# The $I$-method with a Morawetz bootstrap

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# 1. Bourgain's High-Low Fourier Truncation

S. Kuksin's Request: Survey Bourgain's high-low argument.

Afterwards, we return to discuss Morawetz-type estimates.

# 1. Bourgain's High-Low Fourier Truncation

Consider the Cauchy problem for defocusing cubic NLS on $\mathbb{R}^2$:

$$\begin{cases} (i\partial_t + \Delta)u = +|u|^2 u \\ u(0,x) = \phi_0(x). \end{cases} \qquad (NLS_3^+(\mathbb{R}^2))$$

We describe the first result to give global well-posedness below $H^1$.

- $NLS_3^+(\mathbb{R}^2)$ is GWP in $H^s$ for $s > \frac{2}{3}$ [Bourgain 98].
- First use of Bilinear Strichartz estimate was in this proof.
- Proof cuts solution into low and high frequency parts.

# Setting up; Decomposing Data

- Fix a large target time $T$.
- Let $N = N(T)$ be large to be determined.
- Decompose the initial data:

$$\phi_0 = \phi_{low} + \phi_{high}$$

high/low cut

where

$$\phi_{low}(x) = \int_{|\xi|<N} e^{ix\cdot\xi}\widehat{\phi_0}(\xi)d\xi.$$

- Our plan is to evolve:

$$\phi_0 = \phi_{low} + \phi_{high}$$

NLS NLS DE

$$u(t) = u_{low}(t) + u_{high}(t).$$

# Setting up; Decomposing Data

Low Frequency Data Size:

- Kinetic Energy:

$$\begin{aligned}\|\nabla\phi_{low}\|_{L^2}^2 &= \int_{|\xi|<N} |\xi|^2|\widehat{\phi_0}(\xi)|^2 dx \\ &= \int_{|\xi|<N} |\xi|^{2(1-s)}|\xi|^{2s}|\widehat{\phi_0}(\xi)|^2 dx \\ &= N^{2(1-s)}\|\phi_0\|_{H^s}^2 \leq C_0 N^{2(1-s)}.\end{aligned}$$

$\boxed{\text{Big}}$

- Potential Energy: $\|\phi_{low}\|_{L_x^4} \leq \|\phi_{low}\|_{L^2}^{1/2}\|\nabla\phi_{low}\|_{L^2}^{1/2}$

$$\implies H[\phi_{low}] \leq CN^{2(1-s)}.$$

High Frequency Data Size:

$$\|\phi_{high}\|_{L^2} \leq C_0 N^{-s}, \ \|\phi_{high}\|_{H^s} \leq C_0.$$

$\boxed{\text{Small}}$

# LWP of Low Frequency Evolution along NLS

The NLS Cauchy Problem for the low frequency data

$$\begin{cases} (i\partial_t + \Delta)u_{low} = +|u_{low}|^2 u_{low} \\ u_{low}(0,x) = \phi_{low}(x) \end{cases}$$

is well-posed on $[0, T_{lwp}]$ with $T_{lwp} \sim \|\phi_{low}\|_{H^1}^{-2} \sim N^{-2(1-s)}$.

We obtain, as a consequence of the local theory, that

$$\|u_{low}\|_{L^4_{[0,T_{lwp}],x}} \leq \frac{1}{100}.$$

# LWP of High Frequency Evolution along DE

The NLS Cauchy Problem for the low frequency data

$$\begin{cases} (i\partial_t + \Delta)u_{high} = +2|u_{low}|^2 u_{high} + \text{similar} + |u_{high}|^2 u_{high} \\ u_{high}(0,x) = \phi_{high}(x) \end{cases}$$

is also well-posed on $[0, T_{lwp}]$.

**Remark:** The LWP lifetime of *NLS* evolution of $u_{low}$ AND the LWP lifetime of the *DE* evolution of $u_{high}$ are controlled by $\|u_{low}(0)\|_{H^1}$.

# Extra Smoothing of Nonlinear Duhamel Term

The high frequency evolution may be written

$$u_{high}(t) = e^{it\Delta} u_{high} + w.$$

The local theory gives $\|w(t)\|_{L^2} \lesssim N^{-s}$. Moreover, due to smoothing (obtained via bilinear Strichartz), we have that

$$w \in H^1, \ \|w(t)\|_{H^1} \lesssim N^{1-2s+}. \qquad \text{(SMOOTH!)}$$

Let's postpone the proof of (SMOOTH!).

# Nonlinear High Frequency Term Hiding Step!

- $\forall\ t \in [0, T_{lwp}]$, we have

$$u(t) = u_{low}(t) + e^{it\Delta}\phi_{high} + w(t).$$

- At time $T_{lwp}$, we define data for the progressive sheme:

$$u(T_{lwp}) = \underline{u_{low}(T_{lwp}) + w(T_{lwp})} + e^{iT_{lwp}\Delta}\phi_{high}.$$

$$u(t) = u^{(2)}_{low}(t) + u^{(2)}_{high}(t)$$

for $t > T_{lwp}$.

# HAMILTONIAN INCREMENT: $\phi_{low}(0) \longmapsto u^{(2)}_{low}(T_{lwp})$

The Hamiltonian increment due to $w(T_{lwp})$ being added to low frequency evolution can be calcluated. Indeed, by Taylor expansion, using the bound (SMOOTH!) and energy conservation of $u_{low}$ evolution, we have using

$$\begin{aligned} H[u^{(2)}_{low}(T_{lwp})] &= H[u_{low}(0)] + (H[u_{low}(T_{lwp}) + w(T_{lwp})] - H[u_{low}(T_{lwp})]) \\ &\sim N^{2(1-s)} + N^{2-3s+} \sim N^{2(1-s)}. \end{aligned}$$

Moreover, we can accumulate $N^s$ increments of size $N^{2-3s+}$ before we double the size $N^{2(1-s)}$ of the Hamiltonian. During the iteration, Hamiltonian of "low frequency" pieces remains of size $\lesssim N^{2(1-s)}$ so the LWP steps are of uniform size $N^{-2(1-s)}$. We advance the solution on a time interval of size:

$$N^s N^{-2(1-s)} = N^{-2+3s}.$$

For $s > \frac{2}{3}$, we can choose $N$ to go past target time $T$.

# How do we prove (SMOOTH!)?

Recall Bourgain's Bilinear Strichartz Estimate: For (dyadic) $N \leq L$

$$\|e^{it\Delta} f_L e^{it\Delta} g_N\|_{L^2_{t,x}} \leq \frac{N^{\frac{2-1}{2}}}{L^{\frac{1}{2}}} \|f_L\|_{L^2_x} \|g_N\|_{L^2_x}.$$

## Corollary

*For $s \geq \frac{1}{2}$*

$$\begin{aligned}\|D_x^s(u_1 u_2)\|_{L^2_{[0,\delta],x}} \leq & C(\|u_1\|_{X^{s,1/2+}_{[0,\delta]}} \|u_2\|_{X^{0,1/2+}_{[0,\delta]}} \\ & + \|u_1\|_{X^{1/2,1/2+}_{[0,\delta]}} \|u_2\|_{X^{s-1/2,1/2+}_{[0,\delta]}}).\end{aligned}$$

Thus, the Bilinear Estimate allows us move half a derivative off the high frequency part and instead onto of the low frequency part.

Bourgain's Proof

**Lemma 111.** Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^2)$ such that

$$\psi_1 = \Delta_{M_1}\psi_1 \quad \text{and} \quad \psi_2 = \Delta_{M_2}\psi_2,$$

where we denote

$$\Delta_M\psi = \int_{|\xi|\sim M} \widehat{\psi}(\xi)e^{ix\xi}\, d\xi.$$

Then, for $M_1 \le M_2$, the following inequality holds:

$$\boxed{\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_{L^2(\mathbb{R}^2\times\mathbb{R})} \le C\left(\frac{M_1}{M_2}\right)^{1/2} \|\psi_1\|_2\|\psi_2\|_2.} \tag{112}$$

$\square$

Proof. Since the standard Strichartz inequality yields (112) without the

$$\left(\frac{M_1}{M_2}\right)^{\frac{1}{2}} \text{-factor},$$

we may assume $M_2 \gg M_1$.

Writing

$$(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2) = \int \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\xi_2)e^{i[(\xi_1+\xi_2).x+(|\xi_1|^2+|\xi_2|^2)t]}\, d\xi_1\, d\xi_2,$$

it follows from Parseval's identity and Cauchy-Schwarz that

$$\begin{aligned}\|(e^{it\Delta}\psi_1)(e^{it\Delta}\psi_2)\|_2^2 &= \int d\xi d\lambda \left|\int \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\xi-\xi_1)\delta_0(|\xi_1|^2+|\xi-\xi_1|^2-\lambda)\, d\xi_1\right|^2 \\ &\le \|\psi_1\|_2^2\|\psi_2\|_2^2\Big[\sup_{\lambda,|\xi|\sim M_2} \mathrm{mes}_{(1)}[\xi_1 \,|\, |\xi_1| \sim M_1 \\ &\qquad \text{and } |\xi_1|^2 + |\xi-\xi_1|^2 = \lambda]\Big] \\ &< C\frac{M_1}{M_2}.\end{aligned}$$

# Treatment of a typical term in $w$

- Using the controls we have on $u_{low}, u_{high}$ from the local theory on $[0, T_{lwp}]$, we want to prove for

$$w = \int_0^t e^{i(t-t')\Delta} |u_{low}|^2 u_{high}(t') dt'$$

  that $\sup_{t \in [0, T_{lwp}]} \|\nabla w\|_{L^2} < N^{1-2S+}$.
- By Sobolev embedding, we have

$$\|w\|_{L^\infty_{[0,T_{lwp}]} H^1} \leq \|w\|_{X^{1,1/2+}_{[0,T_{lwp}]}}.$$

- The mapping $f \longmapsto \int_0^t e^{i(t-t')\Delta}$ is formally $f \longmapsto (i\partial_t + \Delta)^{-1} f$ which, due to time localization, is essentially $\widehat{f} \longmapsto \langle \tau - |\xi|^2 \rangle \widehat{f}$. It suffices to control $\|D_x |u_{low}|^2 u_{high}\|_{X^{0,-1/2+}}$. Proceed by duality....

# Treatment of a typical term in $w$

$$\|w\|_{L^\infty_{[0,T_{lwp}]}H^1} \leq \sup_{\|g\|_{X^{0,1/2-}} \leq 1} \langle g, D_x(|u_{low}|^2 u_{high})\rangle.$$

$$\lesssim \sup_g \langle g D_x u_{low}, u_{low} u_{high}\rangle + \sup_g \langle g u_{low}, D_x(u_{low} u_{high})\rangle$$

$$= \text{easier} + \sup_g \langle D_x^{1/2}(g u_{low}), D_x^{1/2}(u_{low} u_{high}\rangle.$$

The corollary and the available bounds then give (SMOOTH!).

# 2. Generalized Virial Identities

# Cauchy Problem for Lagrangian NLS

Consider the initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = \pm F'(|u|^2)u \\ u(0,x) = u_0(x) \end{cases} \qquad (NLS_F^{\pm}(\mathbb{R}^d))$$

**Remarks:**

- Assume $F' \geq 0$. The $+$ case is defocusing; $-$ is focusing.
- Generalized NLS with Lagrangian derivation.
- $U(1)$ solution symmetry: $u \to e^{i\theta}u$.

# Time Invariant Quantities

The following quantities do not change with time:

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx \pm F(|u(t)|^2) dx.$$

$\Longrightarrow$ a priori **conservation controls** (defocusing case):

$$\begin{aligned} \|u\|_{L_t^\infty L_x^2} &\leq \|u_0\|_{L^2} \\ \|\nabla u\|_{L_t^\infty L_x^2} &\leq E[u_0]. \end{aligned}$$

These are very useful bounds but do not give any decay in time.

# Local Conservation Laws

We consider an even more general NLS equation.

- Suppose $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{C}$ solves **generalized NLS**

$$(i\partial_t + \Delta)\phi = \mathcal{N} \qquad (GNLS(\mathbb{R}^d))$$

  for $\mathcal{N} = \mathcal{N}(t, x, \phi) : [0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$. Assume $\phi$ is nice.
- Not necessarily Lagrangian; No $U(1)$ symmetry.
- Express mass & momentum (non)conservation for $GNLS$.

Write $\partial_{x_j}\phi = \partial_j\phi = \phi_j$.

# Local mass/momentum (non)conservation

- mass density: $T_{00} = |\phi|^2$
- momentum density/mass current:
  $T_{0j} = T_{j0} = 2\Im(\overline{\phi}\phi_j)$
- (linear part of the) momentum current:
  $L_{jk} = L_{kj} = -\partial_j\partial_k|\phi|^2 + 4\Re(\overline{\phi_j}\phi_k)$
- mass bracket: $\{f, g\}_m = \Im(f\overline{g})$
- momentum bracket: $\{f, g\}_p^j = \Re(f\partial_j\overline{g} - g\partial_j\overline{f})$

**Local mass (non)conservation identity:**

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

**Local momentum (non)conservation identity:**

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j$$

# Local mass/momentum (non)conservation

Consider $\mathcal{N} = F'(|\phi|^2)\phi$ for polynomial $F : \mathbb{R}^+ \to \mathbb{R}$.

- We calculate the mass bracket

$$\{F'(|\phi|^2)\phi, \phi\}_m = \Im(F'(|\phi|^2)\phi\overline{\phi}) = 0.$$

  Thus mass is conserved for these nonlinearities.

- We calculate the momentum bracket

$$\{F'(|\phi|^2)\phi, \phi\}_p^j = -\partial_j G(|\phi|^2)$$

  where $G(z) = zF'(z) - F(z) \sim F(z)$.
  Thus the momentum bracket contributes a divergence and momentum is conserved for these nonlinearities.

# Generalized Virial Identity

Let $a : \mathbb{R}^d \to \mathbb{R}$ **(virial weight)**. Form the **virial potential**

$$V_a(t) = \int_{\mathbb{R}^d} a(x)|\phi(t,x)|^2 dx.$$

Form the **Morawetz action**

$$M_a(t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi) dx.$$

Conservation identities lead to the **generalized virial identities**

$$\partial_t V_a = M_a + \int_{\mathbb{R}^d} a(x)\{\mathcal{N},\phi\}_m(t,x)dx,$$

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) + 2a_j\{\mathcal{N},\phi\}_p^j dx.$$

# Remarks on Virial Identities

- The virial potential is a weighted average of the mass density against the virial weight $a$.
- The Morawetz action is a contraction of the momentum density against $\nabla a$. Vector fields not arising as gradients could also be considered.
- Useful estimates emerge from monotonicity and boundedness of terms in the virial identities.
- Monotone quantities provide dynamical insights.
- **Idea of Morawetz Estimates:** Cleverly choose the weight function $a$ so that $\partial_t M_a \geq 0$ but $M_a \leq C(\phi_0)$ to obtain spacetime control on $\phi$. This strategy imposes various constraints on $a$ which suggest choosing $a(x) = |x|$.

# Variance Identity

[Glassey], [Vlasov-Petrischev-Talanov]

- Consider *GNLS* with $\mathcal{N} = \pm|u|^{4/d}u$. This is the $L^2$ critical focusing equation $NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$.
- Choose $a(x) = |x|^2$. Calculations reveal that

$$\partial_t^2 \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx = 16H[u(t)].$$

- In the focusing case, we can consider initial data $u_0$ with $H[u_0] < 0$ and finite variance. Such data must blow up in finite time.

# 2. A Priori Spacetime Estimates

# [Lin-Strauss] Morawetz identity

Consider $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. Choose $a(x) = |x|$. Observe that $a$ is weakly convex, $\nabla a = \frac{x}{|x|}$ is bounded, and $-\Delta\Delta a = 4\pi\delta_0$. From monotonicity $\partial_t M_a \geq 0$ and the bound $|M_a| \leq \sqrt{H[u_0]}$ emerges the **Lin-Strauss Morawetz identity**

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^3} 4\pi\delta_0(x)|\phi(t,x)|^2 + (\geq 0) + 4\frac{G(|\phi|^2)}{|x|} dxdt.$$

This implies the spacetime control estimate (centered at $x = 0$)

$$(H[u_0])^{1/2} \|u_0\|_{L^2} \gtrsim \int_0^T \int_{\mathbb{R}^3} \frac{G(|\phi|^2)}{|x|} dxdt.$$

$\boxed{\text{Kills Solitons}}$

[Morawetz] Reward & Anchor. [Ginibre-Velo] $H^1$-Scattering.

# [Bourgain] & [Grillakis] truncation

- Let $\chi_{B_R}$ denote a smooth cutoff adapted to $B_R = \{|x| < R\}$.
- Choose cutoff virial weight $a(x) = \chi_{B_R}(x)|x|$. and calculate

$$M_a\Big|_0^T \geq \int_0^T \int_{\mathbb{R}^3} 4\pi\delta_0(x)|\phi(t,x)|^2 + 4\int_0^T \int_{|x|<R/2} \frac{G(|\phi|^2)}{|x|} dxdt$$

- $|M_a\Big|_0^T| \leq R^{-1}TH[u_0] + RH[u_0] \implies$ choose $R \sim T^{1/2} \implies$

$$\int_0^T \int_{|x|<T^{1/2}} \frac{G(|\phi|^2)}{|x|} dx \lesssim T^{1/2}\|\nabla u\|^2_{L^\infty_{[0,T]}L^2_x}.$$

[Bourgain][Grillakis]: Energy critical bubbles sparse along time axis.

# Averaging over [Lin-Strauss] center?

- Translation invariance? Weight $|x|^{-1}$ difficult in proofs.
- Recenter [L-S] at fixed $y \in \mathbb{R}^d$. Set $a(x) = |x-y|$.
- Recentered Morawetz action can be expressed

$$M_y[u](t) = \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|} 2\Im(u\nabla\overline{u})(t,x)dx.$$

- Monotonicity $\partial_t M_y[u] \geq 0$: mass is repelled from any $y \in \mathbb{R}^d$.
- Can we **average with respect to center** $y$ and obtain new translation invariant spacetime control?
- Yes, if we average against the natural density $|u(t,y)|^2$.

# Interaction Morawetz via Averaging

- Define the **Morawetz interaction potential**

$$M[u](t) = \int_{\mathbb{R}^d_y} |u(t,y)|^2 M_y[u](t)dy.$$

It is bounded: $\Big|M[u](t)\Big| \lesssim \|u(t)\|^3_{L^2_x}\|\nabla u(t)\|_{L^2_x}$. We calculate

$$\partial_t M[u] = \int_{\mathbb{R}^d_y} |u(t,y)|^2\{\partial_t M_y[u]\} + \{\partial_t |u(y)|^2\} M_y[u]dy.$$

- Local conservation & [L-S] $\implies$ monotonicity:
$\exists\ I, II, III, IV$ such that $I, III \geq 0$ and $II + IV \geq 0$ and $\partial_t M[u] = I + II + III + IV$. Integrating in time gives

$$\int_0^T \int_{\mathbb{R}^3} |u(t,x)|^4 dxdt \lesssim \|u(t)\|^3_{L^\infty_T L^2_x}\|\nabla u(t)\|_{L^\infty_T L^2_x}.$$

# 2-particle interaction Morawetz

(Hassell 04)

- Suppose $\phi_1, \phi_2$ are two solutions of $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. The "2-particle" wave function

$$\Psi(t, x_1, x_2) = \phi_1(t, x_1)\phi_2(t, x_2)$$

  satisfies an NLS-type equation on $\mathbb{R}^{1+6}$

$$(i\partial_t + \Delta_1 + \Delta_2)\Psi = [F'(|\phi_1|^2) + F'(|\phi_2|^2)]\Psi.$$

- Note that $[F'(|\phi_1|^2) + F'(|\phi_2|^2)] \geq 0$ so defocusing.
- Reparametrize $\mathbb{R}^6$ using center-of-mass coordinates $(\overline{x}, y)$ with $\overline{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3$. Note that $y = 0$ corresponds to the diagonal $x_1 = x_2 = \overline{x}$. Apply the generalized virial identity with the **choice** $a(x_1, x_2) = |y|$. Dismissing terms with favorable signs, one obtains...

# 2-particle interaction Morawetz

$$
\begin{aligned}
\|\nabla u\|_{L^\infty_{[0,T]} L^2_x} \|u_0\|^3_{L^2} &\geq \int_0^T \int_{\mathbb{R}^6} (-\Delta_6 \Delta_6 |y|) |\Psi(x_1, x_2)|^2 dx_1 dx_2 dt \\
&\geq c \int_0^T \int_{\mathbb{R}^6} \delta_{\{y=0\}}(x_1, x_2) |\phi_1(x_1)\phi_2(x_2)|^2 dx_1 dx_2 dt \\
&\geq c \int_0^T \int_{\mathbb{R}^3} |\phi_1(t, \overline{x})\phi_2(t, \overline{x})|^2 d\overline{x} dt.
\end{aligned}
$$

Specializing to $\phi_1 = \phi_2$ gives the **2-particle Morawetz estimate**

$$
\int_0^T \int_{\mathbb{R}^3} |\phi(t, x)|^4 dx dt \leq C \|\nabla u\|_{L^\infty_{[0,T]} L^2_x} \|u_0\|^3_{L^2_x}
$$

valid uniformly for all defocusing NLS equations on $\mathbb{R}^3$.

# "The" 2-particle Morawetz Estimate

Efforts to extend the $L^4(\mathbb{R}_t \times \mathbb{R}^3_x)$ interaction Morawetz to the $\mathbb{R}^2_x$ setting led to...

**Theorem (C-Grillakis-Tzirakis & Planchon-Vega)**

*Finite energy solutions of any defocusing $NLS^+(\mathbb{R}^d)$ satisfy*

$$\| D^{\frac{3-d}{2}} |u|^2 \|^2_{L^2_{t,x}} \lesssim \|u_0\|^3_{L^2_x} \|\nabla u\|_{L^\infty_t L^2_x}.$$

- Independently & simultaneously by [Planchon-Vega].
- Gives simple proof of $H^1$-scattering in mass supercritical case. [Nakanishi]
- Simplified proof extends to $H^s$ for certain $s < 1$.

# 4-particle Morawetz Estimate

(Hassel-Tao) [C-Holmer-Visan-Zhang]

- $\mathbb{R}^4 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : x_i \in \mathbb{R}; i = 1, 2, 3, 4\}$
  $\overline{x} =$ **center of mass** $= \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$.
  Define $y = (x_1 - \overline{x}, x_2 - \overline{x}, x_3 - \overline{x}, x_4 - \overline{x})$.
  $\mathbb{R}^4 \ni \mathbf{x} = (x_1, x_2, x_3, x_4) \iff (\overline{x}, y) \in \mathbb{R} \times \mathbb{R}^3$ $\boxed{\text{3d Miracle}}$
- The 4-particle wave function

$$\Psi(t, \mathbf{x}) = \prod_{i=1}^{4} \phi_1(t, x_i)$$

  satisfies a defocusing NLS equation on $\mathbb{R}^{1+4}$.
- Choice of virial weight $a(\mathbf{x}) = |y|$ spawns

$$\int_0^T \int_{\mathbb{R}} |u|^8 d\overline{x} dt \lesssim \|u\|_{L_T^\infty L_x^2}^7 \|\nabla u\|_{L_T^\infty L_x^2}.$$

- How does this estimate generalize to other dimensions?

# 3. $I$-Method with Morawetz Bootstrap

Application: **Subcritical scattering** for certain $NLS_p^+(\mathbb{R}^d)$.
[CKSTT], [CHVZ], [Fang-Grillakis], [C-Grillakis-Tzirakis]

- 2-particle Morawetz is an $\dot{H}^{1/4}$-critical input.
  4-particle Morawetz is an $\dot{H}^{1/8}$-critical input.
- Scaling invariant $H^s$ for $NLS_p^+(\mathbb{R}^d)$ : $s_c = \frac{d}{2} - \frac{2}{p-1}$
- When $1/4 < s_c < 1$ $\exists$ $s_* \in (s_c, 1)$ and $\forall$ $s \geq s_*$ the $H^s$-solutions of $NLS_p^+(\mathbb{R}^d)$ scatter.
- We obtain scattering for certain energy subcritcal $(s_c < 1)$ NLS for infinite energy data of subcritical regularity $(s_c < s_*)$.
- The *critical scattering conjecture* corresponds to $s_c = s_*$.
  This is known (for general data) only for $s_c = 1$.

# $I$-Method with Morawetz Bootstrap

Consider $NLS^{+}_{2k+1}(\mathbb{R})$ (nonlinearity $+|u|^{2k}u$) for $k = 3, 4, \ldots$.
Note that $s_c = \frac{1}{2} - \frac{1}{k}$.
Define
$$s_k = \frac{8k - 16}{9k - 14} < 1.$$

**Theorem (C-Holmer-Visan-Zhang)**

*$\forall\ s > s_k$, $H^s(\mathbb{R}) \ni u_0 \longmapsto u$ solving $NLS^{+}_{2k+1}(\mathbb{R})$ is global in time and scatters: $\exists\ u_{\pm} \in H^s(\mathbb{R})$ such that*

$$\lim_{t \to \pm\infty} \| u(t) - e^{it\Delta} u_{\pm} \|_{H^s(\mathbb{R})} = 0.$$

Proof treats a family of equations; Wish that $s_k \to 1/2$ as $k \to \infty$.

# Finite Energy Scattering

The 4-particle $L^8(\mathbb{R}_t \times \mathbb{R}_x)$ estimate may be reexpressed:

$$\int_0^T \int_{\mathbb{R}} |u|^8 d\overline{x} dt \lesssim \|u\|^6_{L^\infty_T L^2_x} \|u\|_{L^\infty_T \dot{H}^{1/2}_x},$$

$\implies \|u\|_{L^8_{t,x}} \lesssim \|u\|_{L^\infty_t H^1_x} \implies H^1$-scattering using some interpolation.

# LWP: Data size versus Spacetime slabs

- Recall that, based on a Hölder-in-time step, subcritical local-in-time theory gives

$$T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s-s_c}}.$$

- Bootstrap toward scattering: Hölder-in-time is forbidden.

# $I$-Method & Morawetz: Bootstrap Heuristic

Suppose we have almost conservation of modified energy.

1. RHS of 4-particle $\lesssim$ almost conserved modified energy
2. $\implies L^8_{t,x}$ controlled on long time interval $t \in [0, T]$
3. $\implies$ spacetime slab decomposition:
   $[0, T] \times \mathbb{R} = \cup_{j=1}^{J} [t_j, t_{j+1}) \times \mathbb{R}$ such that
   $$\|u\|_{L^8([t_j, t_{j+1}) \times \mathbb{R})} = \eta \sim \frac{1}{100}$$
4. $\implies$ almost conserved modified energy on $[t_j, t_{j+1}]$
5. $\implies$ RHS of 4-particle.... bootstrap loop!

# Almost Conservation Used In Bootstrap

## Lemma (Almost Conservation on Slab)

*Let* $H^s \ni u_0 \to u$ *solve* $NLS^+_{2k+1}(\mathbb{R}^d)$ *with* $s > s_k$. *Suppose we have a spacetime slab* $[t_-, t_+]$ *on which*

$$\|u\|_{L^8([t_-,t_+]\times\mathbb{R})} \lesssim \eta$$

*and* $\exists\ t_0 \in [t_-, t_+]$ *such that* $H[I_N u(t_0)] \leq 1$. *Then for large* $N$ *we have almost conservation:*

$$\sup_{t\in[t_-,t_+]} H[I_N u(t)] = H[I_N u(t_0)] + O(N^{-1+}).$$

Rescaling and continuity arguments glue it all together.