
- The cascade phenomena of high Sobolev norm explosion should be generic in phase space.
Main Result

We consider the defocusing initial value problem:

\[
\begin{cases}
(-i\partial_t + \Delta)u = |u|^2 u \\
u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2.
\end{cases}
\]  

(\text{NLS}(\mathbb{T}^2))

Theorem (C-Keel-Staffilani-Takaoka-Tao)

Let \( s > 1 \), \( K \gg 1 \) and \( 0 < \sigma < 1 \) be given. Then there exists a global smooth solution \( u(t, x) \) and \( T > 0 \) such that

\[
\|u_0\|_{H^s} \leq \sigma
\]

and

\[
\|u(t)\|_{H^s}^2 \geq K.
\]
2. Overview of Proof
2. Overview of Proof

- NLS (with Gauge; Mass)
- FNLS (Projection to RFNLS)
- RNFLS_L (Projection)
- Toy Model
- Arnold Diffusion

- Stationary Phase
- Nonresonant Part
- Concatenated Sliders
- Restrict Data
Preliminary reductions

- **Gauge Freedom:**
  If $u$ solves NLS then $v(t, x) = e^{-i2Gt} u(t, x)$ solves
  \[
  \begin{cases}
  i \partial_t v + \Delta v = (2G + |v|^2)v \\
  v(0, x) = v_0(x), \quad x \in \mathbb{T}^2.
  \end{cases}
  \]
  \( (NLS_G) \)

- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,
  \[
  v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)}.
  \]

\[
\begin{cases}
  i \partial_t a_n = 2G a_n + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} a_{n_1} \bar{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\
  n_1 - n_2 + n_3 = n \\
  a_n(0) = \hat{u}_0(n),
  \end{cases}
\]
\( (\mathcal{F}NLS_G) \)
Preliminary reductions

- Diagonal decomposition of sum:

\[
\sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

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= \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
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\]

- Choice of $G$: 

\[G = -\|u_0\|_{L^2}^2.\]
Preliminary reductions

- Diagonal decomposition of sum:

\[
\sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n = \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
\quad + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n \quad + \quad \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
\quad + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n \quad - \quad \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
\quad + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 = n_1 = n_3
\]

- Choice of \( G \):

\[
G = -\|u_0\|_{L^2}^2.
\]
**Resonant truncation**

- *NLS* dynamic is recast as

\[-i \partial_t a_n = -a_n|a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t}.\]  

\[(\mathcal{F} NLS)\]

where \(\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2\), and

\[\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.\]

- \(\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}\).

\[= \{\text{Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n) \text{ is a rectangle}\}\]

- The *resonant truncation* of \(\mathcal{F} NLS\) is

\[-i \partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b}_{n_2} b_{n_3}.\]  

\[(R\mathcal{F} NLS)\]
Finite dimensional resonant restriction

- A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if
  $$n_1, n_2, n_3 \in \Gamma_{\text{res}}(n), \quad n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$  

- A finite dimensional resonant restriction of $\mathcal{F}NLS$ is
  $$-i \partial_t b_n = - b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} b_{n_2} b_{n_3}. \quad (R\mathcal{F}NLS_{\Lambda})$$

- $\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $R\mathcal{F}NLS_{\Lambda}$ is an ODE.
- If $\text{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}NLS$-evolution $a_n(0) \mapsto a_n(t)$ is nicely approximated by $R\mathcal{F}NLS_{\Lambda}$-ODE $a_n(0) \mapsto b_n(t)$.
- Given $\epsilon, s, K$, build $\Lambda$ so that $R\mathcal{F}NLS_{\Lambda}$ cascades.
Imagine a resonant-closed \( \Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M \) with properties.

Define a **nuclear family** to be a rectangle \((n_1, n_2, n_3, n_4)\) where the frequencies \(n_1, n_3\) (the 'parents') live in generation \(\Lambda_j\) and \(n_2, n_4\) (‘children’) live in generation \(\Lambda_{j+1}\).

- \(\forall 1 \leq j < M \text{ and } \forall n_1 \in \Lambda_j \exists \text{ unique nuclear family such that } n_1, n_3 \in \Lambda_j \text{ are parents and } n_2, n_4 \in \Lambda_{j+1} \text{ are children.}\)
- \(\forall 1 \leq j < M \text{ and } \forall n_2 \in \Lambda_{j+1} \exists \text{ unique nuclear family such that } n_2, n_4 \in \Lambda_{j+1} \text{ are children and } n_1, n_3 \in \Lambda_j \text{ are parents.}\)
- The sibling of a frequency is never its spouse.
- Besides nuclear families, \(\Lambda\) contains no other rectangles.
- The function \(n \mapsto a_n(0)\) is constant on each generation \(\Lambda_j\).
Cartoon Construction of $\Lambda$
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The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $R\mathcal{F}NLS_\Lambda$ simplifies to the toy model ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\bar{b}_j(t) [b_{j-1}(t)^2 + b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 (\sum_{n \in \Lambda_j} |n|^{2s}).$$

We also want $\Lambda$ to satisfy Wide Diaspora Property

$$\sum_{n \in \Lambda_M} |n|^{2s} \gg \sum_{n \in \Lambda_1} |n|^{2s}.$$
Conservation laws for the ODE system

\[
\text{Mass} = \sum_j |b_j(t)|^2 = C_0
\]

\[
\text{Momentum} = \sum_j |b_j(t)|^2 \sum_{n \in \Lambda_j} n = C_1,
\]

\[
\text{Energy} = K + P = C_2,
\]

where

\[
K = \sum_j |b_j(t)|^2 \sum_{n \in \Lambda_j} |n|^2,
\]

\[
P = \frac{1}{2} \sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2.
\]

Conservation laws for ODE do not involve Fourier moments!
3. Concatenated Sliders for Toy Model ODE
Using dynamical systems methods, we construct a Toy Model ODE evolution such that:
3. **Concatenated Sliders for Toy Model ODE**

Using dynamical systems methods, we construct a Toy Model ODE evolution such that:

A travelling wave through the generations.
Solution of the Toy Model is a vector flow \( t \rightarrow b(t) \in \mathbb{C}^M \)

\[
b(t) = (b_1(t), \ldots, b_M(t)) \in \mathbb{C}^M; \quad b_j = 0 \quad \forall \ j \leq 0, j \geq M + 1.
\]

Local Well-Posedness; Let \( S(t) \) denote associated flowmap.

Mass Conservation: \( |b(t)|^2 = |b(0)|^2 \implies \)

- Toy Model ODE is Globally Well-Posed.
- Invariance of the sphere: \( \Sigma = \{ x \in \mathbb{C}^M : |x|^2 = 1 \} \)

\[
S(t)\Sigma = \Sigma.
\]
**Properties of the Toy Model ODE**

- **Support Conservation:**

\[
\partial_t |b_j|^2 = 2 \text{Re}(\overline{b_j} \partial_t b_j) = 4 \text{Re}(i b_j^2 [b_{j-1}^2 + b_{j+1}^2]) \leq 4 |b_j|^2.
\]

Thus, if \(b_j(0) = 0\) then \(b_j(t) = 0\) for all \(t\).

- **Invariance of coordinate tori:**

\[
\mathcal{T}_j = \{(b_1, \ldots, b_M \in \Sigma) : |b_j| = 1, b_k = 0 \forall k \neq j\}
\]

Mass Conservation \(\implies S(T)\mathcal{T}_j = \mathcal{T}_j\).
Dynamics on the invariant tori is easy:

\[
b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \forall k \neq j.
\]
Consider $M = 2$. Then ODE is of the form

\[
\begin{align*}
\partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1}b_2^2 \\
\partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2}b_1^2.
\end{align*}
\]

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

\[
\begin{align*}
   b_1(t) &= \frac{e^{-it}}{\sqrt{1 + e^{2\sqrt{3}t}}}\omega, \\
   b_2(t) &= \frac{e^{-it}}{\sqrt{1 + e^{-2\sqrt{3}t}}}\omega^2.
\end{align*}
\]

- As $t \to -\infty$, $(b_1(t), b_2(t)) \to (e^{-it}\omega, 0) \in \mathbb{T}_1$.
- As $t \to +\infty$, $(b_1(t), b_2(t)) \to (0, e^{-it}\omega^2) \in \mathbb{T}_2$. 

**Explicit Slider Solutions**
TWO EXPLICIT SOLUTION FAMILIES

\[ T_j \]

\[ T_1 \]

\[ T_2 \]
Concatenated Sliders: Idea of Proof

$T_3 \rightarrow T_4 \rightarrow T_5 \rightarrow T_6$
The Perfect Shot?

*Off the expressway, over the river, off the billboard, through the window, nothin but net.* —Michael Jordan
Diffusion for Toy Model Statement

**Theorem**

Let $M \geq 6$. Given $\epsilon > 0$ there exist $x_3$ within $\epsilon$ of $T_3$ and $x_{M-2}$ within $\epsilon$ of $T_{M-2}$ and a time $t$ such that

$$S(t)x_3 = x_{M-2}.$$ 

**Remark**

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time $t_0$ and then arbitrarily concentrated at mode $j = M − 2$ at later time $t$. 

Let $O, D$ denote points in our phase space $\Sigma$. Can we flow along $S(t)$ from *nearby* the origin point 0 to *nearby* the destination point $D$? More generally, suppose $O$ and $D$ are subsets of $\Sigma$.

The notion of a **target** quantifies this question.
Let $\mathcal{M}$ denote a subset of $\Sigma$. Let $d$ be a (pseudo)metric on $\Sigma$. Let $R > 0$ be a radius.

The **Target** $(\mathcal{M}, d, R) := \{x \in \Sigma : d(x, \mathcal{M}) < R\}$.

Given $x, y \in \Sigma$.
We say $x$ **hits** $y$ if $y = S(t)x$ for some $t \geq 0$. 
Given an initial target \((M_1, d_1, R_1)\) and a final target \((M_2, d_2, R_2)\). We say \((M_1, d_1, R_1)\) can cover \((M_2, d_2, R_2)\) and write

\[
(M_1, d_1, R_1) \implies (M_2, d_2, R_2)
\]

if:
\[
\forall x_2 \in M_2 \exists x_1 \in M_1 \text{ such that } \forall y_1 \in \Sigma \text{ with } d_1(x_1, y_1) < R_1 \exists y_2 \in \Sigma \text{ with } d(x_2, y_2) < R_2 \text{ such that } y_1 \text{ hits } y_2.
\]

- The flowout of \((M_1, d_1, R_1)\) is **surjective** onto \((M_2, d_2, R_2)\).
- Covering also includes a notion of stability.
Strategy of Proof

- **Transitivity of Covering:** If \((M_1, d_1, r_1) \implies (M_2, d_2, r_2)\) and \((M_2, d_2, r_2) \implies (M_3, d_3, r_3)\), then \((M_1, d_1, r_1) \implies (M_3, d_3, r_3)\).

- \(\forall j \in 3, \ldots, M - 2\) we define 3 targets close to \(T_j\):
  - Incoming Target \((M^-_j, d^-_j, R^-_j)\)
  - Ricochet Target \((M^0_j, d^0_j, R^0_j)\)
  - Outgoing Target \((M^+_j, d^+_j, R^+_j)\)

- \(\forall j = 3, \ldots, M - 2\) with appropriate \(d^-_j, 0, +, R^-_j, 0, +\), prove:
  - \((M^-_j, d^-_j, R^-_j) \implies (M^0_j, d^0_j, R^0_j)\)
  - \((M^0_j, d^0_j, R^0_j) \implies (M^+_j, d^+_j, R^+_j)\)
  - \((M^+_j, d^+_j, R^+_j) \implies (M^-_{j+1}, d^-_{j+1}, R^-_{j+1})\)
Targets Around $T_j$
4. **Construction of Resonant Set** $\Lambda$
The task is to construct a finite set $\Lambda \subset \mathbb{Z}^2$ satisfying the properties that led to the Toy Model ODE. We do this in two steps:

1. Build combinatorial model of $\Lambda$ called $\Sigma \subset \mathbb{C}^{M-1}$.
2. Build a map $f : \mathbb{C}^{M-1} \rightarrow \mathbb{R}^2$ which gives

$$f(\Sigma) = \Lambda \subset \mathbb{Z}^2$$

satisfying the properties.
Construction of Combinatorial Model $\Sigma$

- Standard Unit Square: $S = \{0, 1, 1 + i, i\} \subset \mathbb{C}$, $S = S_1 \cup S_2$
  where $S_1 = \{1, i\}$ and $S_2 = \{0, 1 + i\}$

- $\mathbb{Z}^2 \equiv \mathbb{Z}[i]; \ (n_1, n_2) \equiv n_1 + in_2$
Construction of Combinatorial Model $\Sigma$

- We define

$$\Sigma_j = \{(z_1, z_2, \ldots, z_{M-1}) : z_1, \ldots, z_{j-1} \in S_2, z_j, \ldots, z_{M-1} \in S_1\}$$

with the properties
  - $\Sigma_j = S_2^{j-1} \times S_1^{M-j} \subset \mathbb{C}^{M-1}$
  - $|\Sigma_j| = 2^{M-1}$

- Next, we define

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M.$$

- $|\Sigma| = M 2^{M-1}$.
- $\Sigma_j$ is called a generation.
Consider the set $F = \{F_0, F_1, F_{1+i}, F_i\} \subset \Sigma$ defined by

$$F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n)$$

with $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_n \in S_2$ and $w \in S$.

- The elements $F_0, F_{1+i} \in \Sigma_{j+1}$ are called *children*.
- The elements $F_1, F_i$ are called *parents*.
- The four element set $F$ is called a *combinatorial nuclear family* connecting the generations $\Sigma_j$ and $\Sigma_{j+1}$.

$\forall j \exists 2^{M-2}$ combinatorial nuclear families connecting generations $\Sigma_j$ and $\Sigma_{j+1}$.

- The set $\Sigma$ satisfies
  - Existence and uniqueness of spouse and children (of sibling and parents).
  - Sibling is never also a spouse.
We need to map $\Sigma \subset \mathbb{C}^{M-1}$ into the frequency lattice $\mathbb{Z}^2$.

- We first define $f_1 : \Sigma_1 \to \mathbb{C}$.
- $\forall 1 \leq j \leq M$ and each combinatorial nuclear family $F$ connecting generations $\Sigma_j$ and $\Sigma_{j+1}$, we associate an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.
- Given $f_1$ and the angles of all the families, we define placement functions $f_j : \Sigma_j \to \mathbb{C}$ recursively by the rule:
  Suppose $f_j : \Sigma_j \to \mathbb{C}$ has been defined. We define $f_{j+1} : \Sigma_{j+1} \to \mathbb{C}$:

$$f_{j+1}(F_{1+i}) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) + \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

$$f_{j+1}(F_0) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) - \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

for all combinatorial nuclear families connecting $\Sigma_j$ to $\Sigma_{j+1}$.
Theorem: Good Placement Function

Let $M \geq 2$, $s > 1$, and let $N$ be a sufficiently large integer (depending on $M$). \exists an initial placement function $f_1 : \Sigma_1 \to \mathbb{C}$ and choices of angles $\theta(F)$ for each nuclear family $F$ (and thus an associated complete placement function $f : \Sigma \to \mathbb{C}$) with the following properties:

- **(Non-degeneracy)** The function $f$ is injective.
- **(Integrality)** We have $f(\Sigma) \subset \mathbb{Z}[i]$.
- **(Magnitude)** We have $C(M)^{-1}N \leq |f(x)| \leq C(M)N$ for all $x \in \Sigma$.
- **(Closure/Faithfulness)** If $x_1, x_2, x_3$ are distinct elements of $\Sigma$ are such that $f(x_1), f(x_2), f(x_3)$ form a right-angled triangle, then $x_1, x_2, x_3$ belong to a combinatorial nuclear family.
- **(Wide Diaspora/Norm Explosion)** We have

$$\sum_{n \in f(\Sigma_M)} |n|^{2s} > \frac{1}{2^{2(s-1)(M-1)}} \sum_{n \in f(\Sigma_1)} |n|^{2s}.$$