# Low Regularity Aspects of NLS Blowup

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# 1. Blowup Solutions Exist

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We consider the Cauchy problem for $L^2$ critical focusing NLS:

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^-(\mathbb{R}^2))$$

The solution has an $L^2$-invariant dilation symmetry

$$u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

Time invariant conserved quantities:

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t)\nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx - \frac{1}{2}|u(t)|^4 dx.$$

# $NLS_3^-(\mathbb{R}^2)$ $H^1$-GWP Theory

- Weinstein's $H^1$-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u, T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- $Q$ is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

# Pseudoconformal Symmetry

- Pseudoconformal transformation:

$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$

- $\mathcal{PC}$ is $L^2$-critical $NLS$ solution symmetry:
  Suppose $0 < t_1 < t_2 < \infty$. If

$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$

  then

$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$

  solves

$$i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v.$$

- $\mathcal{PC}$ is an $L^2$-Strichartz isometry:
  If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then

$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

# Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t}+\frac{i}{t}},$$

  is an explicit blowup solution.
- $S$ has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in $S$ is conically concentrated into a point.
- Minimal mass $H^1$ blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

# Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

# Lemma (Subcritical Scaling Lower Bound):

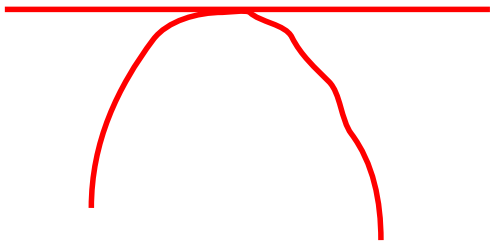
If $H^s \ni u_0 \longmapsto u(t)$ with $s > 0$ solving $NLS_3^-(\mathbb{R}^2)$ for all $t$ near $T^*$ in the maximal finite (forward) interval of existence $[0, T^*)$ then

$$\frac{c}{(T^* - t)^{s/2}} \leq \|D^s u(t)\|_{L^2_x}.$$

Scaling invariance and LWP theory:

- $v(\tau, y) := \frac{1}{\lambda} u(t + \frac{\tau}{\lambda^2}, \frac{y}{\lambda}) \implies \|D^s v(0)\|_{L^2} = \frac{1}{\lambda^s} \|D^s u(t)\|_{L^2}$.
- Choose $\lambda$ so that $\|D^s v(0)\|_{L^2} = 1 \implies \lambda = \|D^s u(t)\|_{L^2}^{\frac{1}{s}}$.
- LWP $\implies v(0) \longmapsto v(t)$ for $\tau \in [0, 1] \iff t + \frac{1}{\lambda^2} < T^*$.
- $\lambda^2 > \frac{1}{T^* - t} \implies$ claim.

# Mass Concentration Property: $H^1$ theory



## $H^1$ Theory of Mass Concentration

- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\liminf_{t \nearrow T^*} \int_{|x|<(T^*-t)^{1/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

  [Merle-Tsutsumi]
- $H^1$ blowups parabolically concentrate at least the ground state mass. Explicit blowups $S$ concentrate mass much faster.
- Fantastic recent progress on the $H^1$ blowup theory. [Merle-Raphaël]

# Mass Concentration Property: $L^2$ Theory

## $L^2$ Theory of Mass Concentration

- $L^2 \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\limsup_{t \nearrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^*-t)^{1/2}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

  [Bourgain]

  $L^2$ blowups parabolically concentrate some mass.
- For large $L^2$ data, do there exist tiny concentrations?
- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
- Upgrading lim sup into lim inf appears challenging.

# $NLS_3^-(\mathbb{R}^2)$: Conjectures/Questions

- **Scattering Below the Ground State Mass?**

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies^{???} u_0 \longmapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$

- **Minimal Mass Blowup Characterization?**

$$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \longmapsto u, T^* < \infty \implies^{???} u = S,$$

  modulo symmetries. Intermediate step: Characterize in $H^s$?
- **Concentrated mass amounts are quantized?**
  Ground and excited state profiles are only asymptotic profiles?
- **Are there any general upper bounds?** $\limsup$ **vs.** $\liminf$ **?**
- **What are the possible "singular sets" for NLS blowups?**

# $L^2$ Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

  $H^s$-blowup solutions concentrate ground state mass. [C-Raynor-C.Sulem-Wright]
- $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $u_0 \in H^s$, $\sim 0.86 < s < 1$, $T^* < \infty \implies$ $\exists\ t_n \nearrow T^*$ s.t. $u(t_n) \to Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence). For $H^s$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}$, $u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an $H^s$ analog of an $H^1$ result of [Weinstein] which preceded the minimal $H^1$ blowup solution characterization.

# $L^2$ Critical Case: Partial Results

- Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

$$\limsup_{t \nearrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

[C-Roudenko]

# 2. Ground State Mass Concentration for $H^s$

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**Theorem (C-Raynor-Sulem-Wright 05)**

*For* $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1, H^s \cap \{radial\} \ni u_0 \longmapsto u, T^* < \infty \implies$

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- $\{radial\}$ removed by concentration compactness. [Tzirakis] $NLS_5^-(\mathbb{R})$
- Higher dimension generalization $NLS^-_{1+\frac{4}{d}}(\mathbb{R}^d)$. [Visan-Zhang]

# Ground State Mass Concentration for $H^1$

Recall [Merle-Tsutsumi]. $H^1 \cap \{radial\} \ni u_0 \longmapsto u$ with $T^* < \infty$.

- **Rescalings (weakly) converge to asymptotic profile.**

  Consider $\{u(t_n, \cdot)\}_{n\in\mathbb{N}} = \{u_n(\cdot)\}_{n\in\mathbb{N}}$ along $t_n \nearrow T^*$. Form

  $$v_n(\cdot) = \lambda_n^{-1} u_n(\lambda_n^{-1}(\cdot))$$

  with $\lambda_n = \|\nabla u_n\|_{L^2} \gtrsim (T^* - t_n)^{-1/2}$ so that $\|\nabla v_n\|_{L^2} = 1$.
  Thus, $\exists\ v \in H^1$ such that $v_n \rightharpoonup v$ in $H^1$ along subsequence.

- **Compactness and energy of rescaled asymptotic object.**

  Radial & Rellich Compactness $\implies v_n \to v$ strongly in $L^4$.
  $|E[v_n]| = \lambda_n^{-2}|E[u(t_n)]| \to 0 \implies E[v] \leq 0$.

- $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$**; undo scaling.**

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# Ground State Mass Concentration for $H^s$

We imitate the [Merle-Tsutsumi] result using modified energy.

- **Blowup Parameter:**

$$\lambda(t) = \|u(t)\|_{H^s};\ \Lambda(t) = \sup_{\tau\in[0,t]} \lambda(\tau).$$

lambda(t) vs. lambda_n

- **Modified Blowup Parameter:**

$$\sigma(t) = \|I\langle\nabla\rangle u(t)\|_{L^2};\ \Sigma(t) = \sup_{\tau\in[0,t]} \sigma(\tau).$$

Recall,

$$\|f\|_{H^s} \leq \|I\langle\nabla\rangle f\|_{L^2} \leq N^{1-s}\|f\|_{H^s}.$$

Thus, $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$.

# Ground State Mass Concentration for $H^s$

**Lemma ( Modified Kinetic $\gg$ Modified Total Energy)**

*$\forall\ s > 0.86$ if $H^s \ni u_0 \longmapsto u$ on maximal $[0, T^*)$ then $\forall T < T^*\ \exists\ N = N(T)$ such that*

$$|E[I_{N(T)}u(T)]| \leq C_0 \Lambda(T)^{p(s)}$$

*with $p(s) < 2$ and $C_0 = C_0(s, T^*, \|u_0\|_{H^s})$.*

- Modified Kinetic Energy $\gg$ Modified Total Energy.
- $N(T) = C\Lambda(T)^{\frac{p(s)}{2(1-s)}}$.
- Proof based on almost conservation; multilinear analysis.

Cancellations!

# Ground State Mass Concentration for $H^s$

1. **Rescale by modified kinetic energy.**
   Choose any *maximizing sequence* $t_n \nearrow T^*$ satisfying $\|u(t_n)\|_{H^s} = \Lambda(t_n)$. Define $v_n(y) = \sigma_n^{-1} I_{N(t_n)} u(t_n, \sigma_n^{-1} y)$ where $N(t_n)$ is as in the Lemma.
2. **Weak convergence and $L^4$ compactness.**
   Rescaling $\implies \|\nabla v_n\|_{H^1} \to 1$ so $\exists\ v \in H^1$ s.t. $v_n \rightharpoonup v$ along subsequence. Radial & Rellich $\implies v_n \to v$ strongly $L^4$.
3. **Energy of asymptotic object.**
   $|E[v_n]| = \sigma_n^{-2} |E[I_N u_n]| \leq \sigma_n^{-2} \Lambda^{p(s)}(t_n) \leq (\Lambda(t_n))^{p(s)-2} \to 0.$
4. **Undo the rescaling.**
   Unravelling scaling using lower bound $\sigma_n \gtrsim (T^* - t_n)^{-s/2}$ completes proof.

# 3. Concentration & Strichartz Explosion

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Idea!

- Ground state soliton $u(t,x) = e^{it}Q(x)$ satisfies

$$\|u\|_{L^4([j,j+1]_t \times \mathbb{R}^2_x)} = \eta = O(1), \ \forall\, j \in \mathbb{N}.$$

- $L^4$-isometry & explicit $S = \mathcal{PC}[e^{it}Q] \sim |\tau|^{-1}Q(\tau^{-1}y)e^{i\cdots}$,

$$\|S\|_{L^4([-\frac{1}{j},-\frac{1}{j+1}]_\tau \times \mathbb{R}^2_y)} = \eta, \ \forall\, j \in \mathbb{N}.$$

- Thus, $\|S\|_{L^4([-1,t]\times\mathbb{R}^2)} \sim \frac{1}{|t|}$; Mass concentrated in $|y| \lesssim |t|$.
- Contrast with [Merle-Tsutsumi], [Bourgain] Concentration: $\|u\|_{L^4([-1,t]\times\mathbb{R}^2)} \nearrow \infty \implies$ Mass concentrated in $|y| \lesssim |t|^{1/2}$.
- Observation?
  Strichartz explosion rate $= f$(concentration window size).

# Heuristic: Window size & $L^4$ Explosion

- When $\|u\|_{L^4([t_n,t_{n+1}]\times\mathbb{R}^2)} \sim \eta$ [Bourgain] shows parabolic concentration: $\exists\ t_n^* \in [t_n, t_{n+1}]$ and $x_0 \in \mathbb{R}^2$ where

$$\int_{|x-x_0|\lesssim|t_{n+1}-t_n|^{1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}.$$

- In [C-Roudenko], we observe (overstated!):

$$\|u\|_{L^4_{[0,T^*-t]\times\mathbb{R}^2}} := f(T^*-t) \nearrow \infty \text{ as } t \nearrow T^*$$

$$\Updownarrow$$

$$\sup_{x_0\in\mathbb{R}^2} \int_{|x-x_0|\lesssim[-\partial_t f(T^*-t)]^{-1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}$$

- Why? By first order Taylor approximation, we have $\eta \sim f(T^*-t_{n+1}) - f(T^*-t_n) \sim [-\partial_t f(T^*-t_n)](t_{n+1}-t_n)$.

# Ideas in Bourgain's Proof

- Decompose $[0, T^*)$ into $\bigcup [t_n, t_{n+1})$ on which

$$\|u\|_{L^4([t_n,t_{n+1}]\times\mathbb{R}^2)} = \eta \sim \frac{1}{100}.$$

- For $t \in [t_n, t_{n+1})$, we have $u \sim e^{i(t-t_n)\Delta} u(t_n)$.

Duhamel Small

- Strichartz Refinements and the conditions

$$\|f\|_{L^2} < \|u_0\|_{L^2}; \ \|e^{it\Delta} f\|_{L^4} > \eta$$

  spawn a spacetime tube decomposition of $e^{it\Delta} f$.
- $\exists$ **concentration time** $t_n^* \in [t_n, t_{n+1})$ $\forall$ $n$.
  Thus, proof is more refined than the lim sup claim.
- Taylor expansion connects $(t_{n+1} - t_n)$ with $T^* - t_n$.

# Squares Lemma

**Lemma (Bourgain)**

*$\forall\ \epsilon > 0$ and $\forall\ f \in L^2(\mathbb{R}^2)\ \exists\ \{\widehat{f_r}\}_{1 \le r \le R(\epsilon)}$ such that*

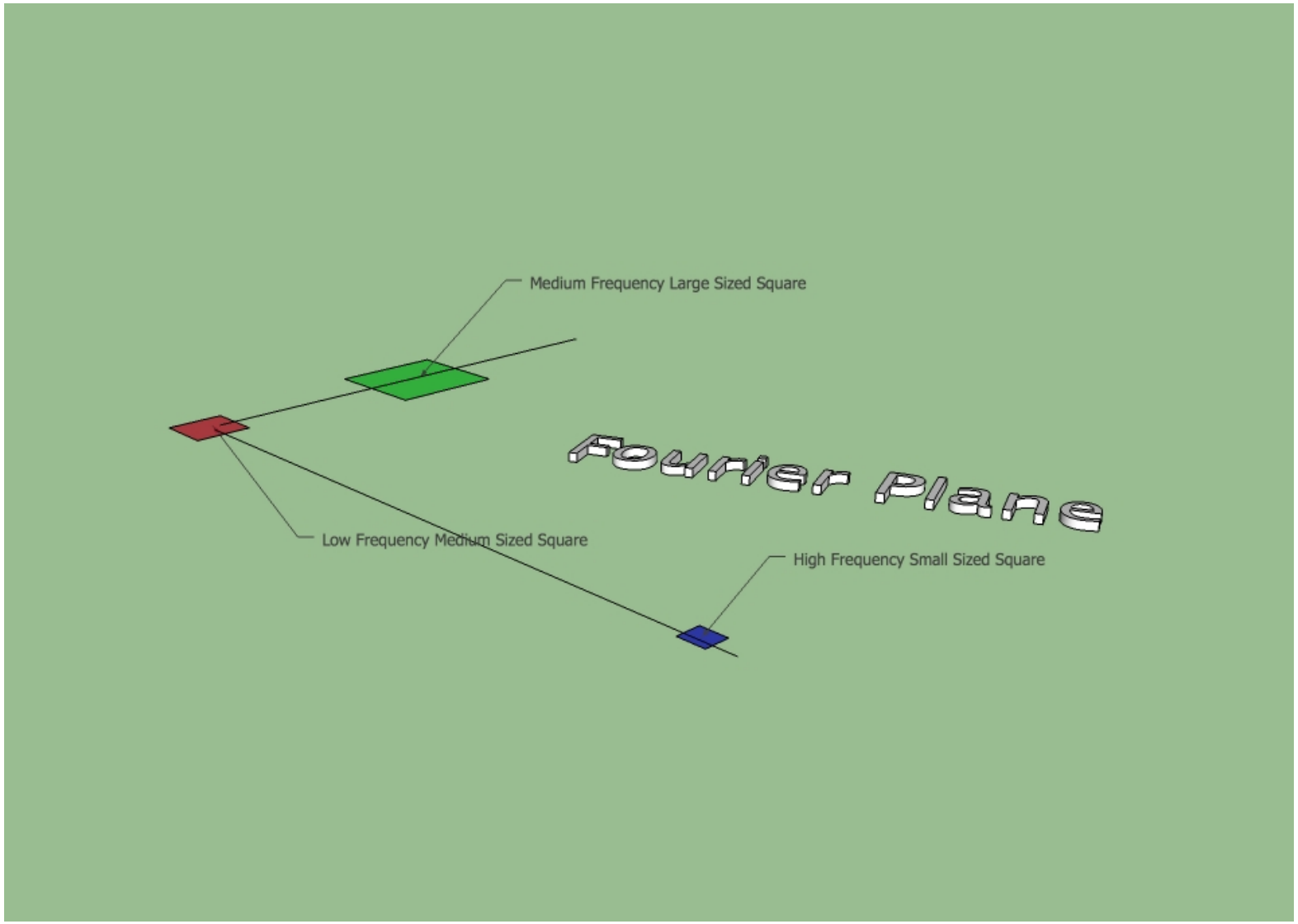
- *spt $\widehat{f_r} \subset \tau_r \subset \mathbb{R}^2$ with $\tau_r$ a square of side $A_r$ centered at $\xi_r$*
- $|\widehat{f_r}| \le \frac{1}{A_r}$
- $\|\widehat{f_r}\|_{L^2} \ge \delta(\epsilon) > 0$

*and*

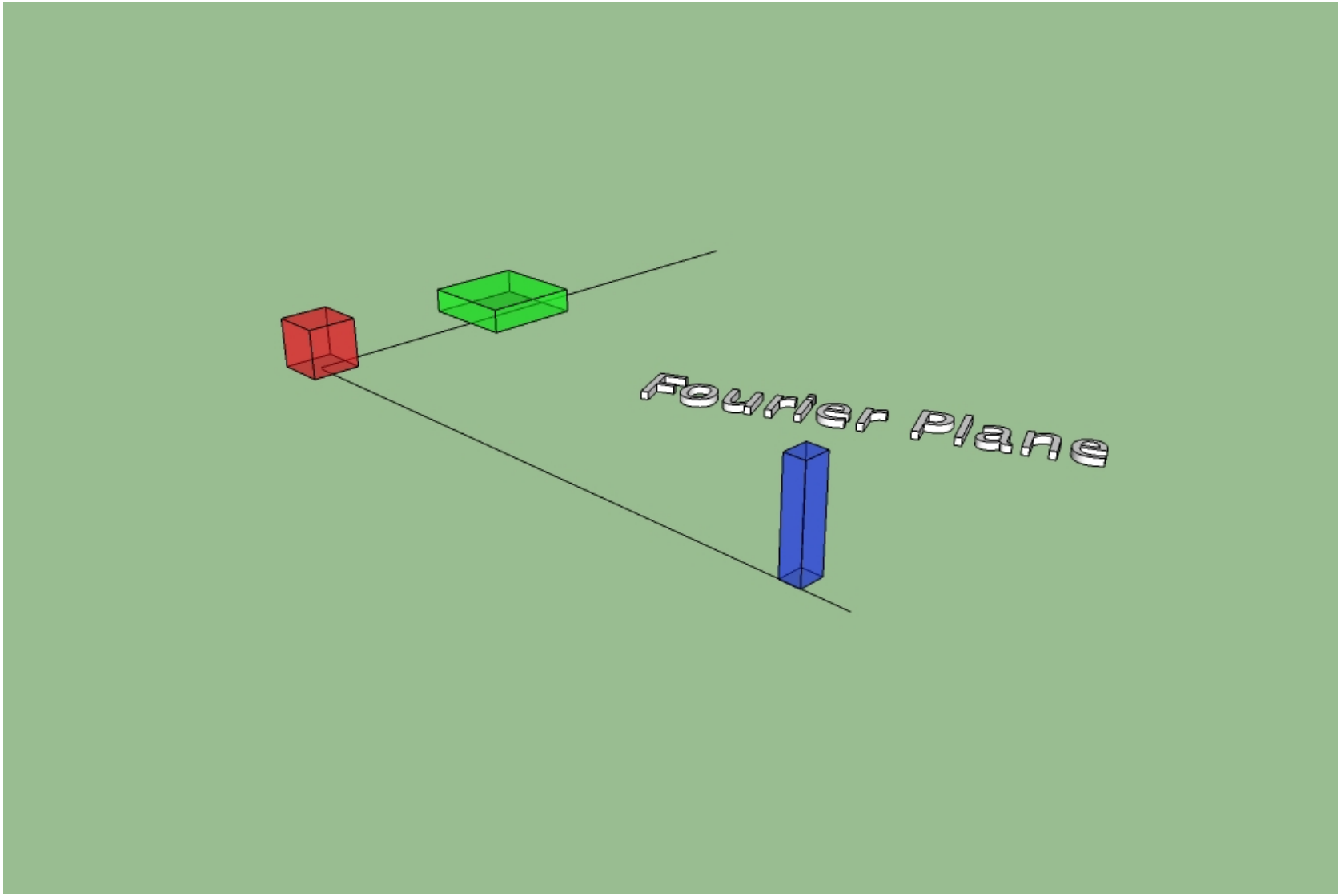
$$\|e^{it\Delta} f - \sum_{r=1}^{R(\epsilon)} e^{it\Delta} f_r\|_{L^4_{t,x}} \le \epsilon.$$

The linear Schrödinger evolution of any $L^2$ function is approximated by the evolution of a function with Fourier support on a system of squares and bounded Fourier transform.

# Squares Lemma



# Squares Lemma



# Tubes Lemma

## Lemma (Bourgain)

*Consider a function $g$ satisfying: (Think of $g$ as one of the $f_r$.)*

- *spt* $\widehat{g} \subset \tau \subset \mathbb{R}^2$ *with* $\tau$ *a square of side* $A$ *centered at* $\xi_0$
- $|\widehat{g}| \leq \frac{1}{A}$.

$\forall\ \epsilon > 0\ \exists$ *spacetime tubes* $\{Q_s\}_{1 \leq s \leq S(\epsilon)}$ *of form*

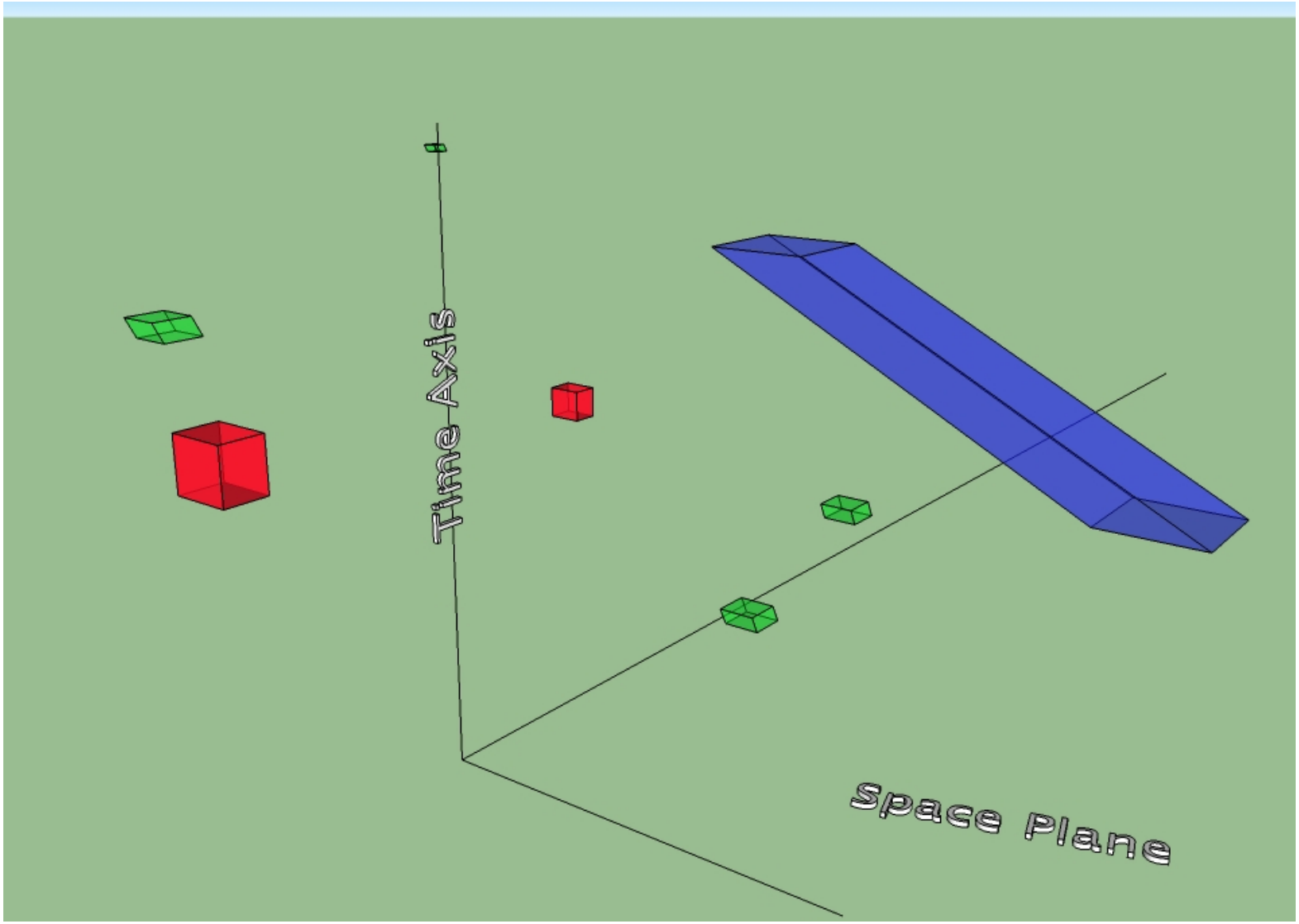
- $Q_s = \{(t, x) \in \mathbb{R}^3 : x - 2t\xi_0 \in \tau_s, t \in J_s\}$
- $\tau_s$ *is a (dual sized to* $\tau$*) cube of side* $\frac{1}{A}, |J_s| = \frac{1}{A^2}$

*and*

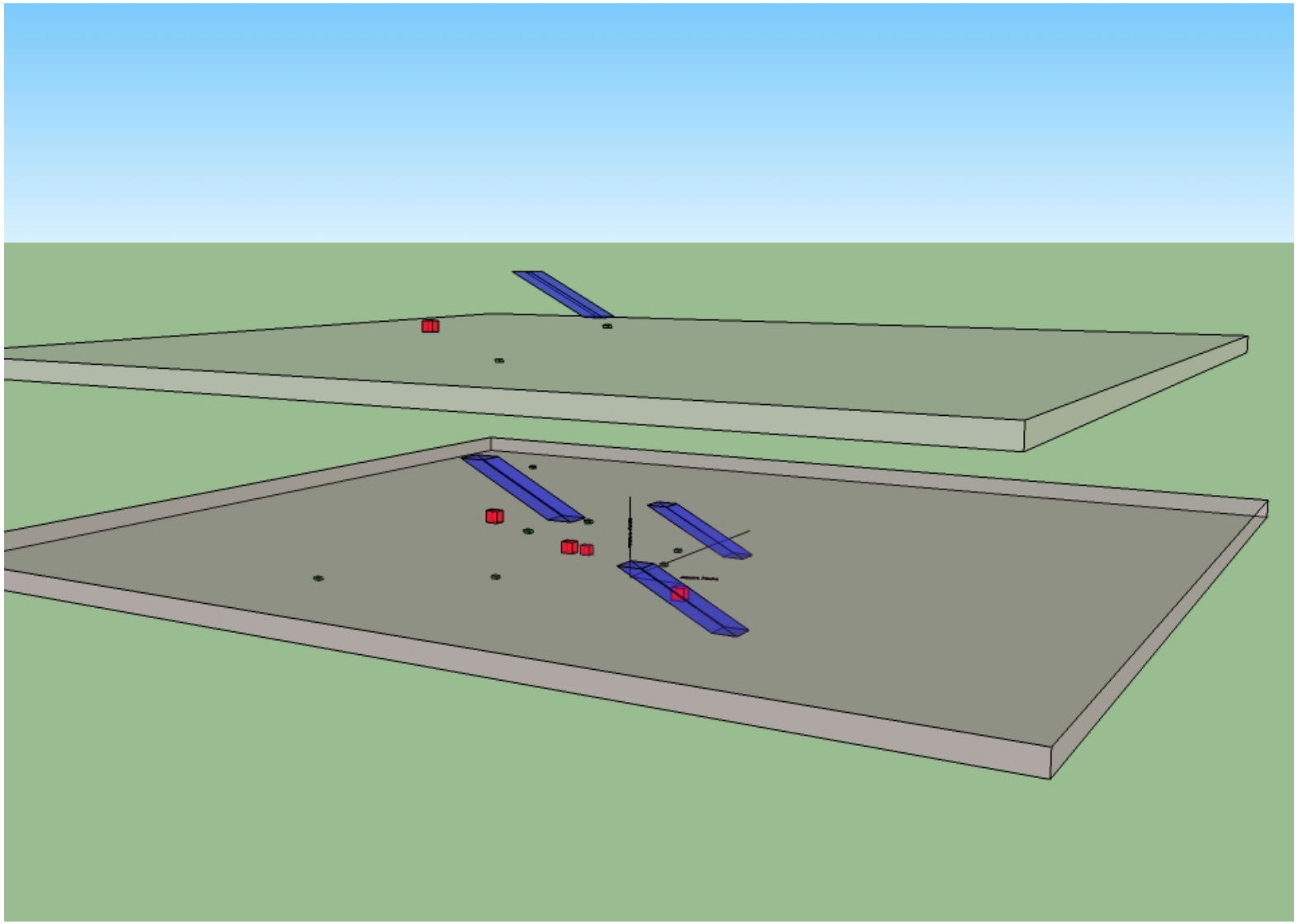
$$\left( \int_{\mathbb{R}^3 \setminus \cup_s Q_s} |e^{it\Delta} g|^4 dxdt \right)^{1/4} < \epsilon.$$

There is just dust outside the tubes!

# Tubes Lemma



# Tubes Lemma with Time Slices



# Strichartz Explosion $\implies$ Tight Window

**Theorem (C-Roudenko)**

*Suppose* $T^* < \infty$ *and* $\|u\|_{L^{\frac{2(d+2)}{d}}([0,t]\times\mathbb{R}^d)} \gtrsim (T^* - t)^{-\beta}$. *Then*

$$\limsup_{t \nearrow T^*} \sup_{\substack{\text{cubes } J \in \mathbb{R}^d : \\ l(J) < (T^* - t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |u(t,x)|^2 \, dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

*Furthermore,* $\forall\ t \in (0, T^*)\ \exists$ *a cube* $\tau(t) \subseteq \mathbb{R}^d_\xi$ *of size* $l(\tau(t)) \gtrsim (T^* - t)^{-(\frac{1}{2}+\frac{\beta}{2})}$ *such that*

$$\limsup_{t \nearrow T^*} \sup_{\substack{\text{cubes } J \in \mathbb{R}^d : \\ l(J) < (T^* - t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |P_{\tau(t)} u(t,x)|^2 \, dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

# Thickened Time Interval of Concentration

**Lemma (Frequency localized waves persist)**

*Let $f \in L^2_x(\mathbb{R}^d)$ and spt $\hat{f} \subset [0,1]^d$ and suppose*

$$\int_{[0,1]^d} |f(x)|^2 \, dx \geq c_1 > 0.$$

*Then for $|t| < c(c_1, \|f\|_{L^2})$ concentration persists*

$$\int_{[0,1]^d} |e^{it\Delta} f(x)|^2 \, dx \geq \frac{c_1}{2}.$$

- Frequency localization in conclusion shows concentration persists for $t$ in an interval containing $t_n^*$ of size $(T^* - t)^{1+\beta}$.
- Thickened concentration interval may not cover $[t_n, t_{n+1}]$.

# Tight Window $\implies$ Strichartz Explosion

Let $F(t) = \|u\|^4_{L^4([0,t]\times\mathbb{R}^2)}$ and $P_{L(t)} = P_{\{|\xi|\leq L(t)\}}$.

**Lemma (Pointwise Derivative Lower Bound)**

*Suppose* $\exists\ \alpha \geq \frac{1}{2}, \epsilon > 0$ *such that*

$$\limsup_{t\nearrow T^*} \sup_{\substack{\text{cubes } J\subset\mathbb{R}^d: \\ l(J) < (T^*-t)^\alpha}} \int_J |P_{L(t)}u(t,x)|^2\,dx \geq \epsilon.$$

*Then* $\exists\ t_n \nearrow T^*$ *such that*

$$F'(t_n) \gtrsim (T^* - t_n)^{-2\alpha}.$$

On thickened concentration time intervals, we integrate the derivative lower bound to get a Strichartz lower bound.

# Cautious Remark Concerning lim inf

- Consider $NLS_3^-(\mathbb{R}^2)$ posed at time $t = -\epsilon$ with data

$$\phi_\epsilon(x) = e^{i\epsilon^{-1}|x|^2} e^{i\epsilon^{-1}} Q(x).$$

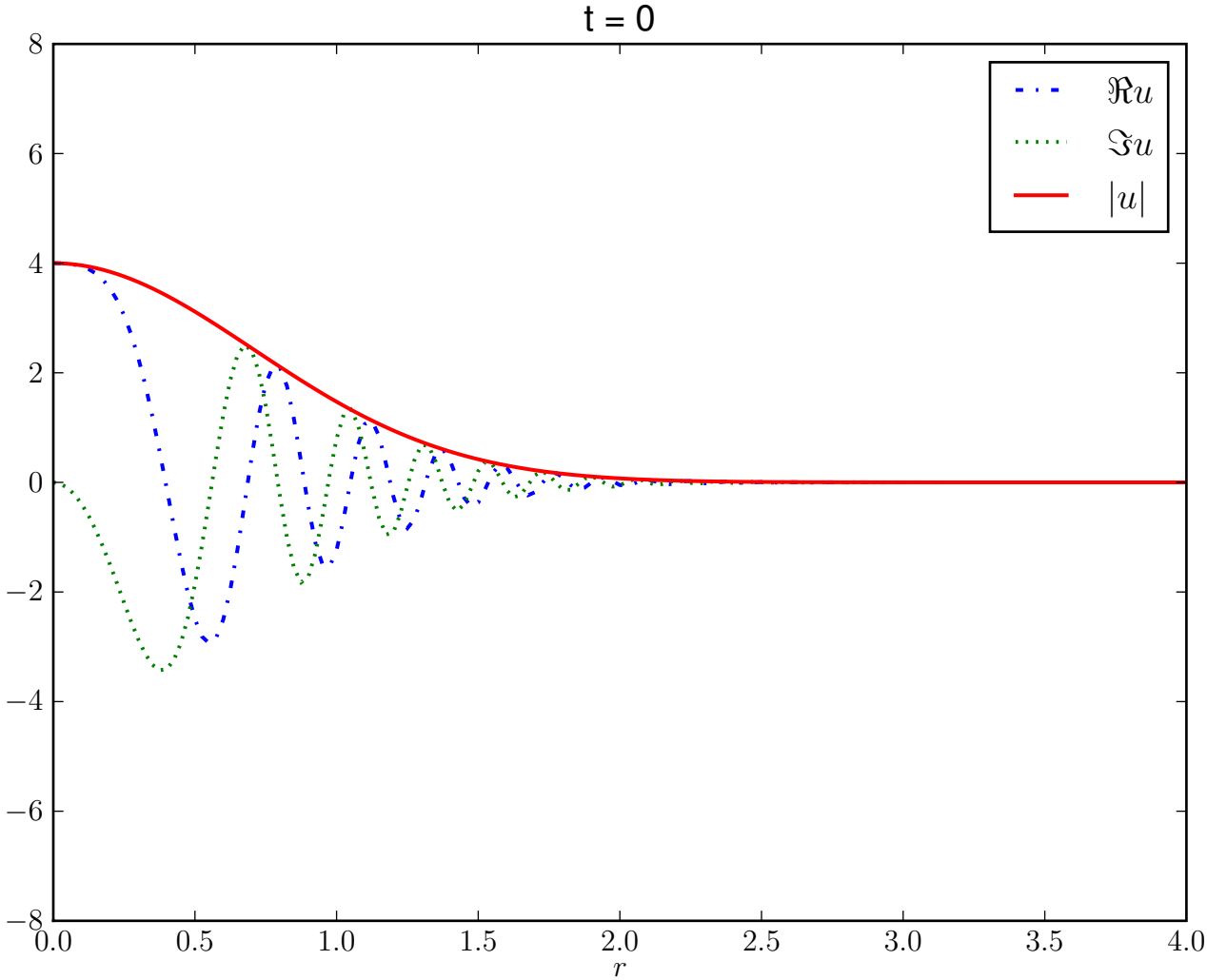
- Dilated explicit solution which blows up at $t = 0 = T^*$!
- The parabolic scale related to distance to blowup time is $\sqrt{\epsilon}$. For $\tau$ a cube of side $\sqrt{\epsilon}$, observe that $\phi_\epsilon$ is non-concentrated

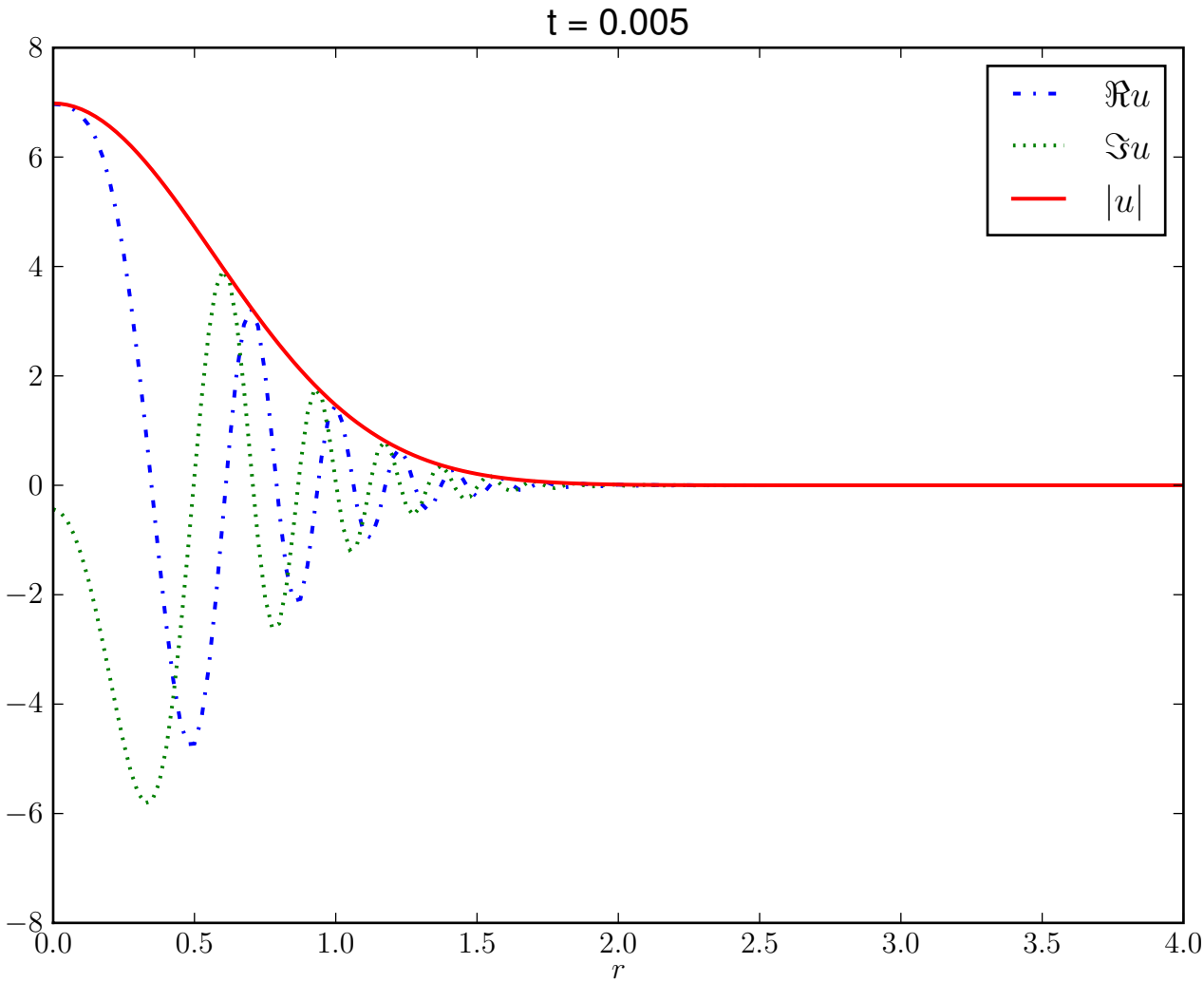
$$\int_\tau |\phi_\epsilon|^2 dx \lesssim \epsilon.$$

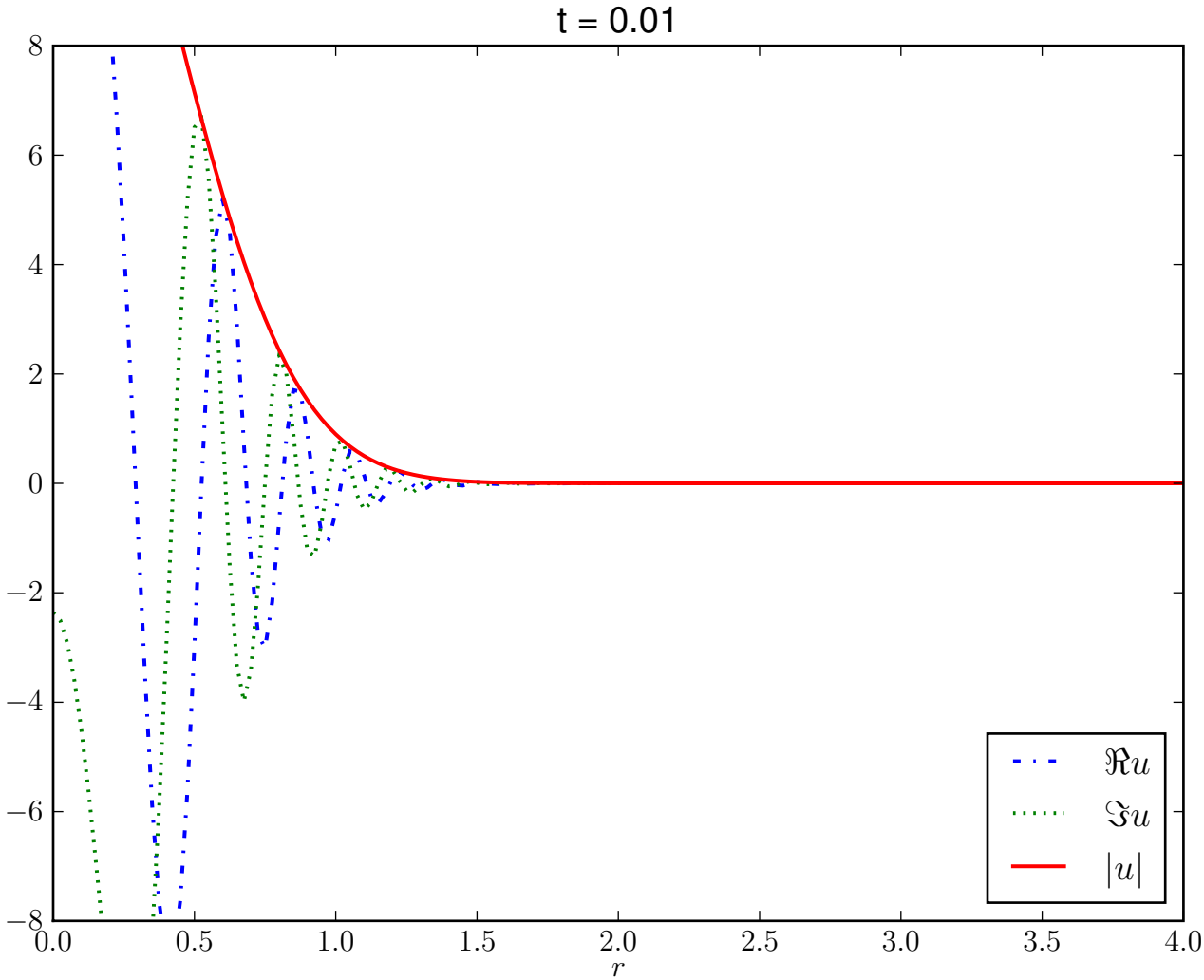
- Consider data $(1-\delta)\phi_\epsilon$....

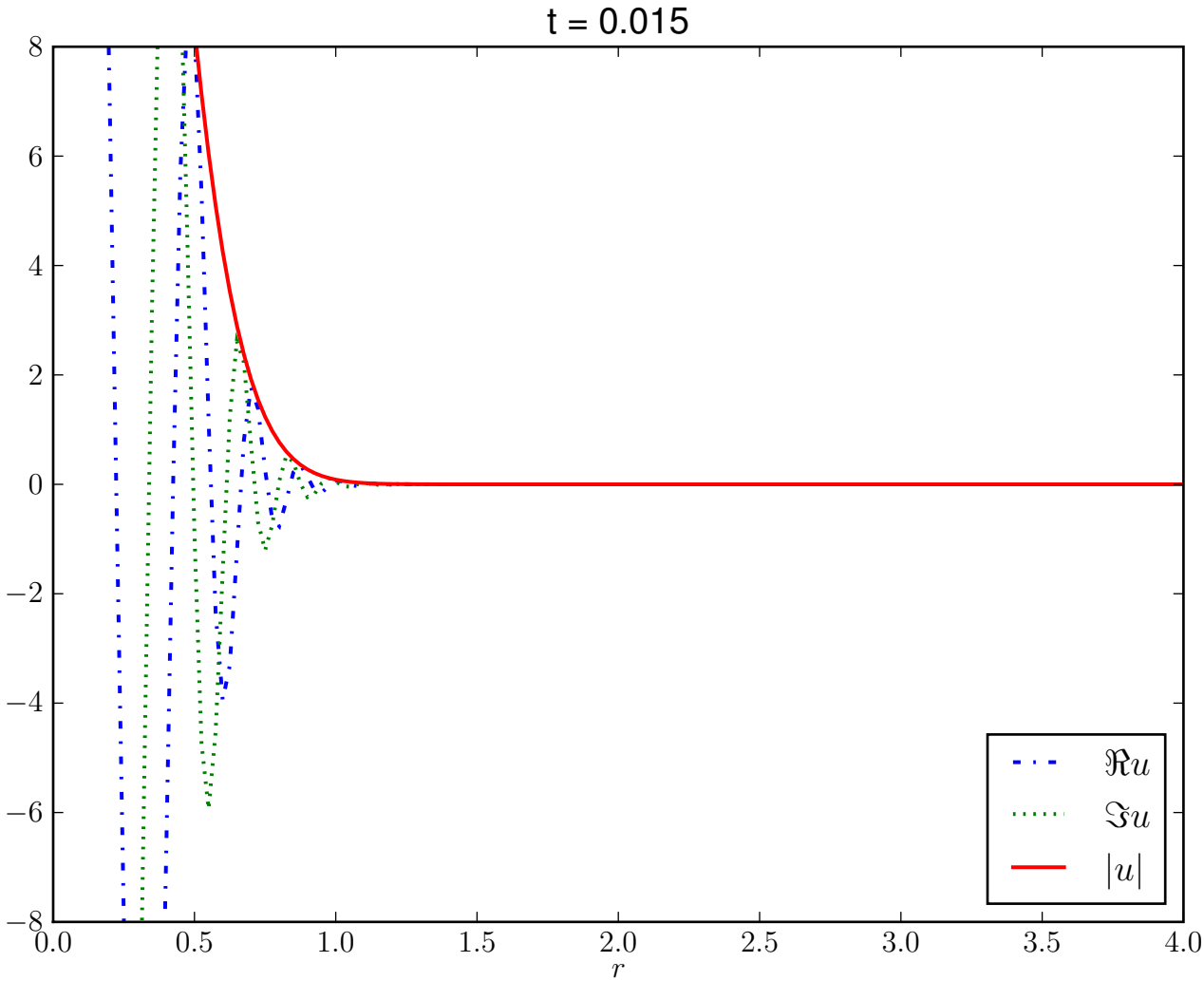
Phase oscillations violently influence $L^2$ blowup behavior.

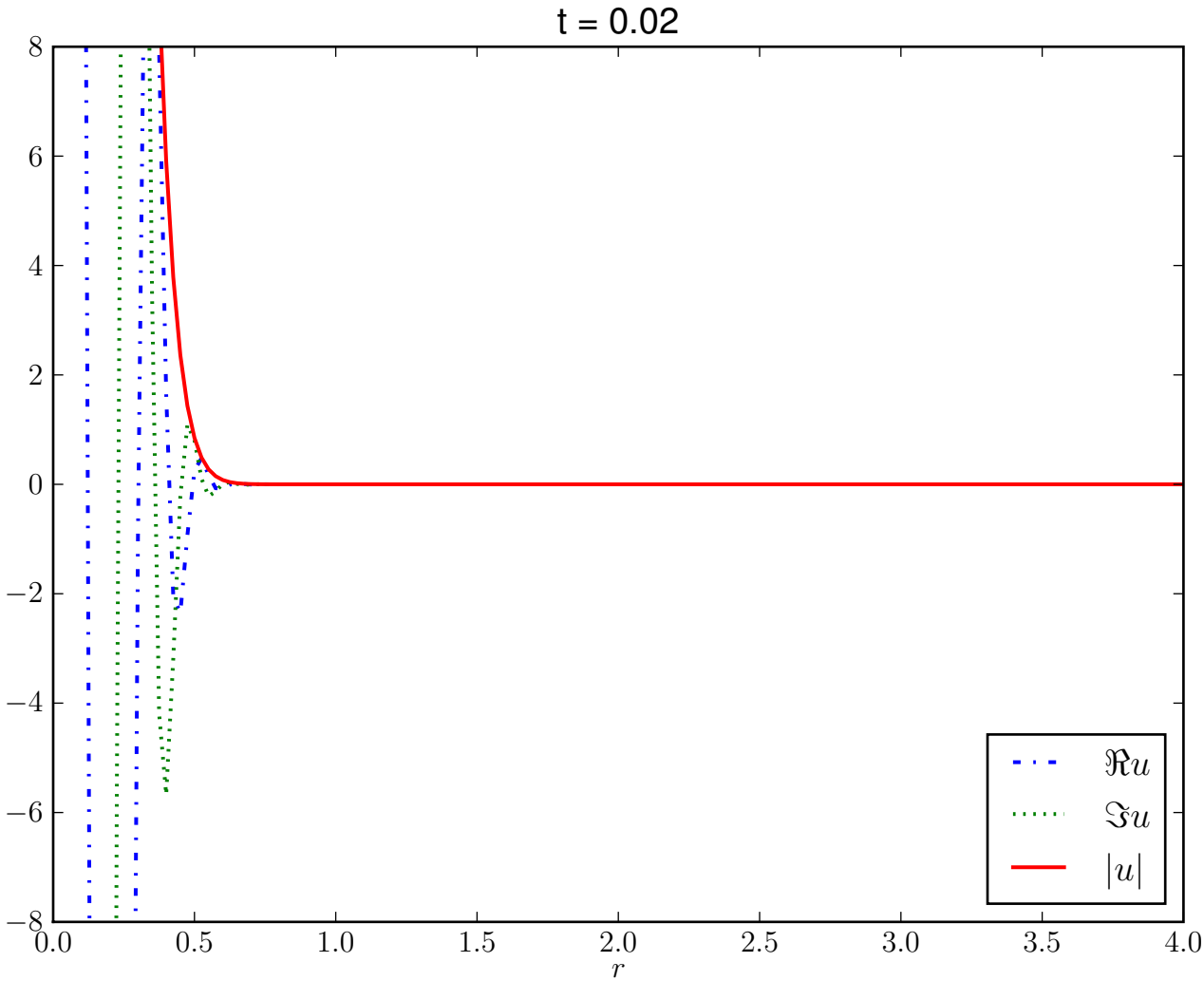
Phased Centered Gaussian Initial Data

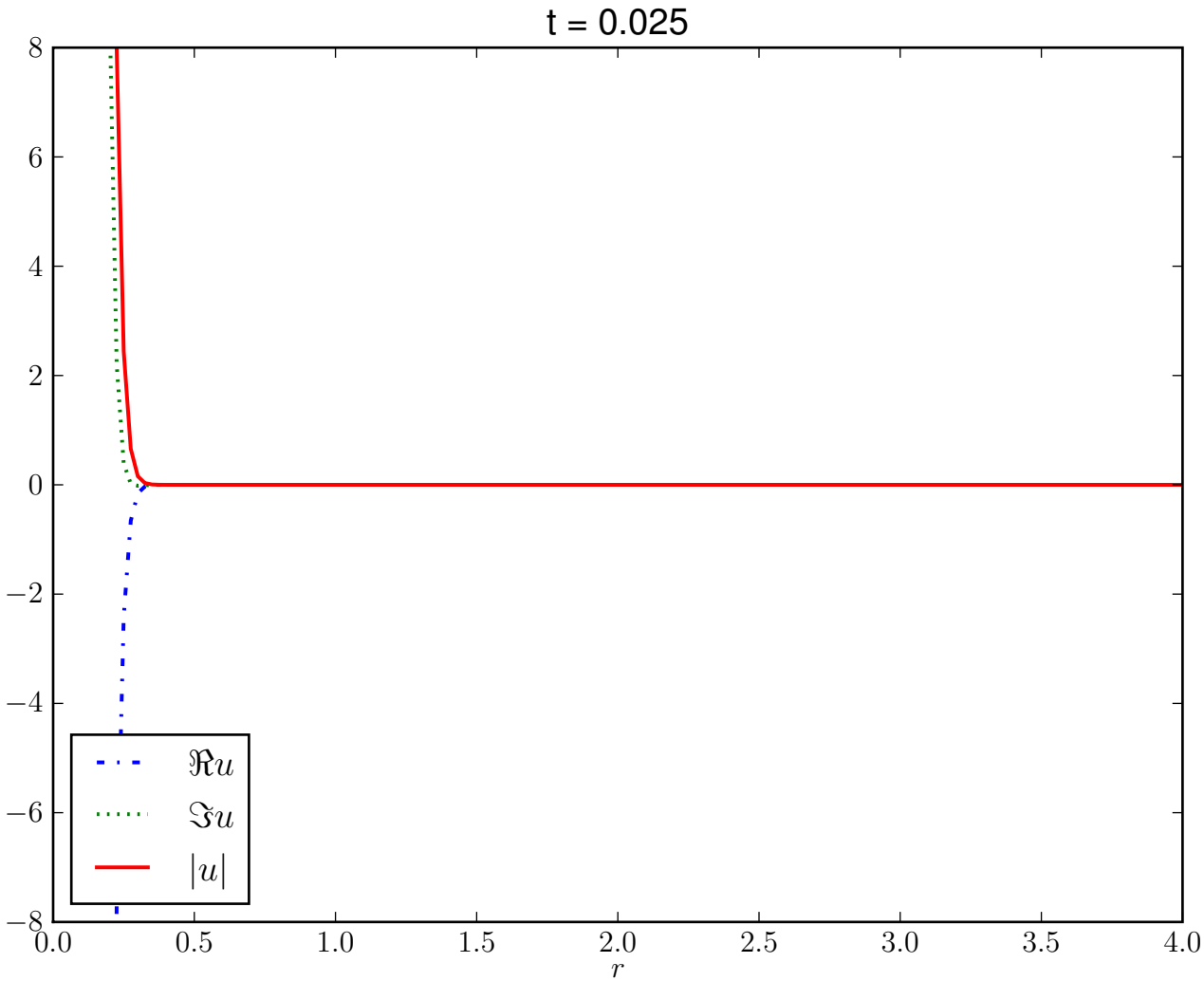


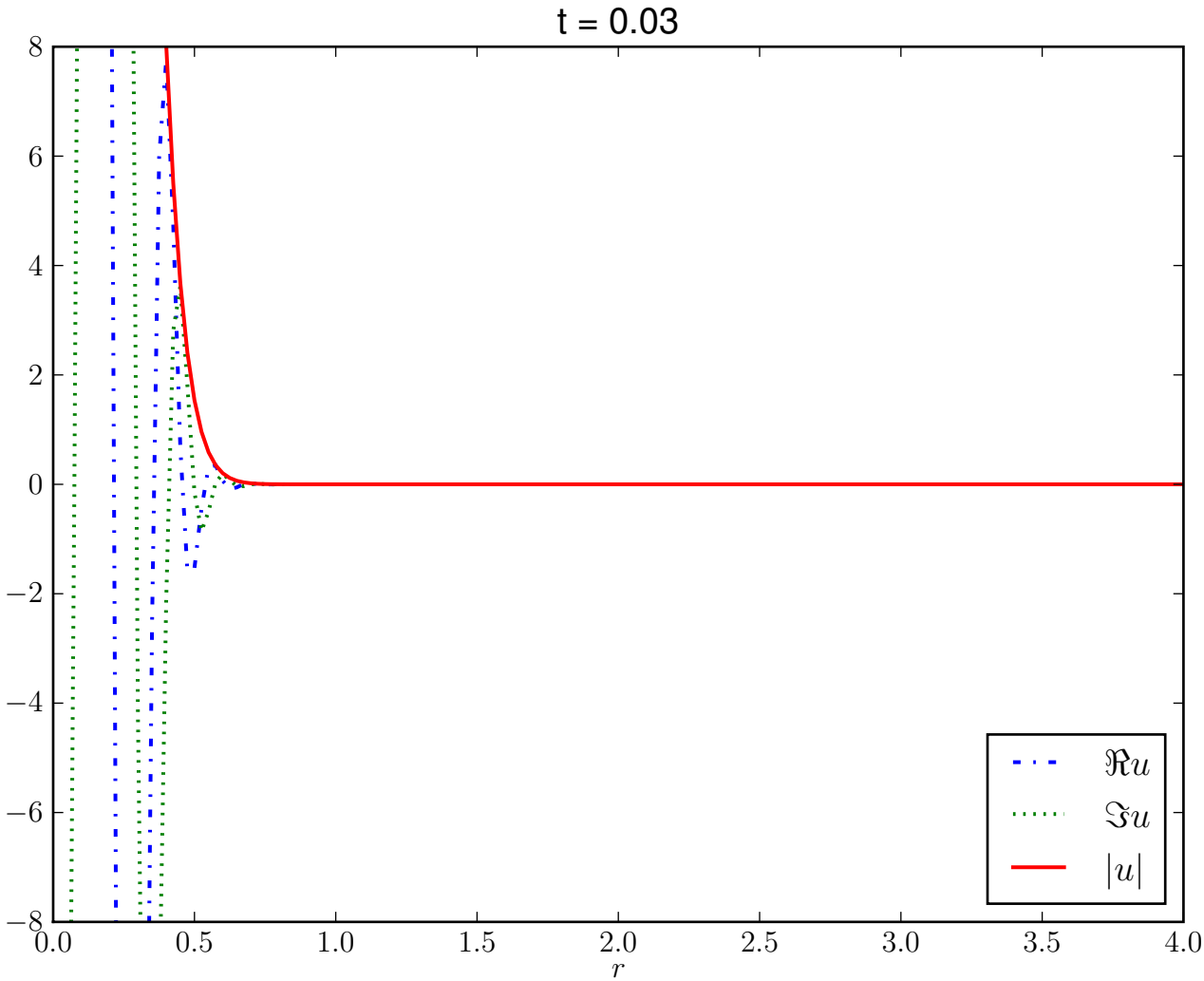

t = 0.005
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

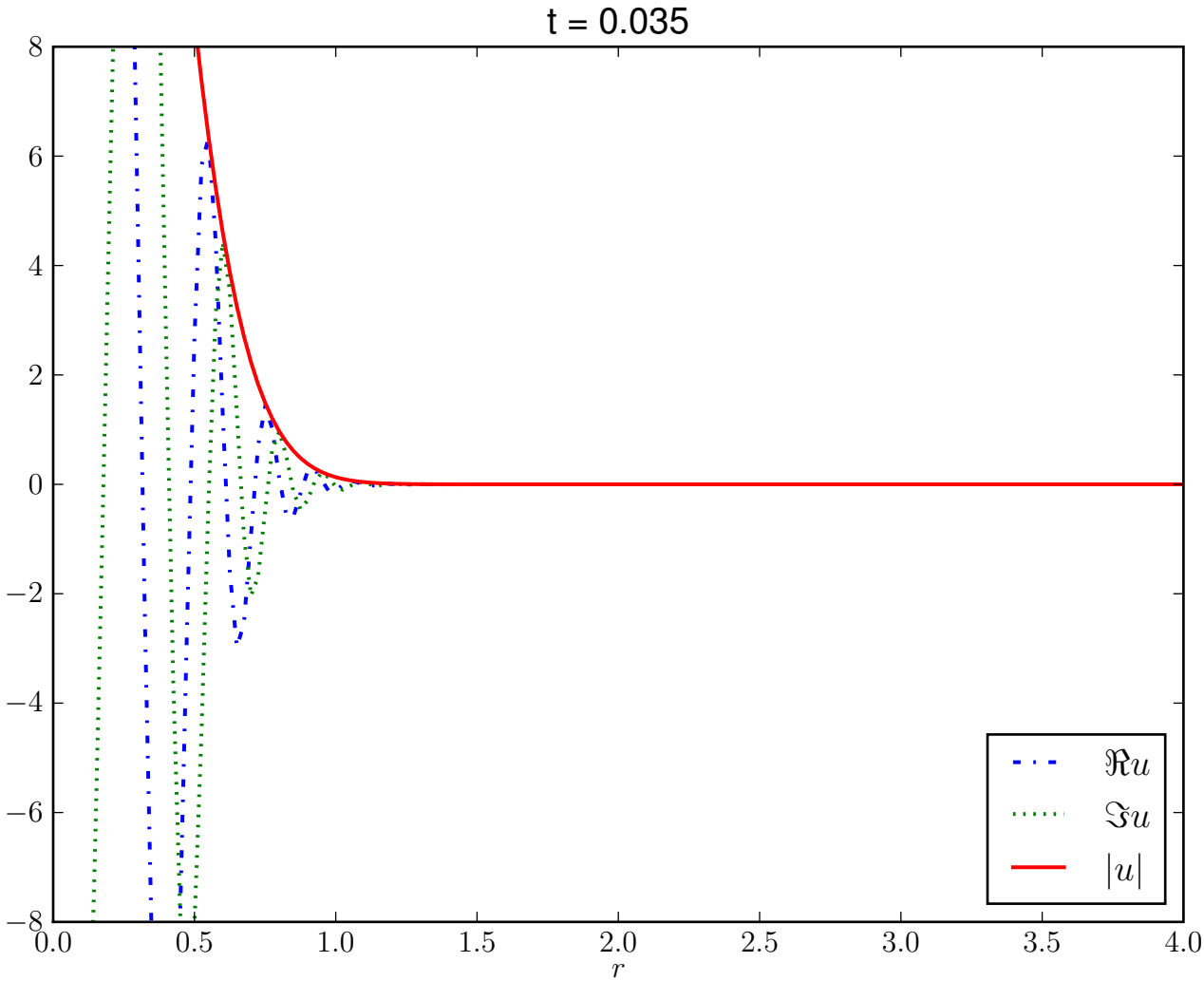

t = 0.01
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

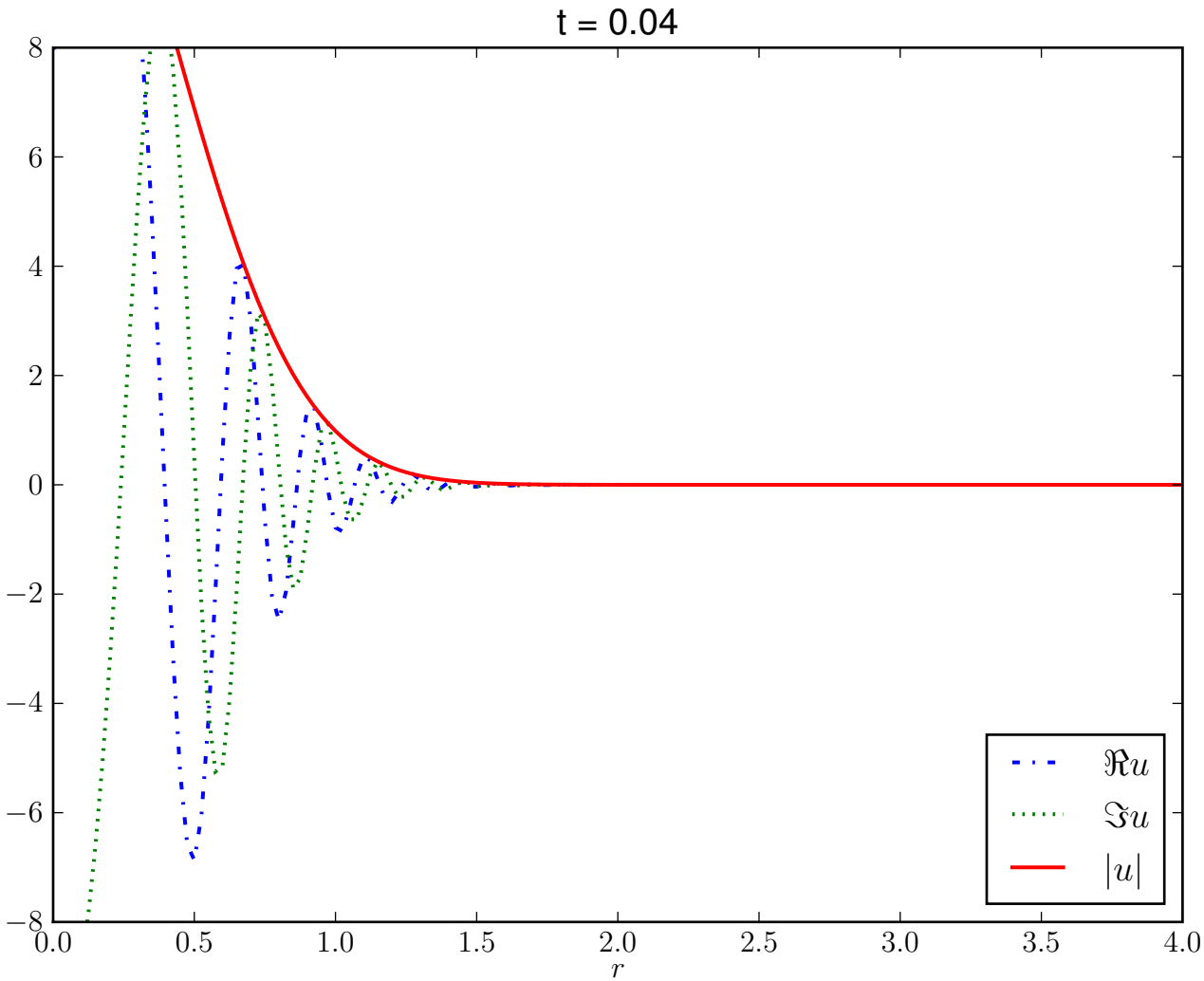

t = 0.015
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

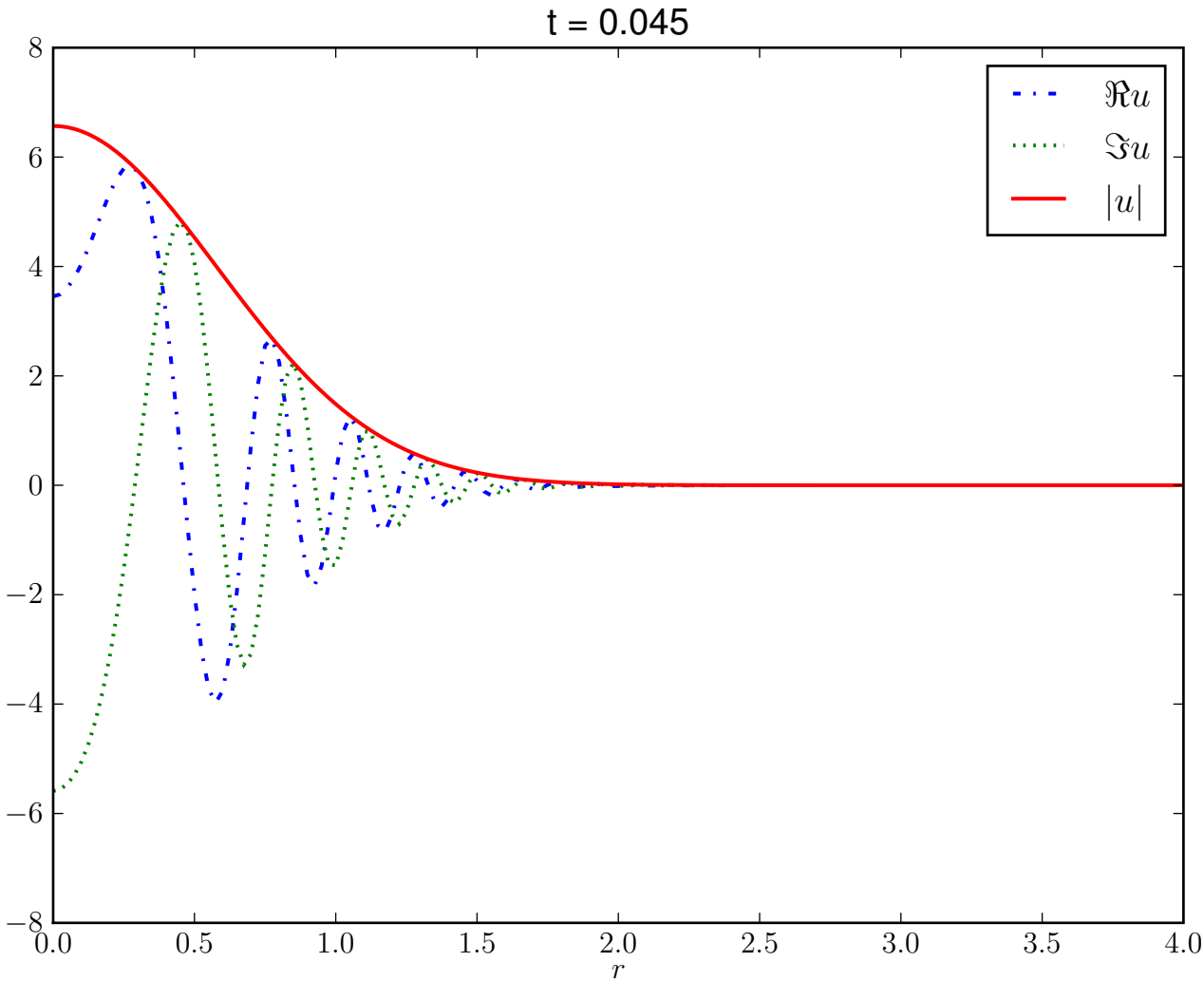

t = 0.02
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

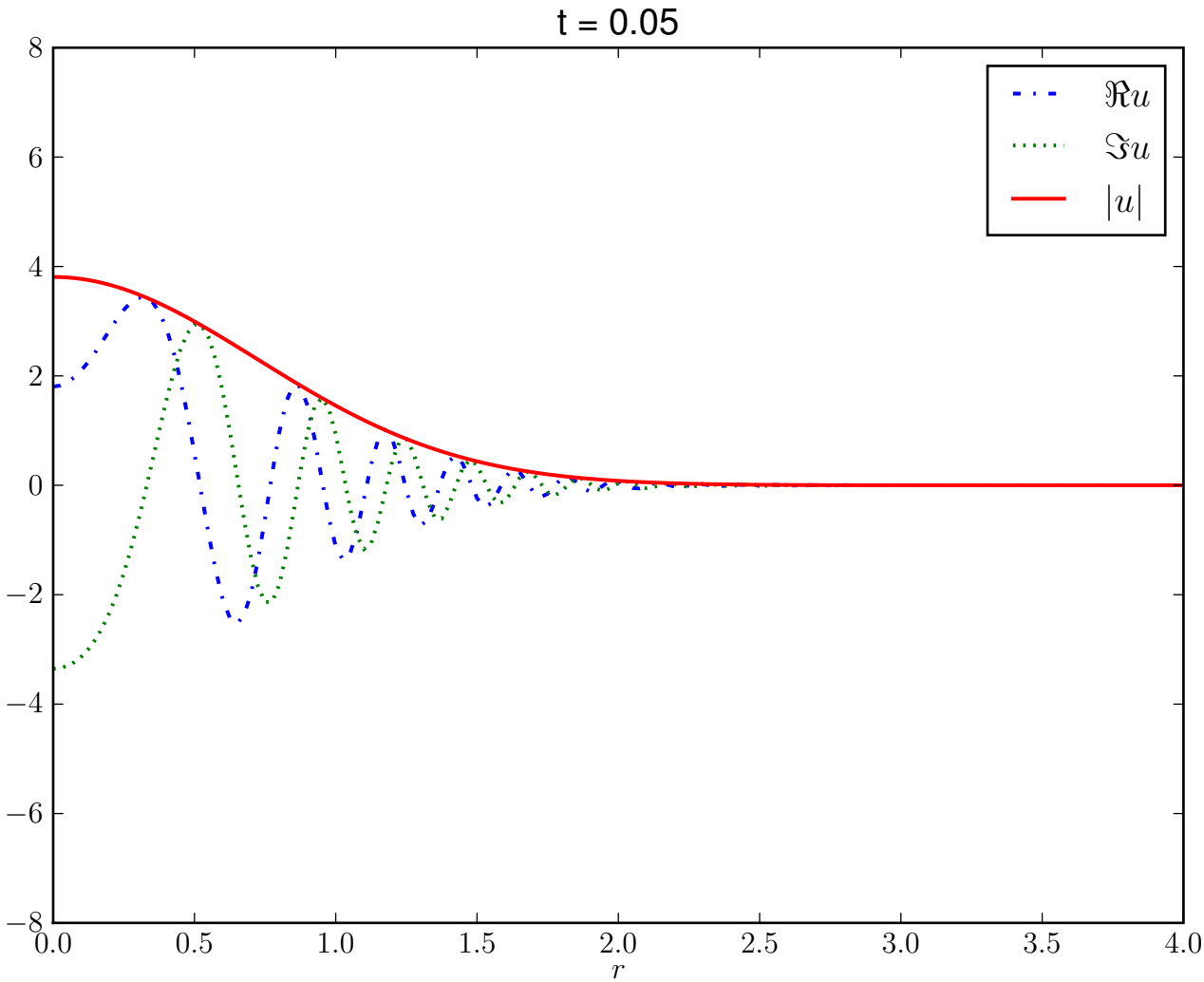

t = 0.025
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

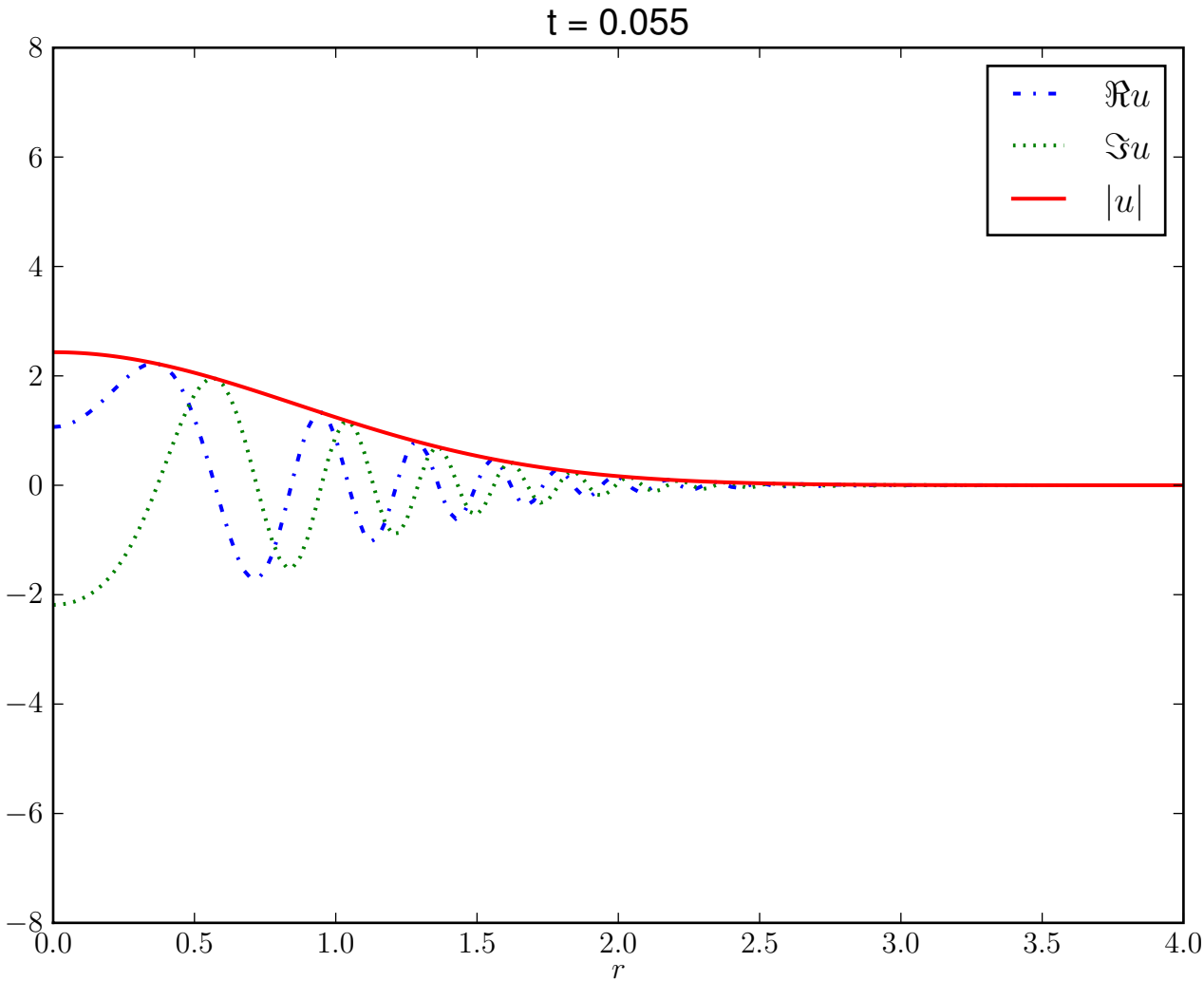

t = 0.03
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

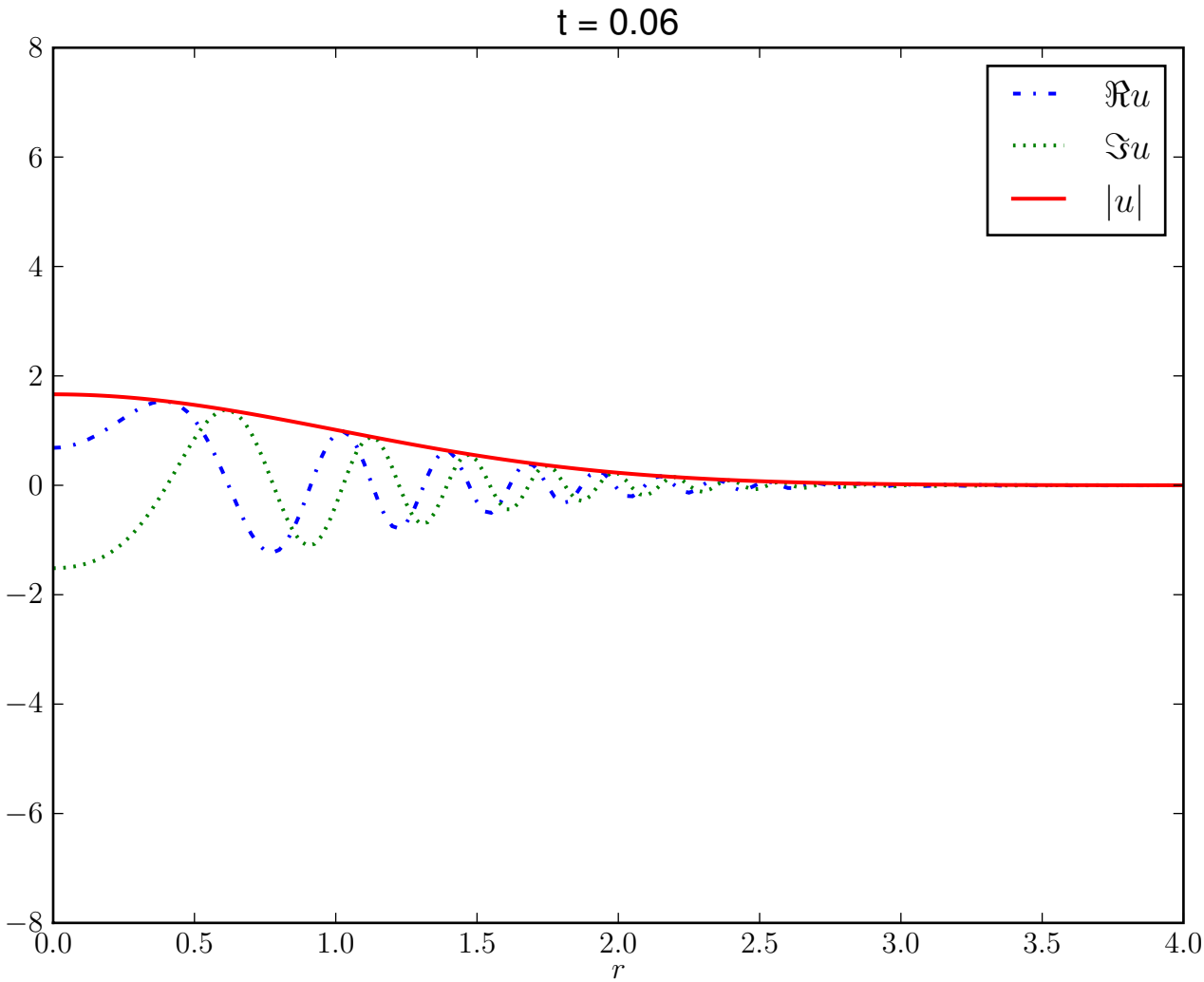

t = 0.035
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

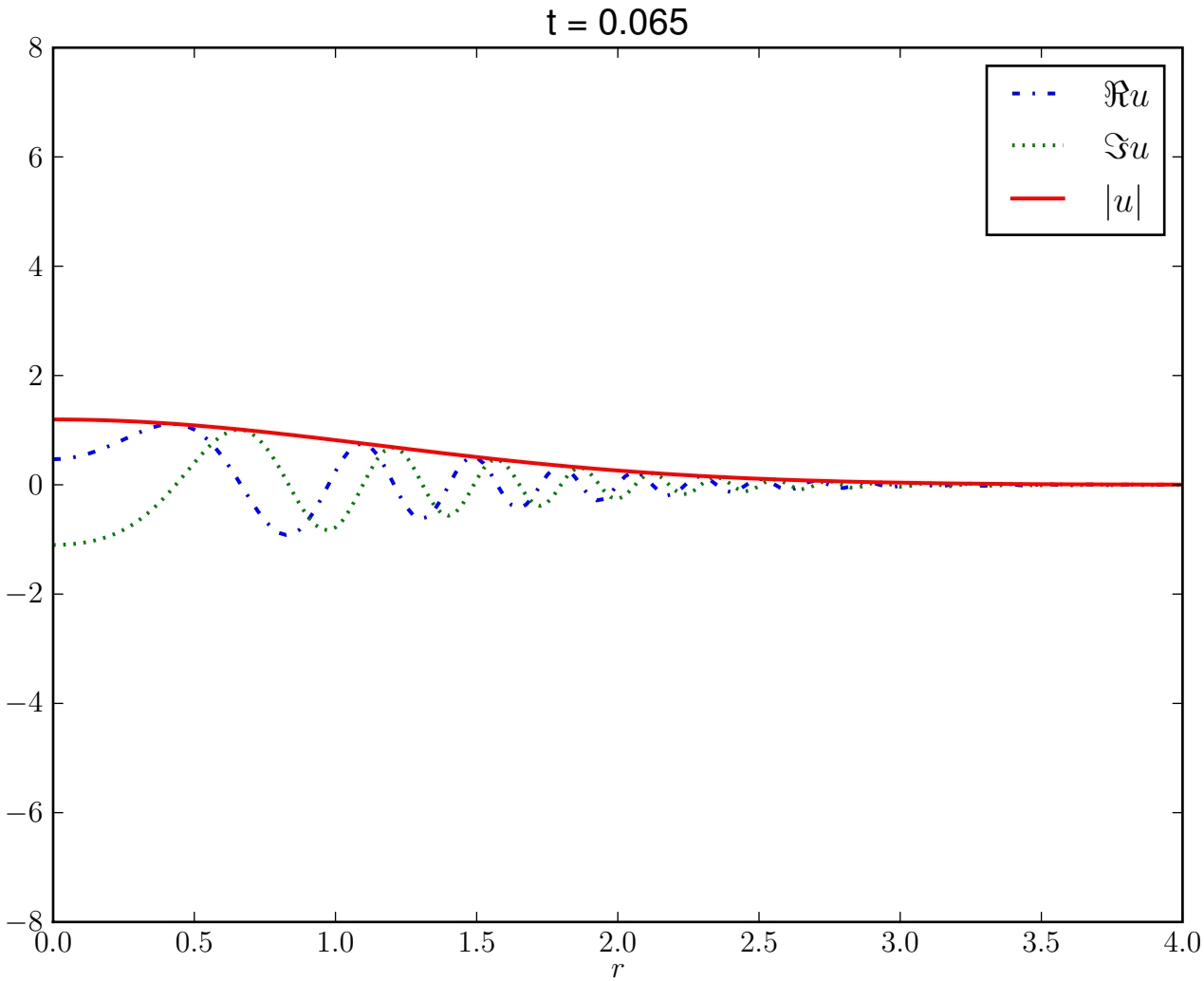

t = 0.04
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

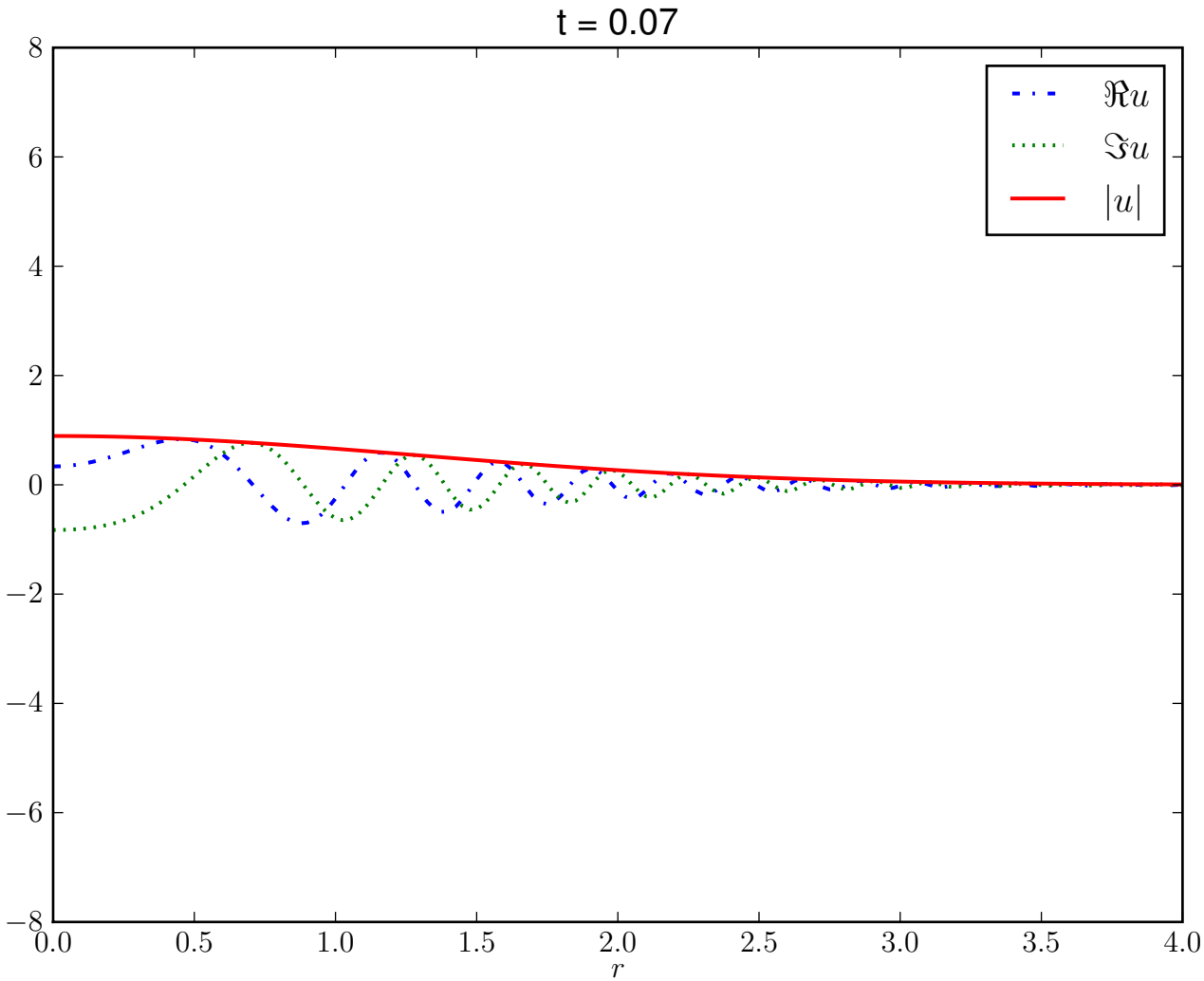

t = 0.045
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r$

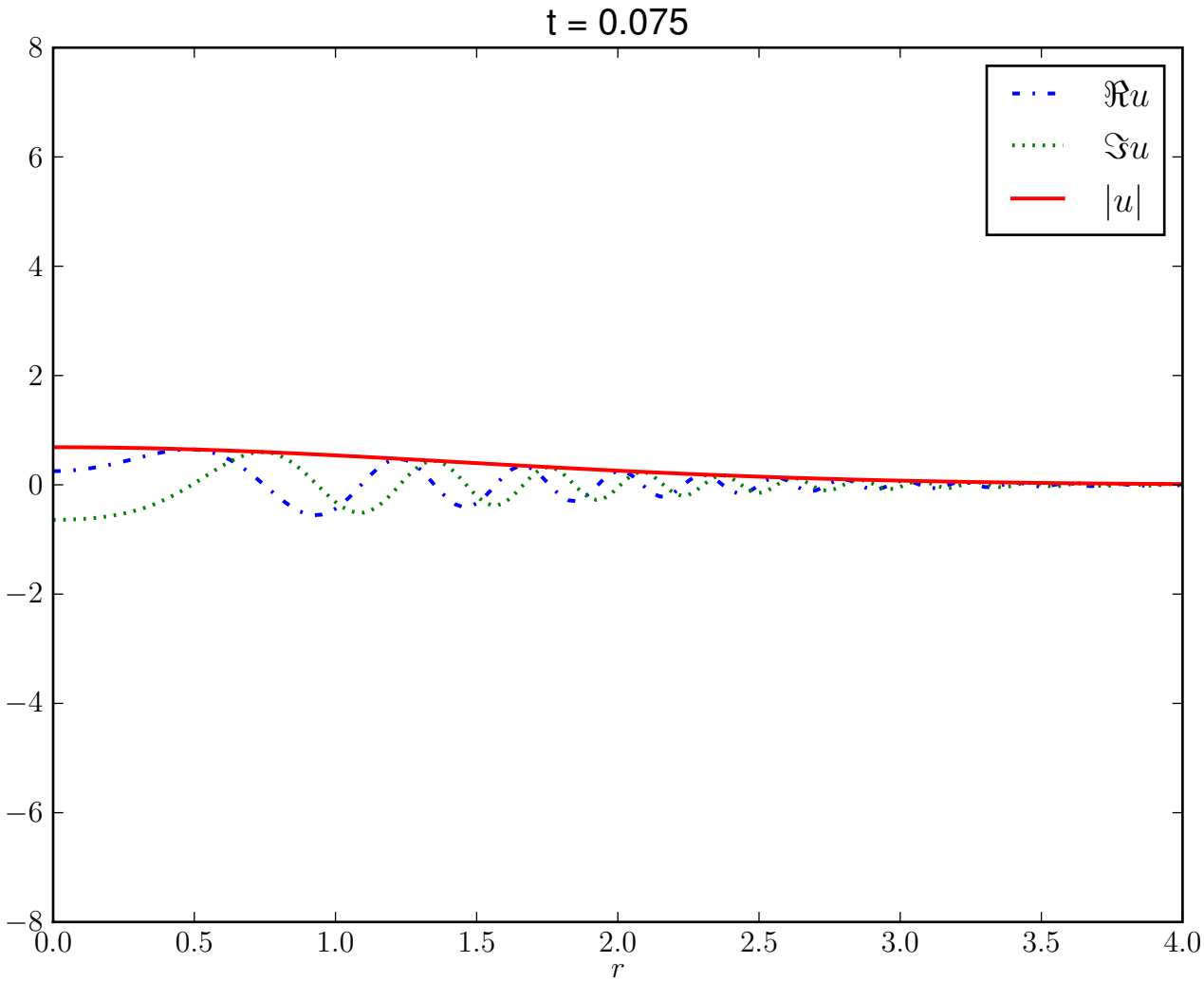

t = 0.05
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r$

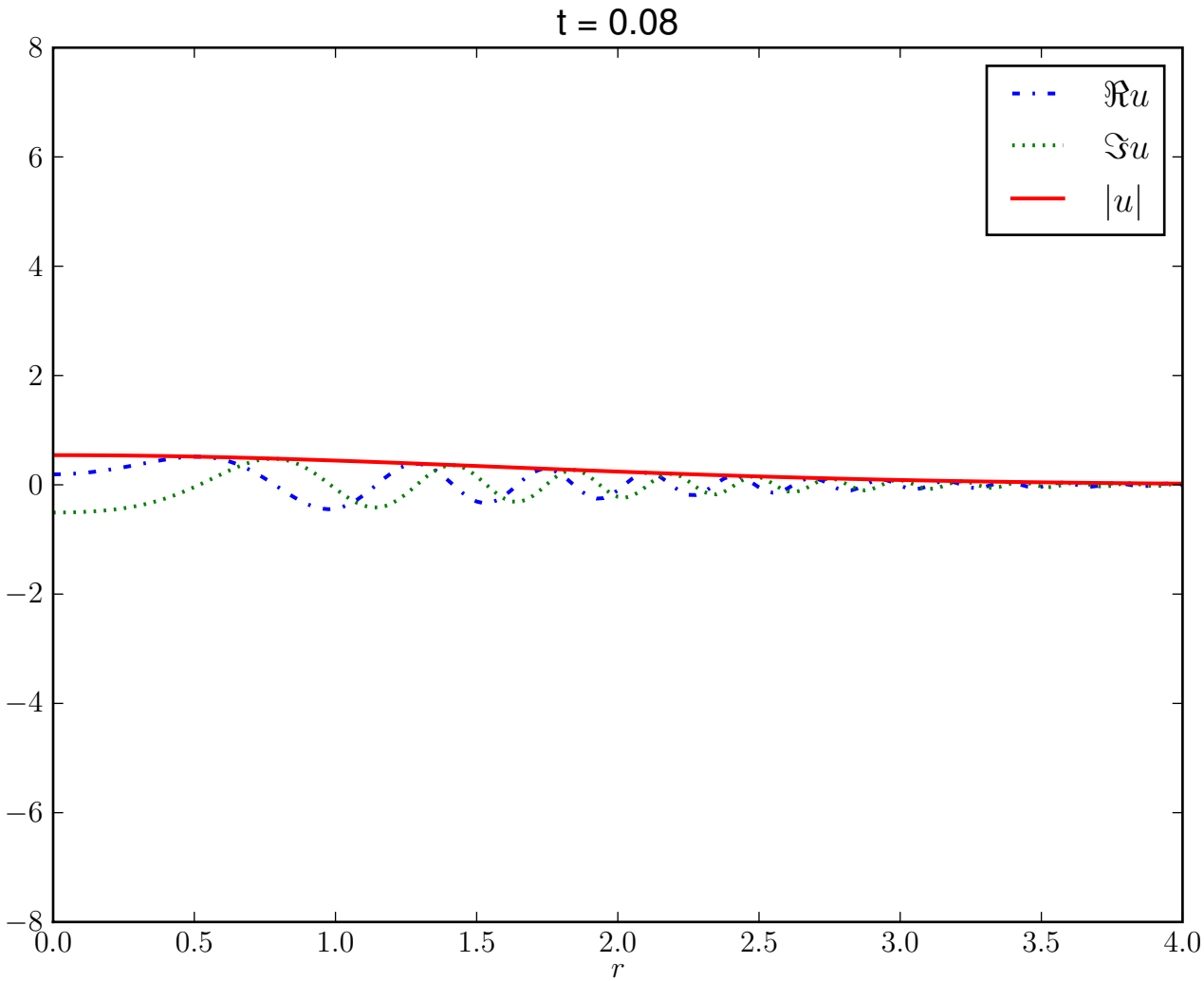

t = 0.055
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

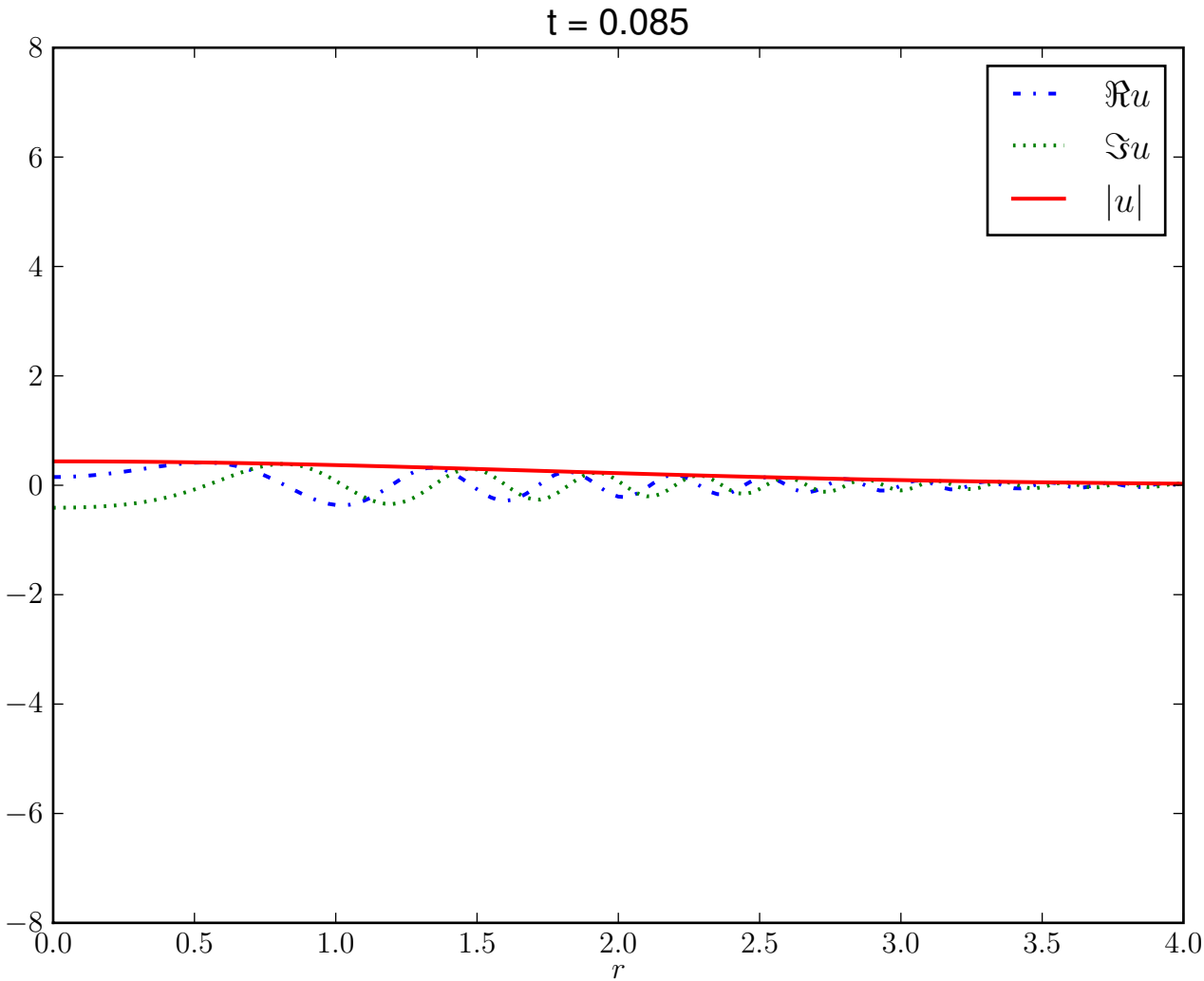

t = 0.06
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

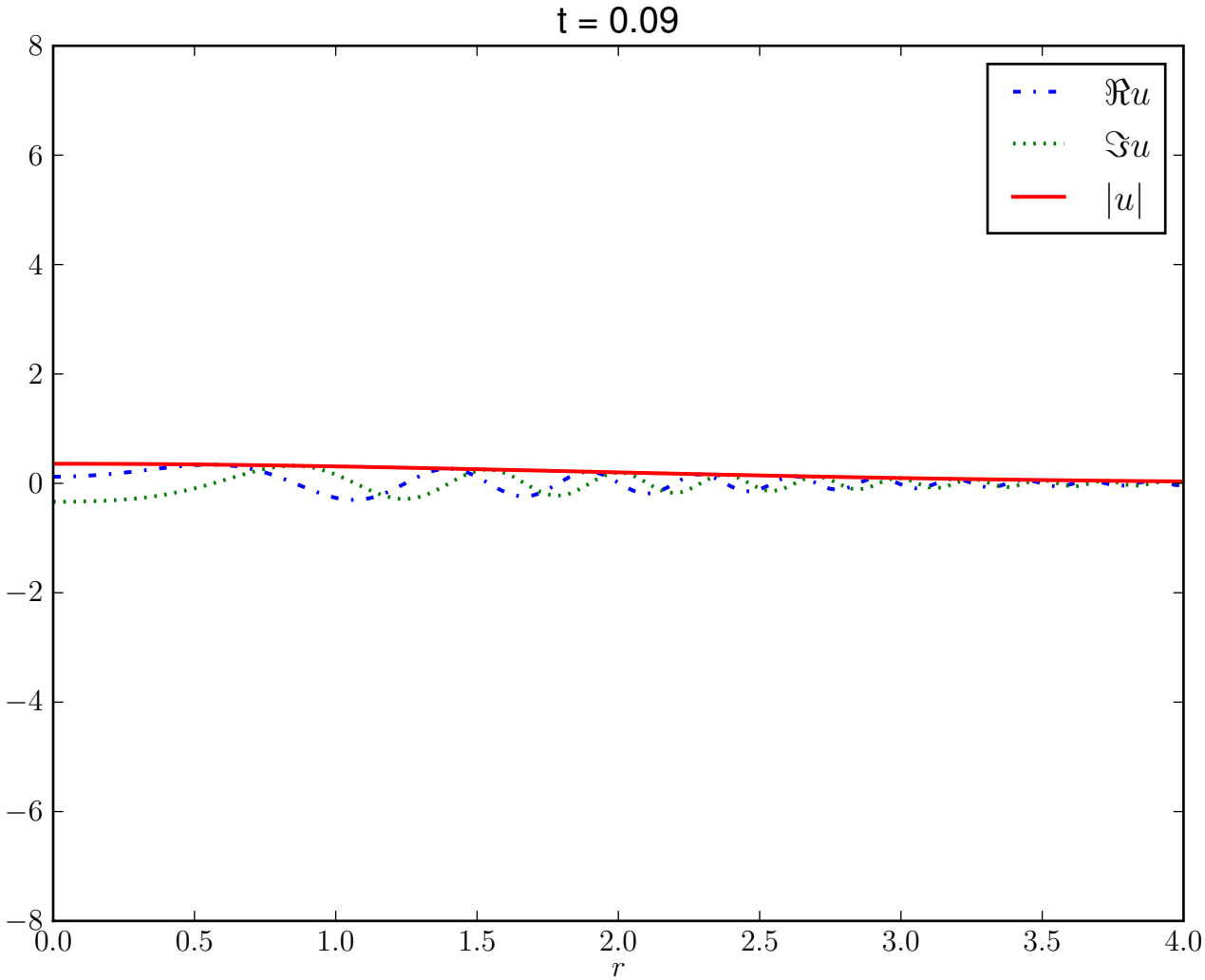

t = 0.065
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

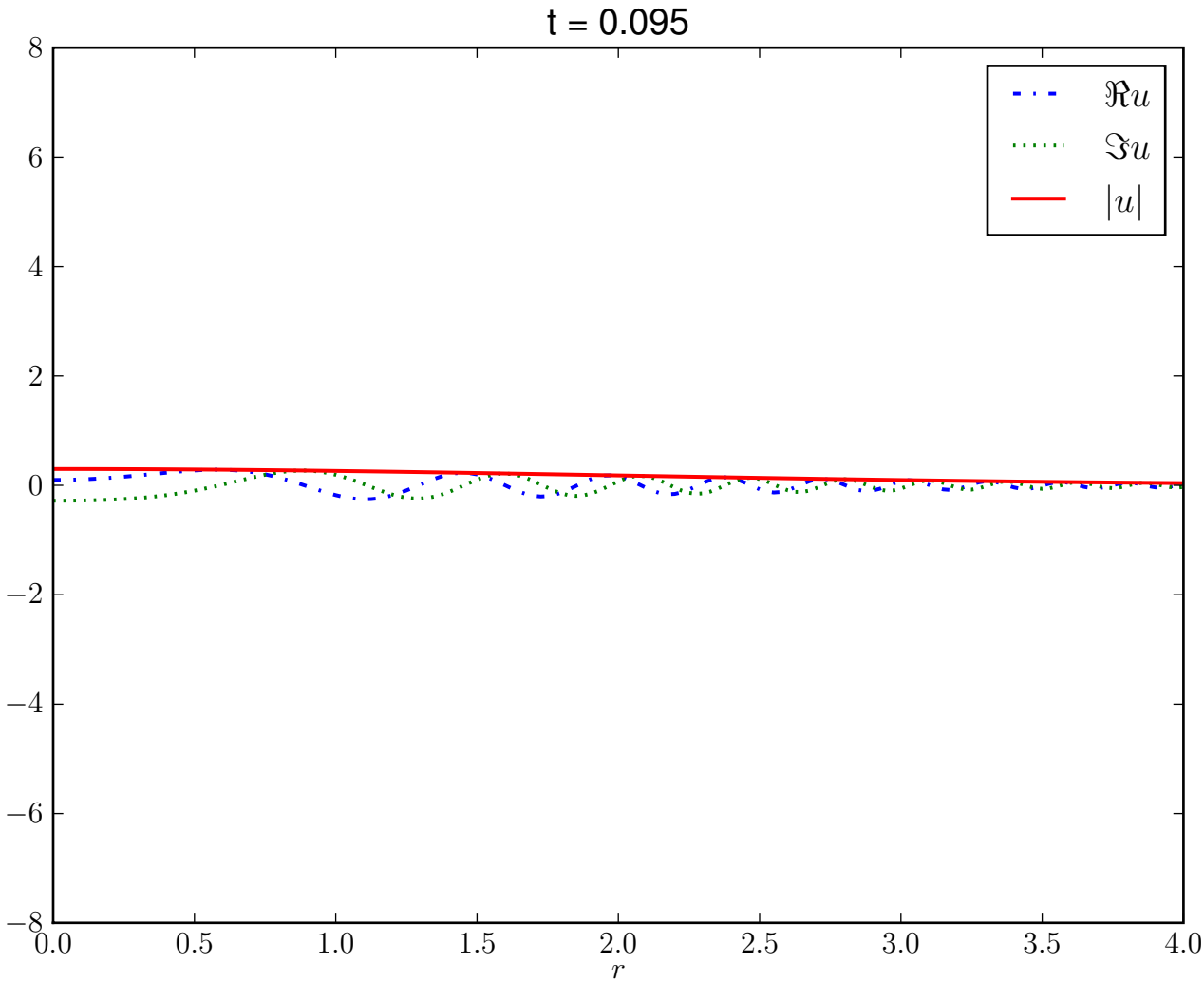

t = 0.07
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r$

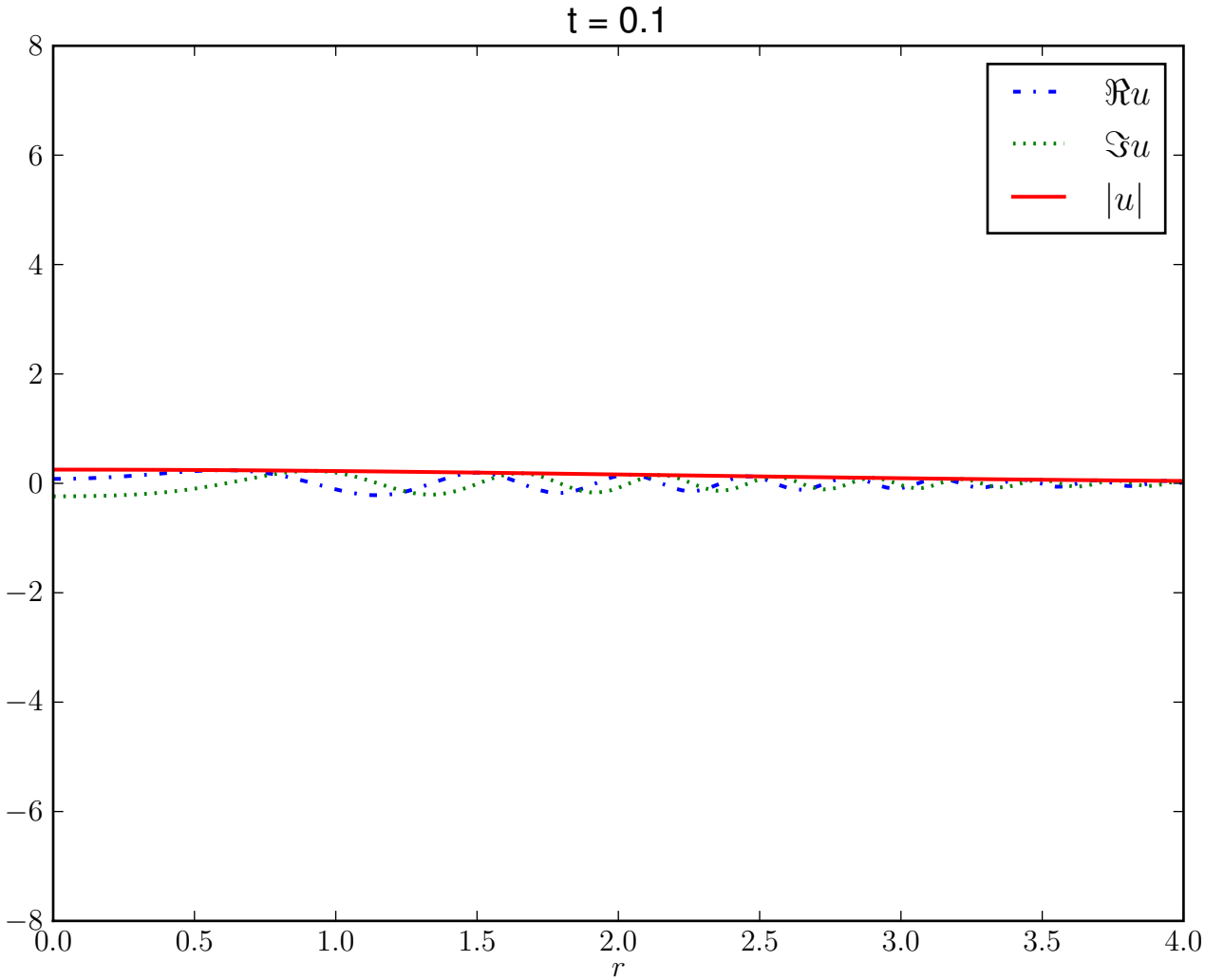

t = 0.075
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

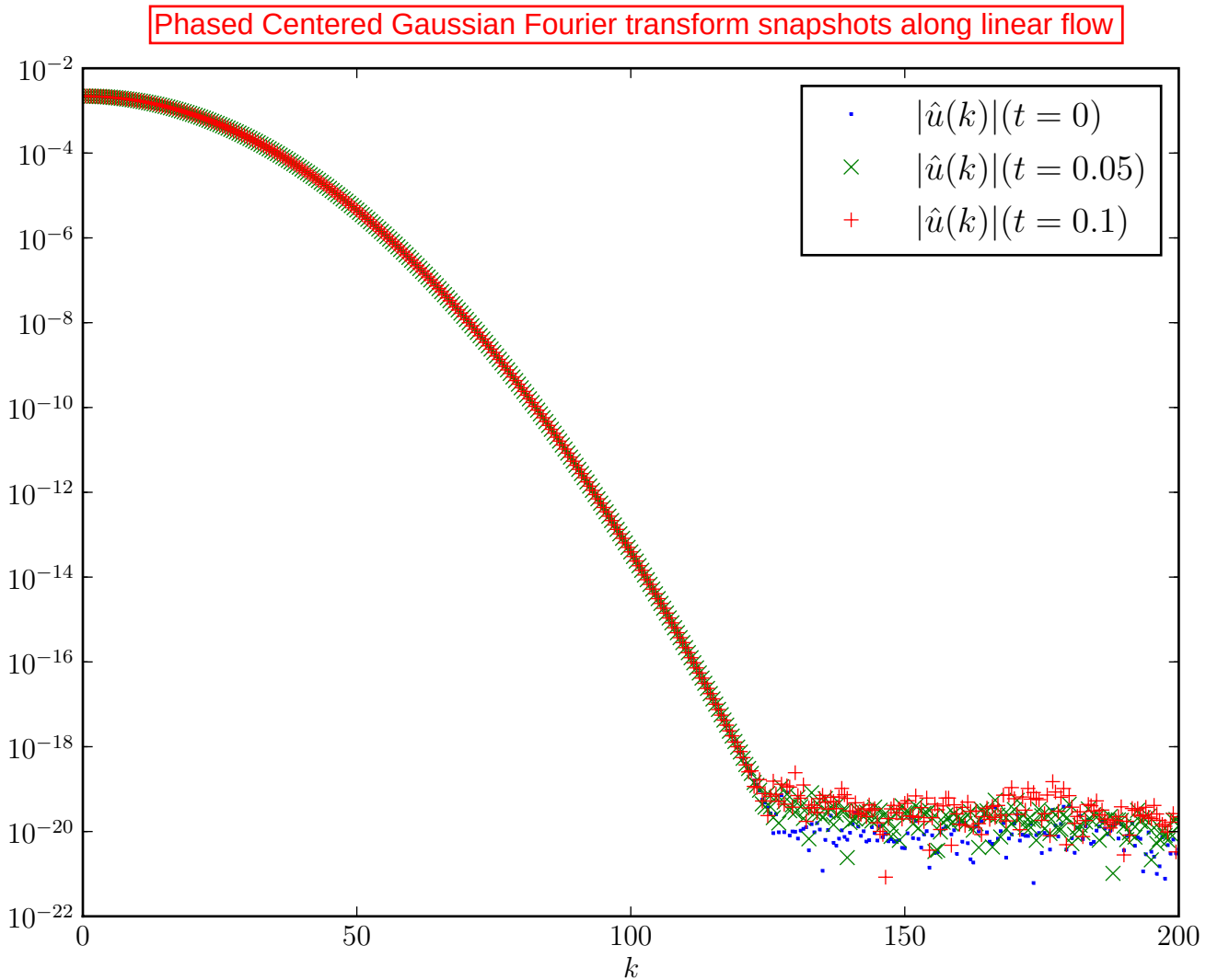

t = 0.08
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.085
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

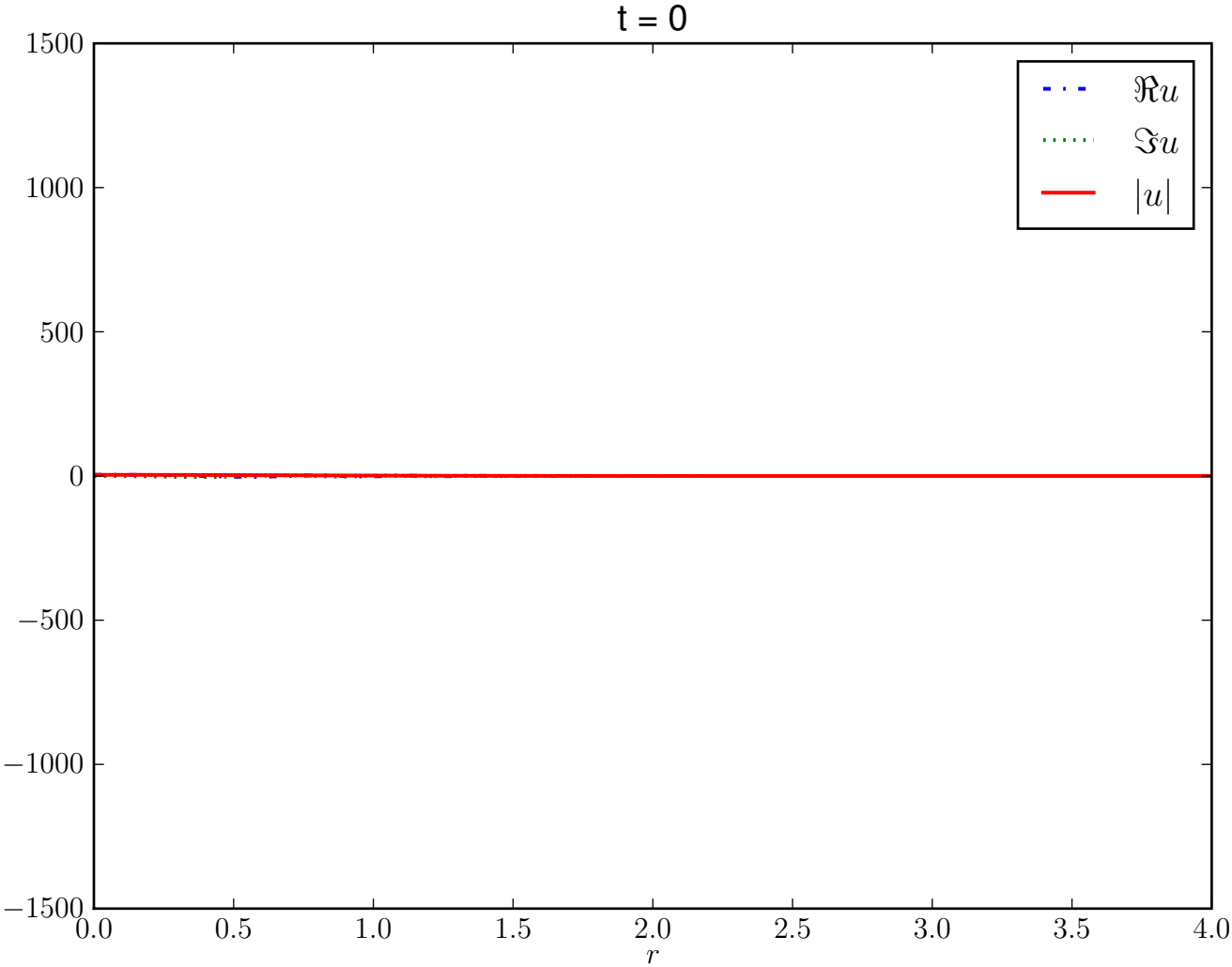

t = 0.09
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

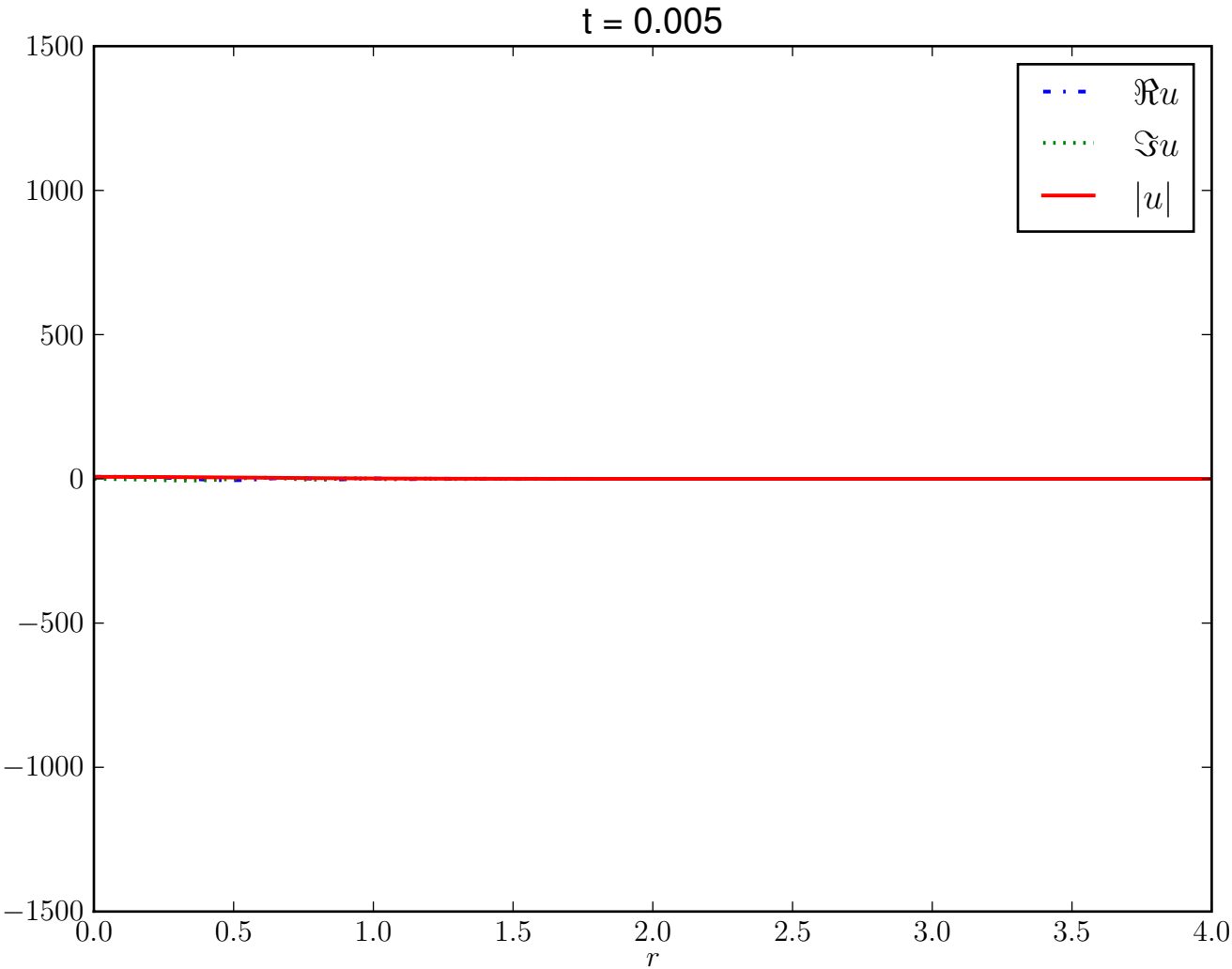

t = 0.095
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

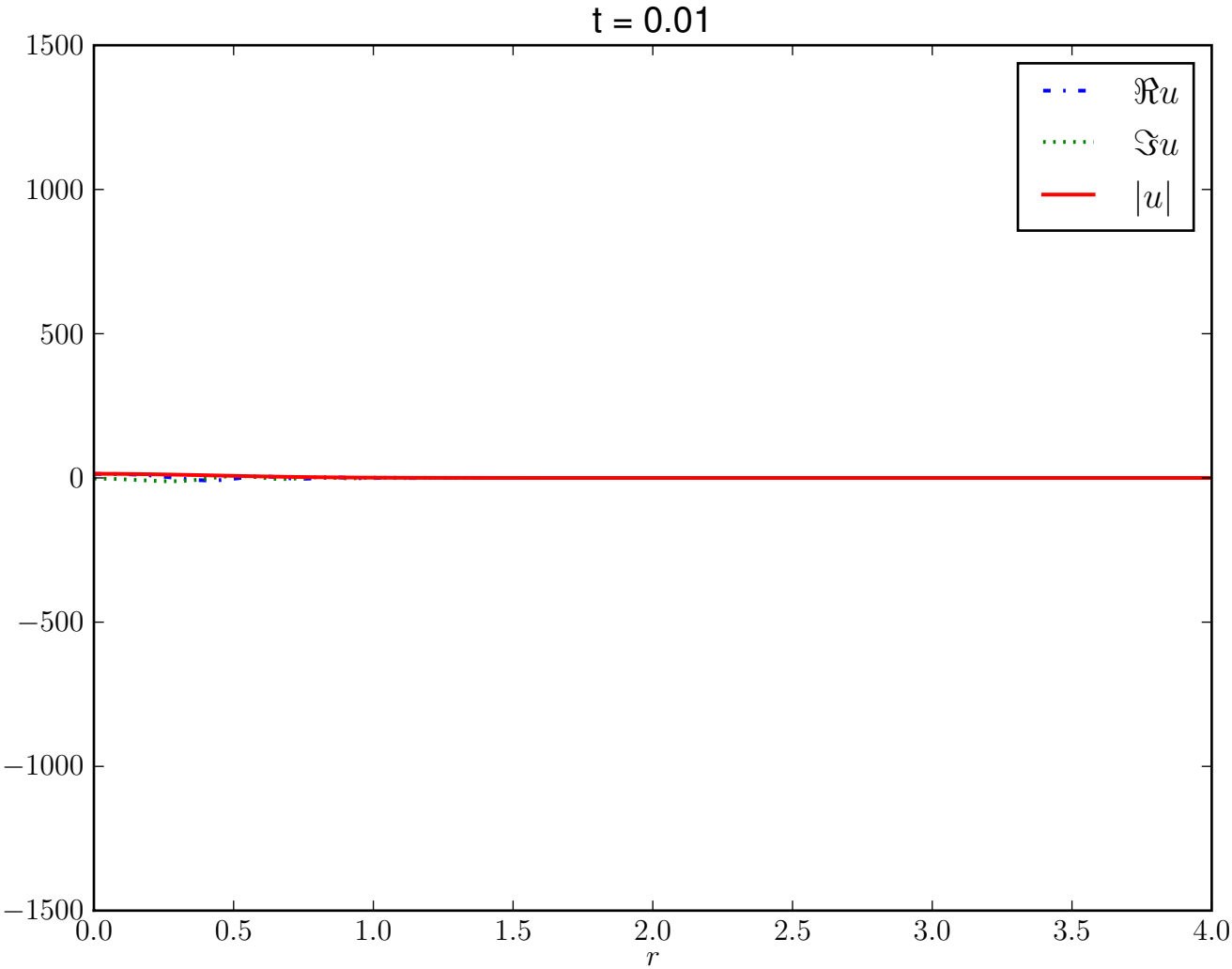

t = 0.1
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r$
$\Re u$
$\Im u$
$|u|$

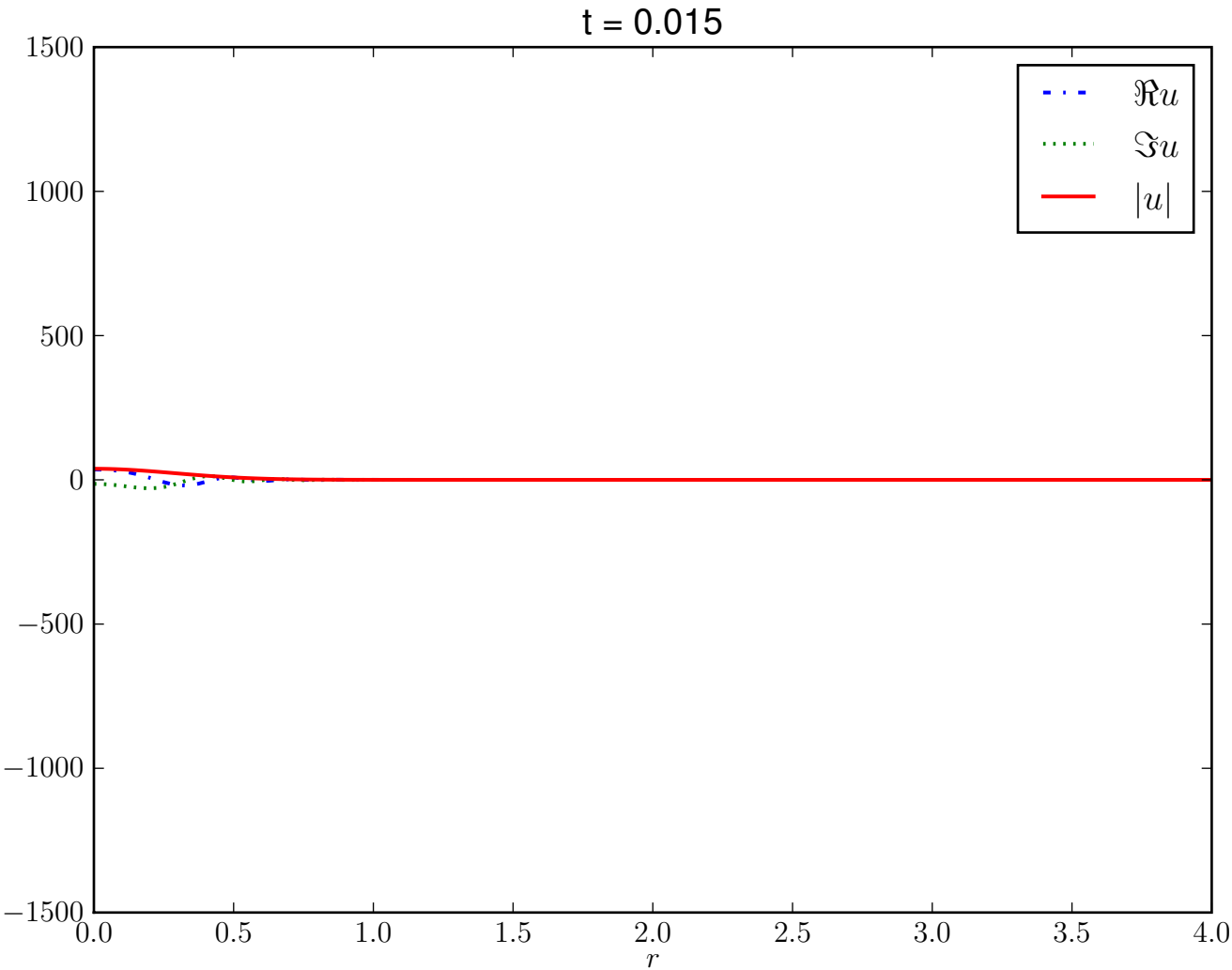

Phased Centered Gaussian Fourier transform snapshots along linear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.05)$
$|\hat{u}(k)|(t=0.1)$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
0
50
100
150
200
$k$

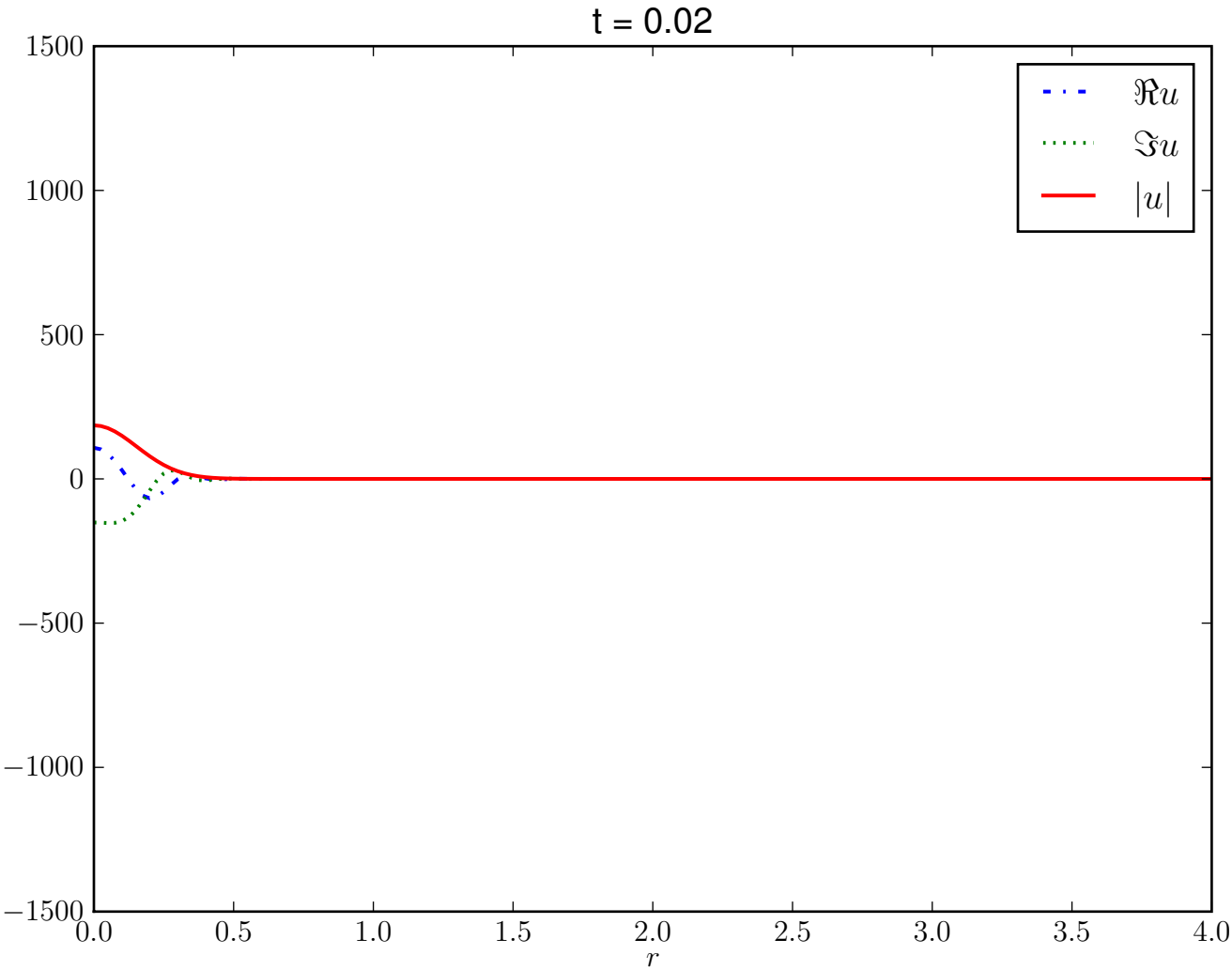
Phased Centered Gaussian under linear flow; bigger vertical axis

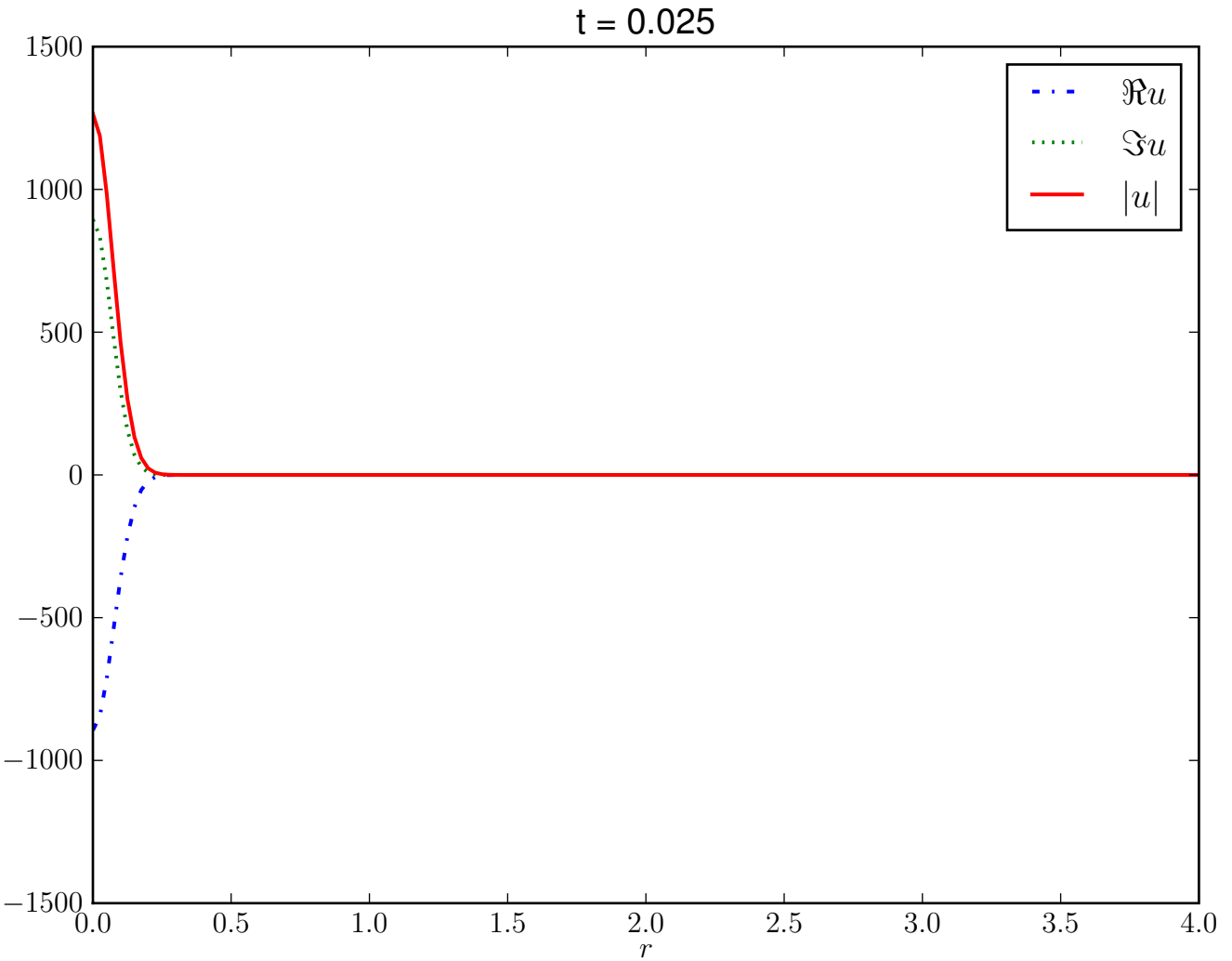


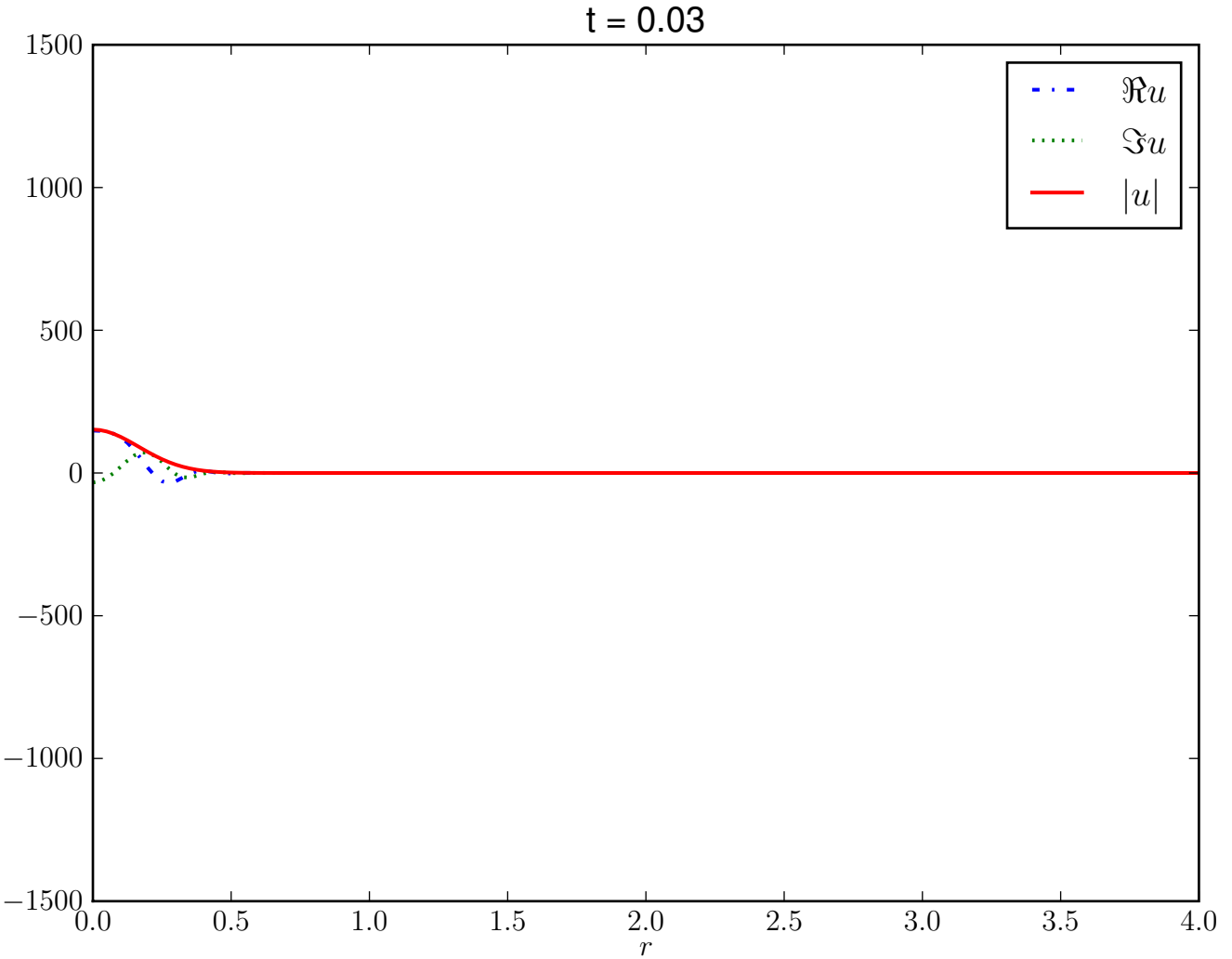

t = 0.005
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

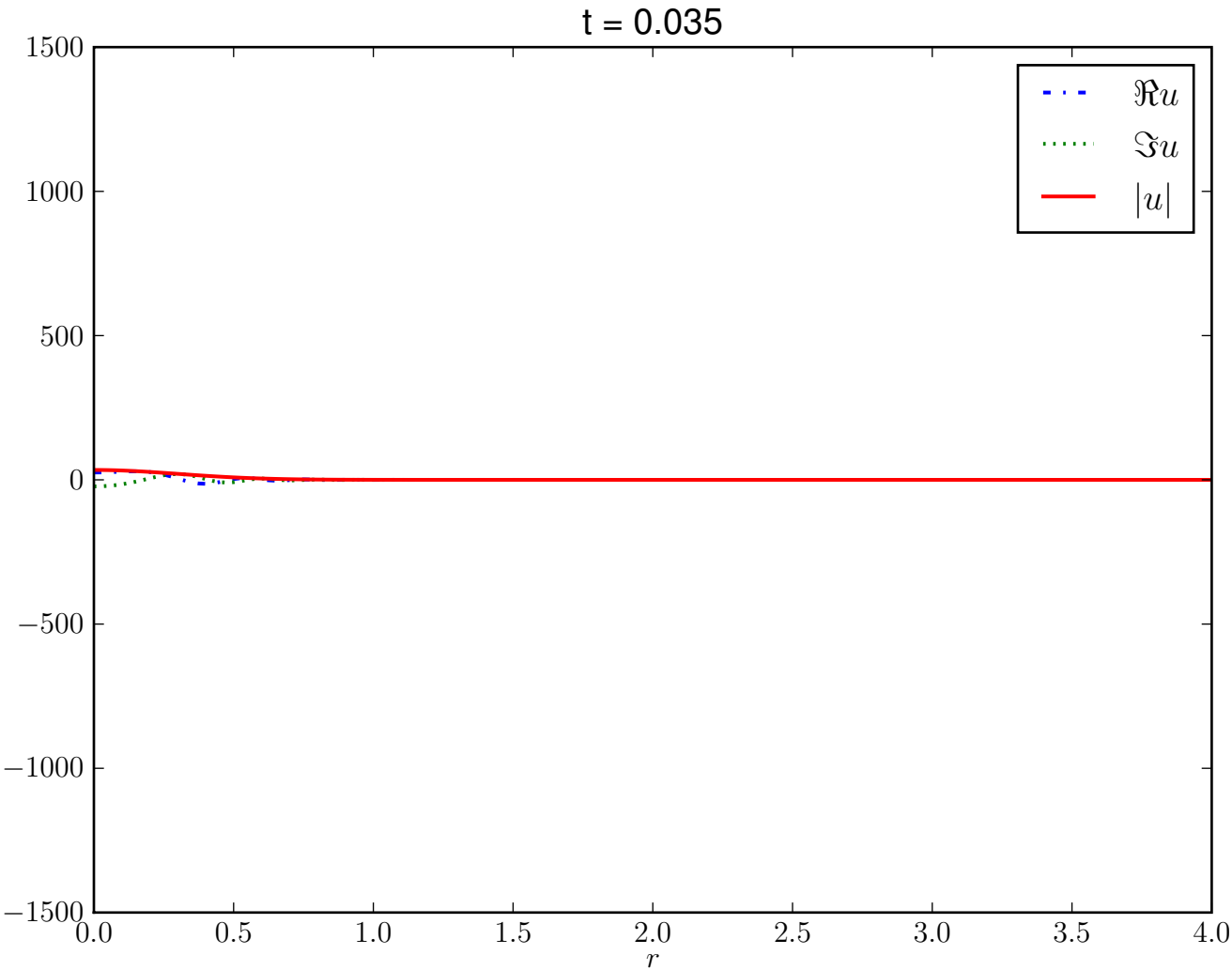

t = 0.01
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

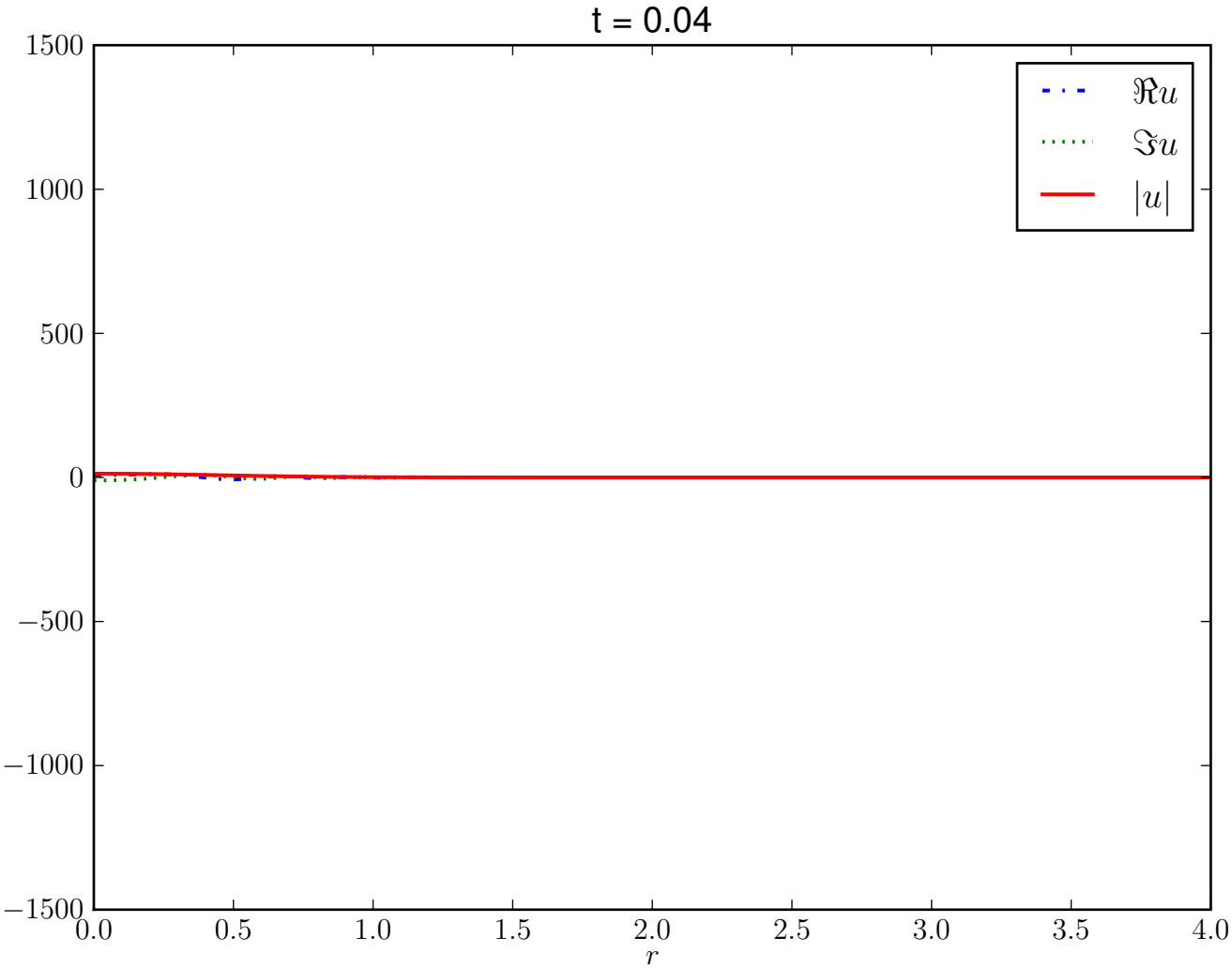

t = 0.015
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

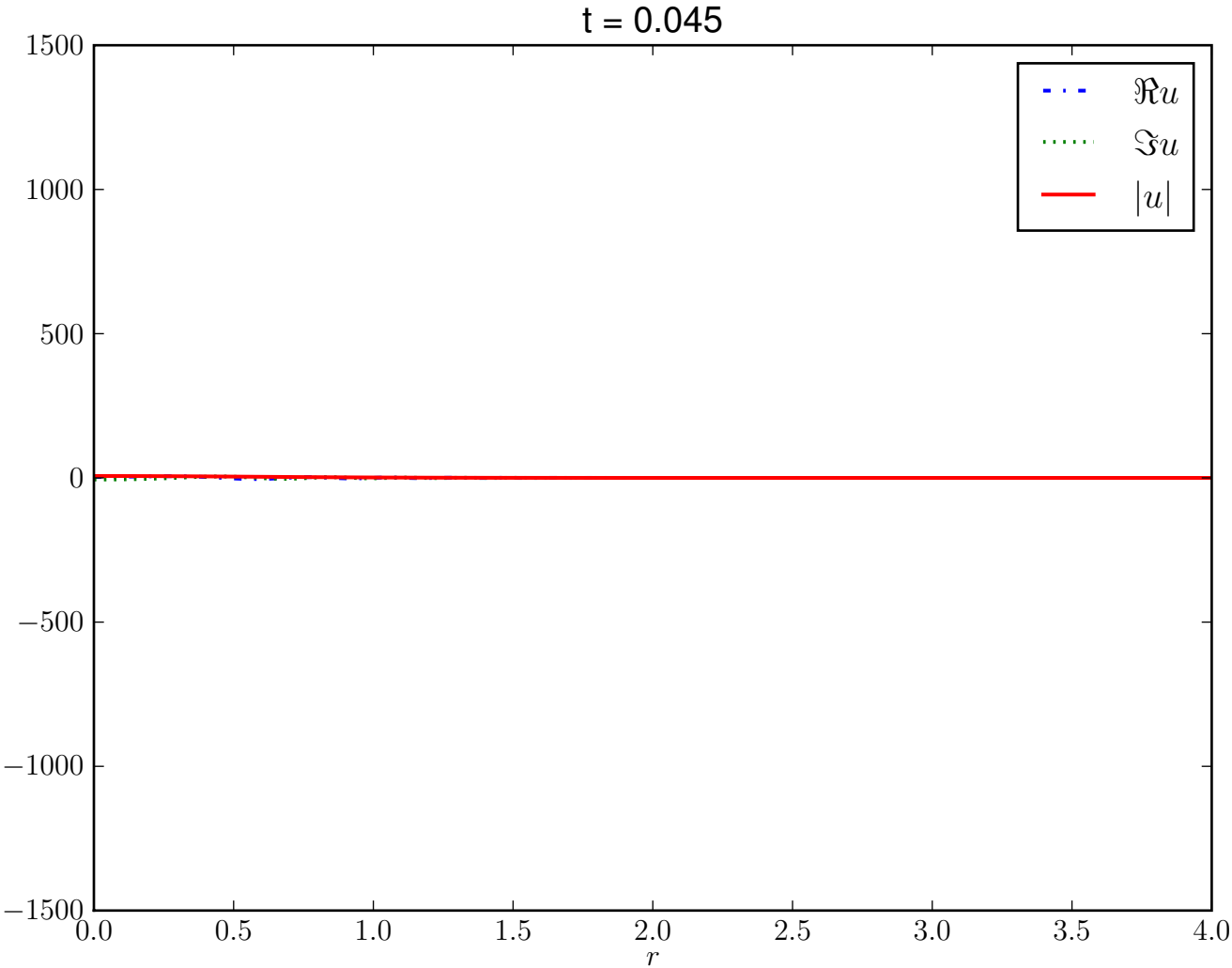

t = 0.02
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

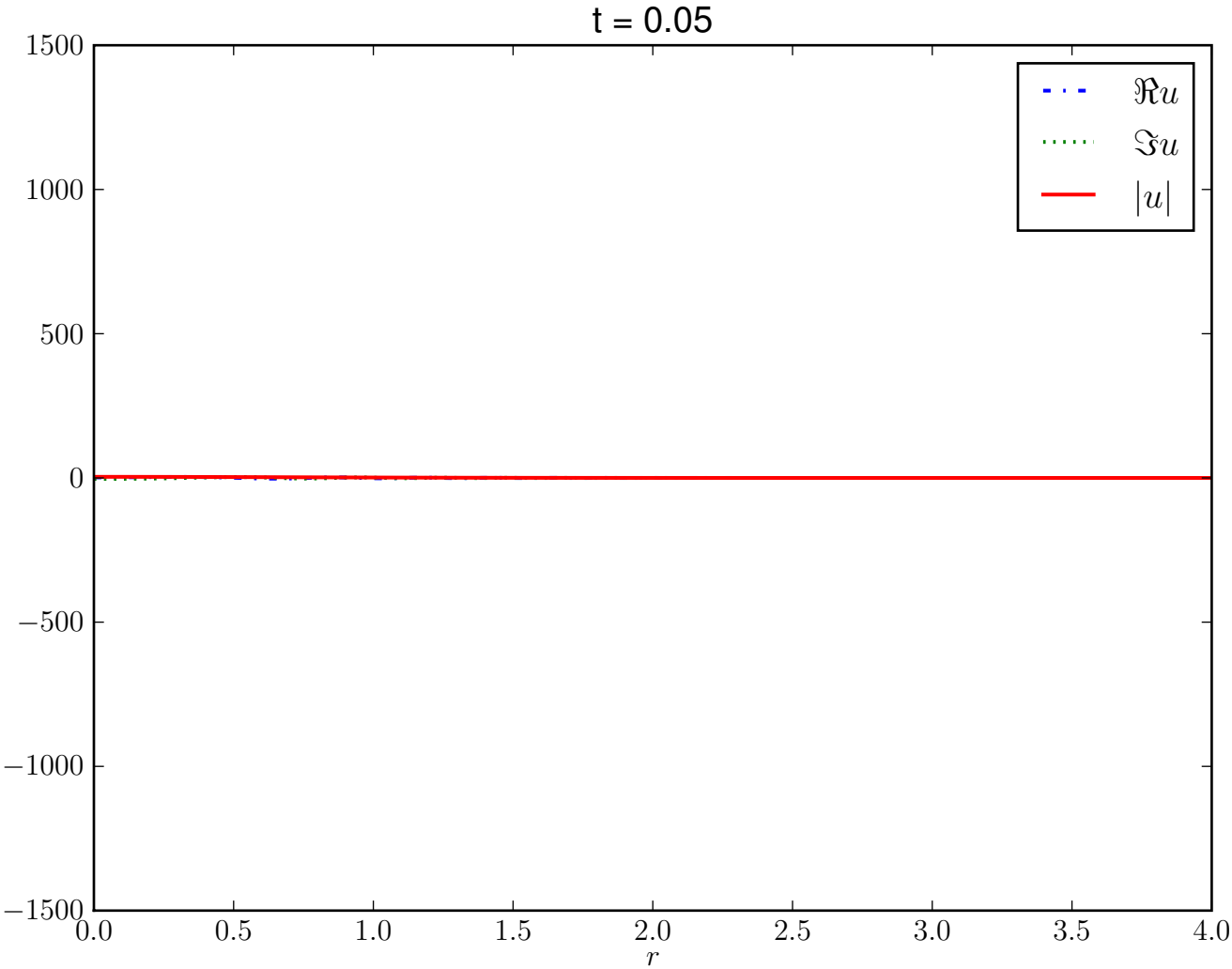

t = 0.025
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

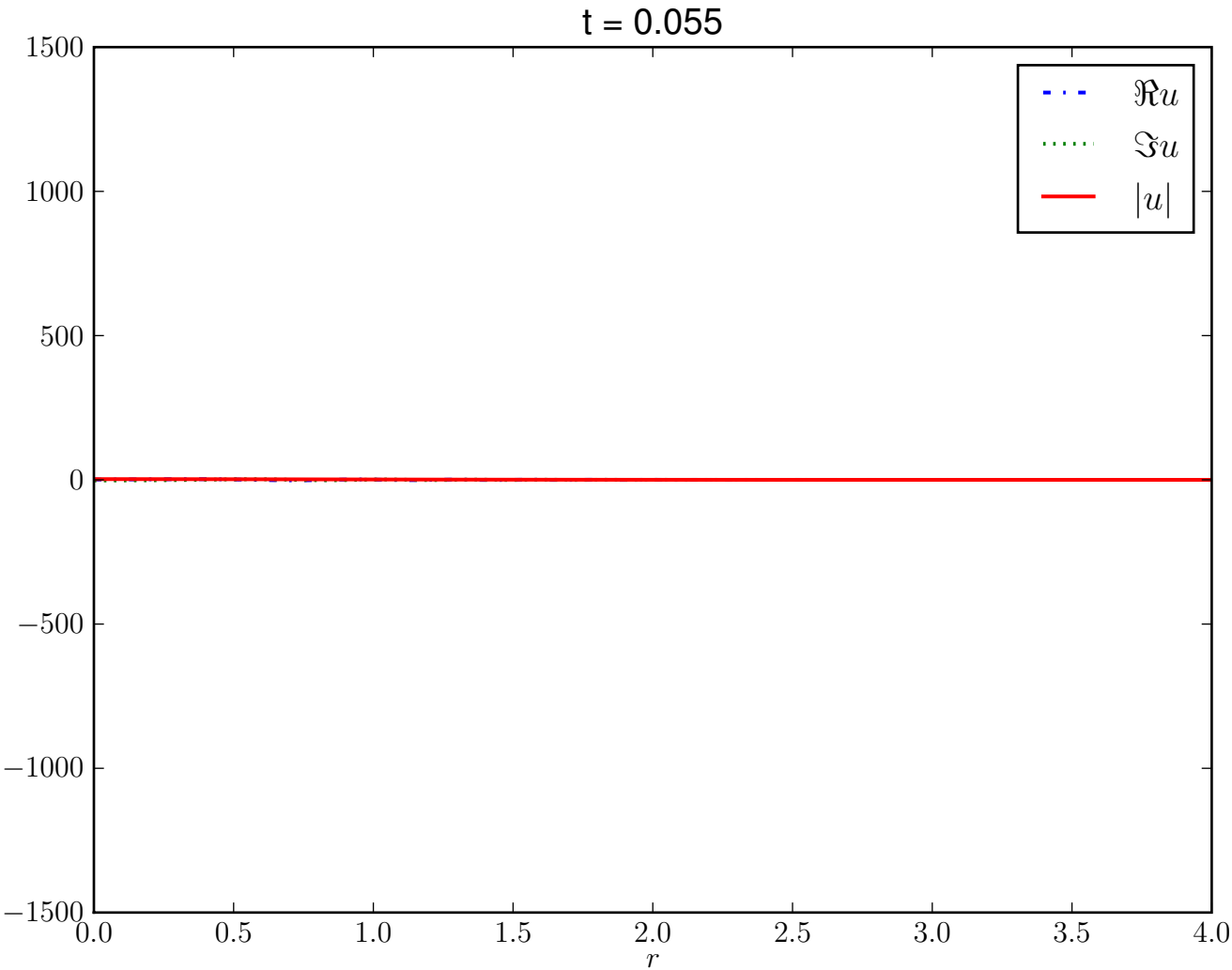

t = 0.03
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

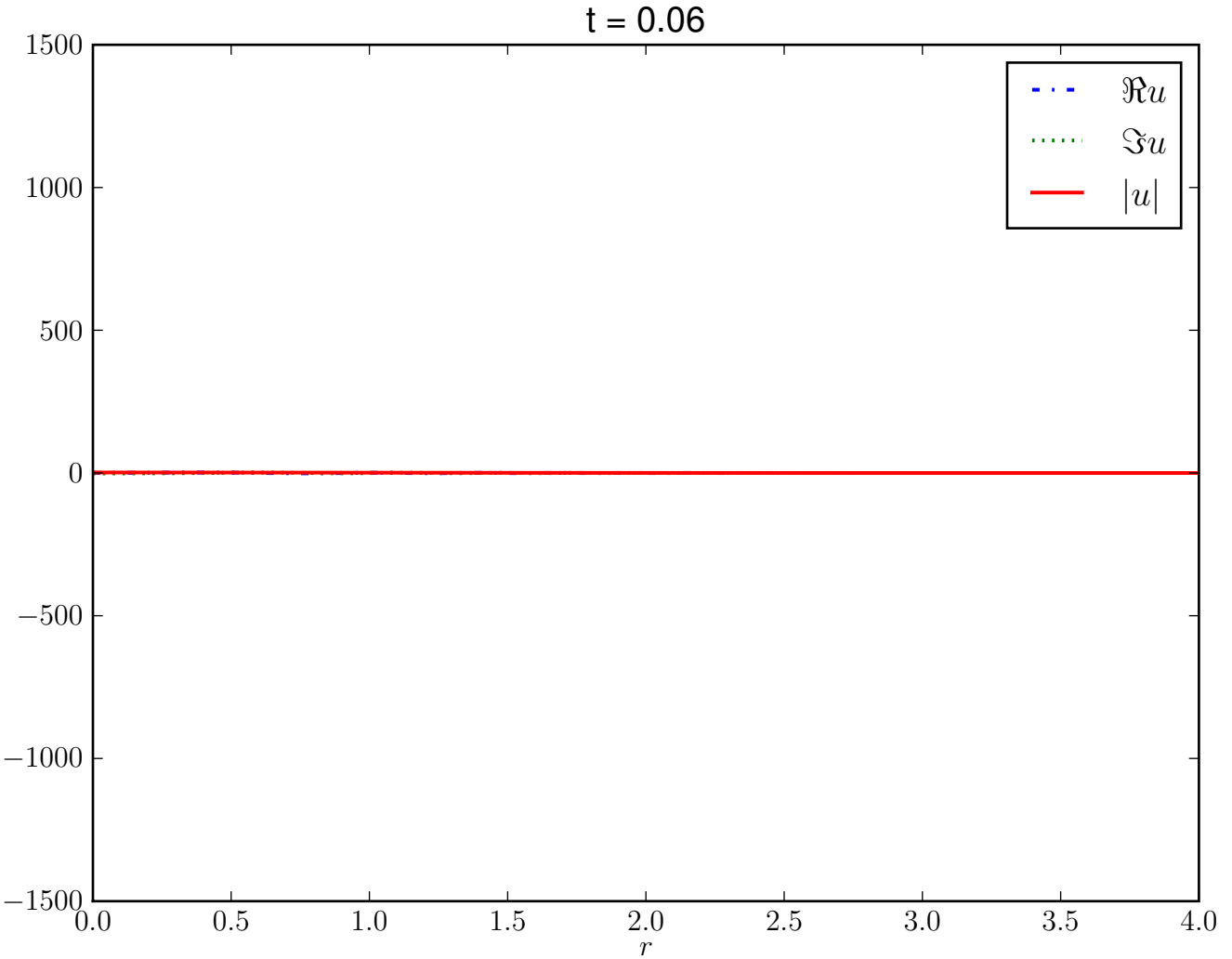

t = 0.035
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

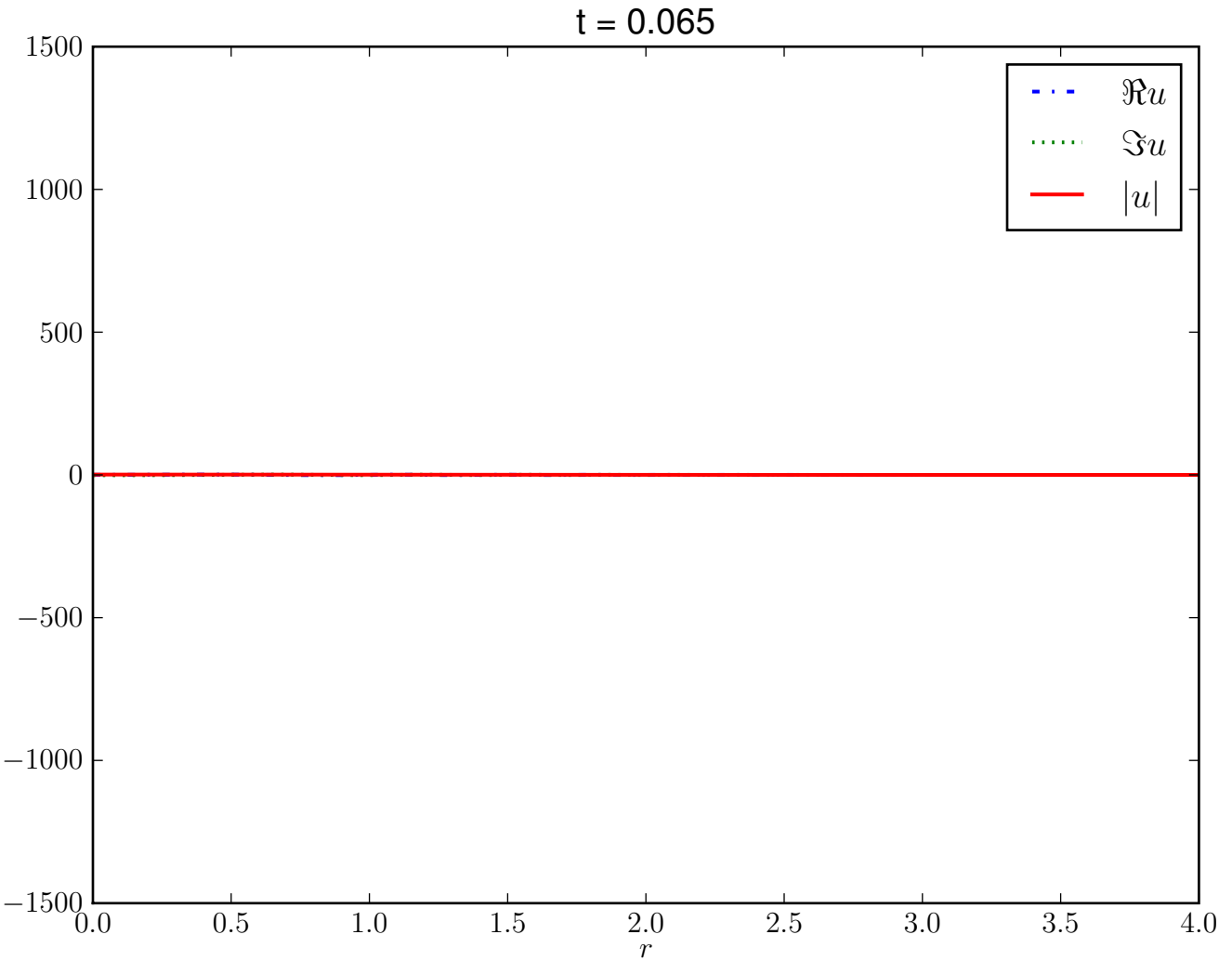

t = 0.04
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

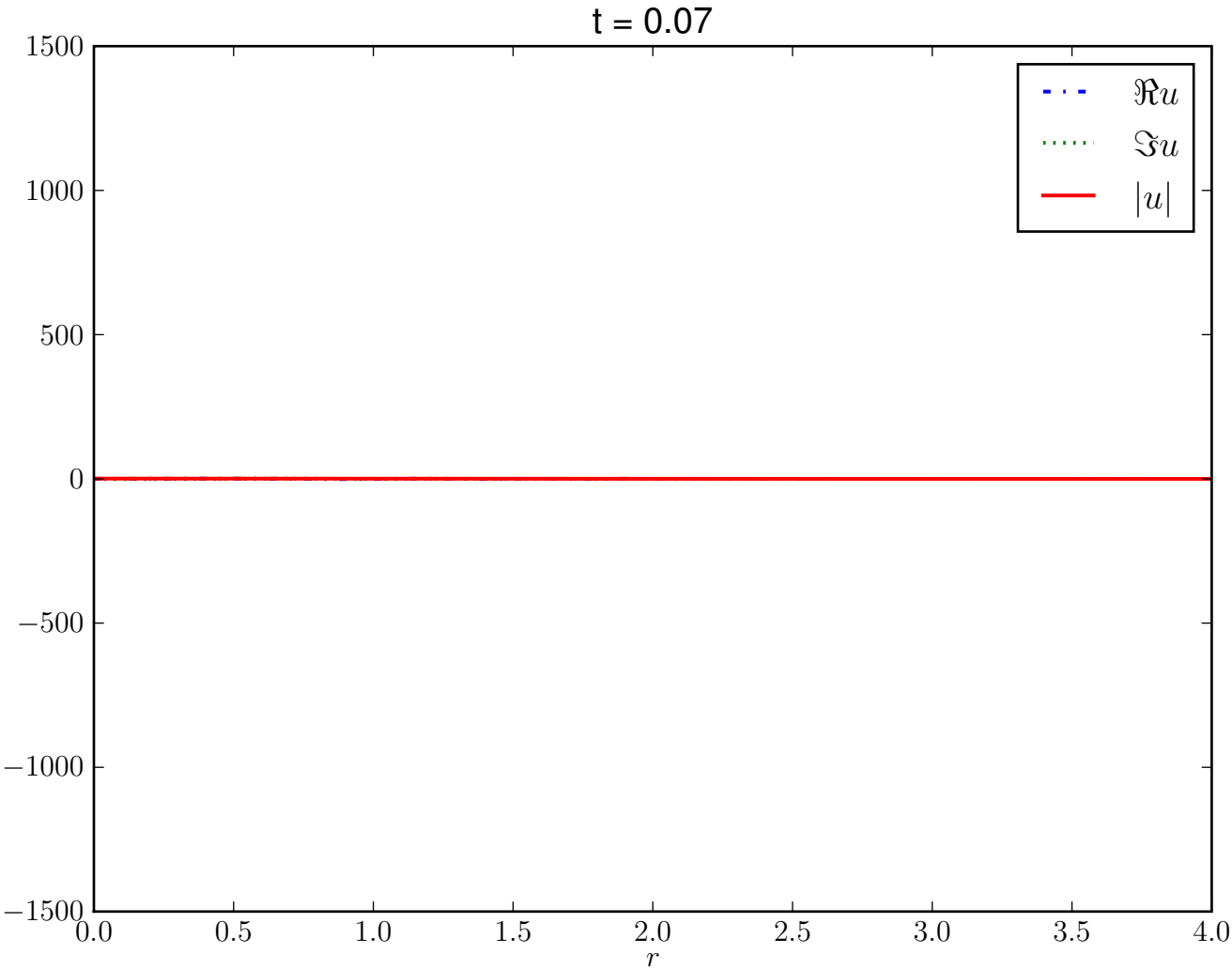

t = 0.045
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

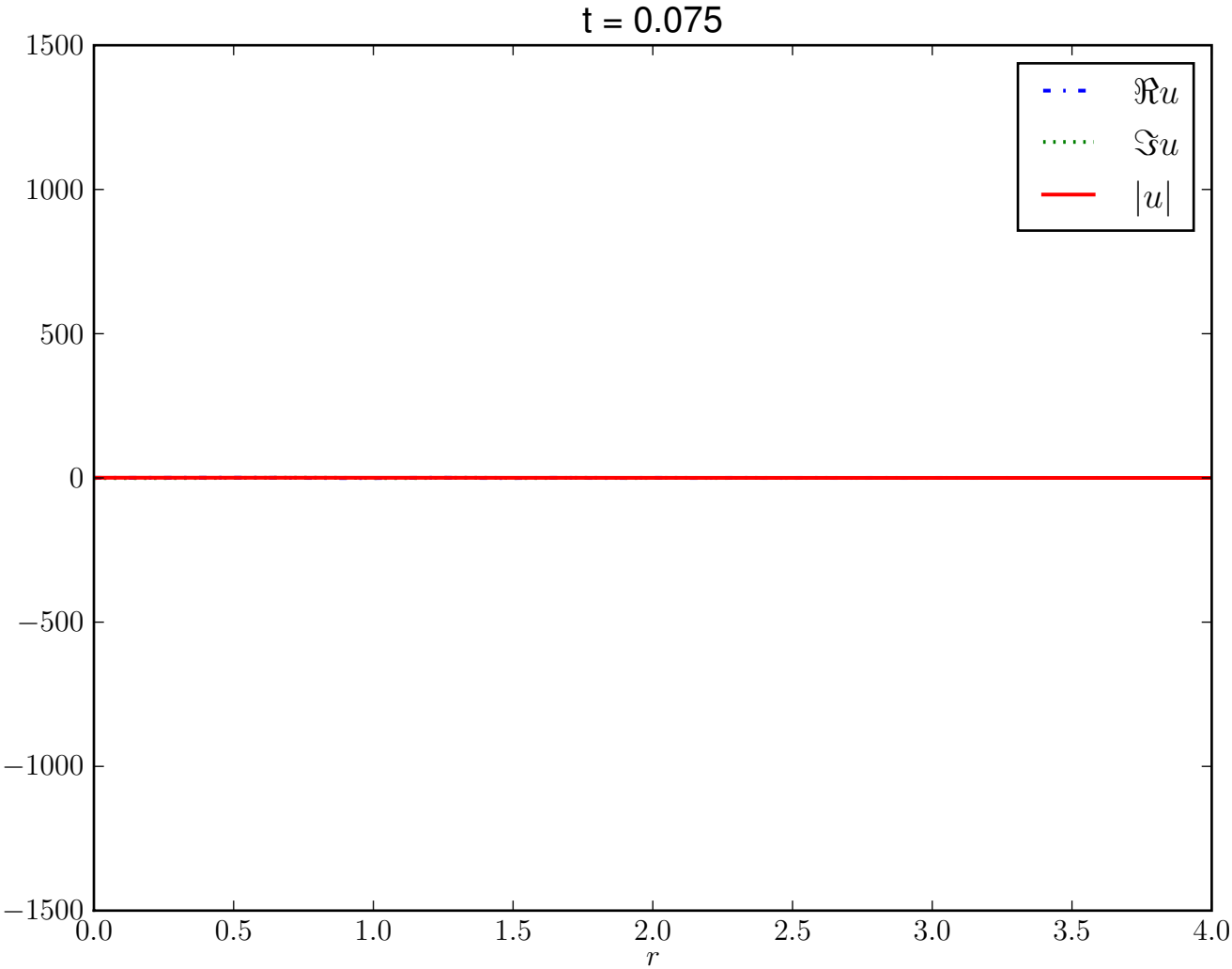

t = 0.05
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

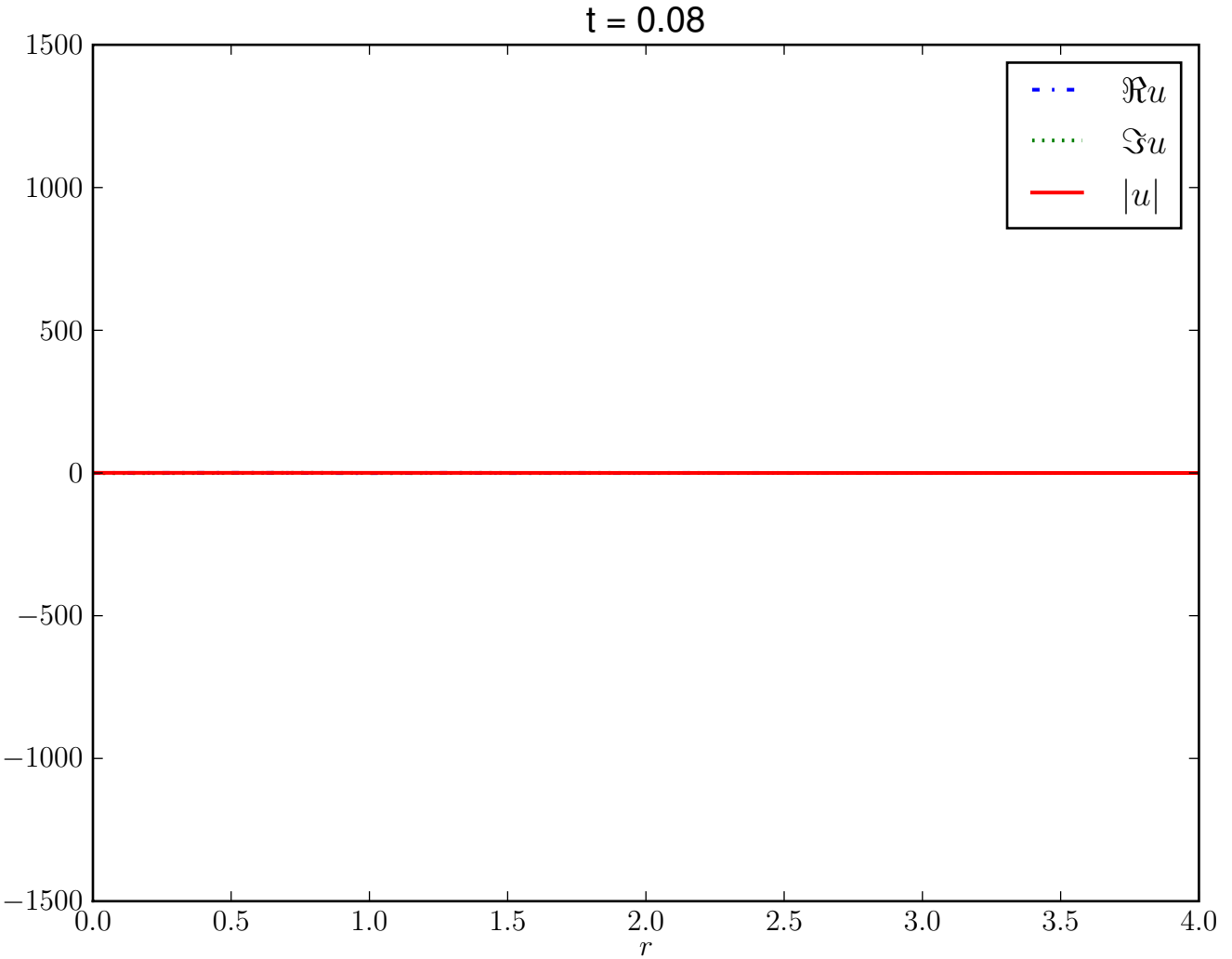

t = 0.055
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r$
$\Re u$
$\Im u$
$|u|$

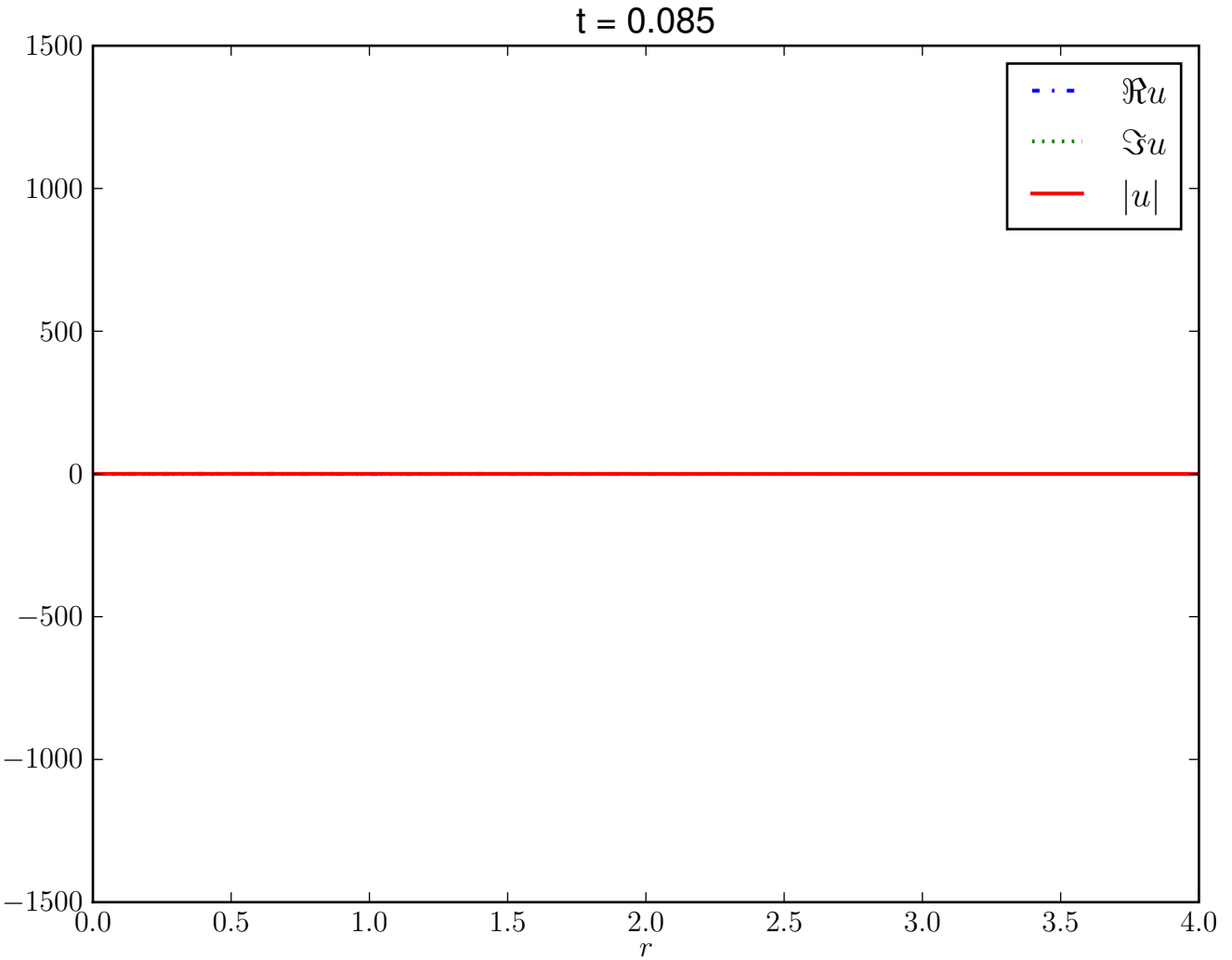

t = 0.06
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

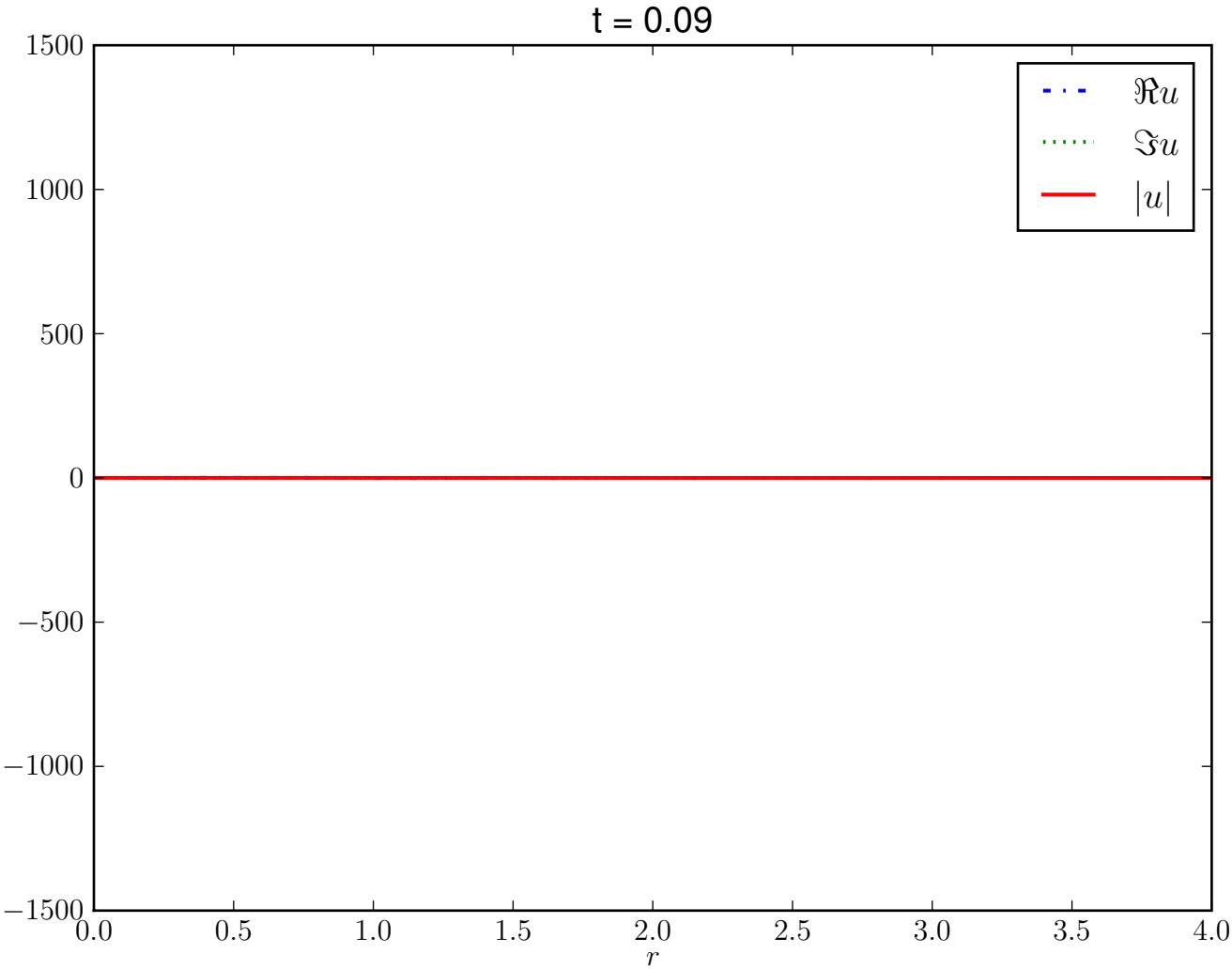

t = 0.065
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

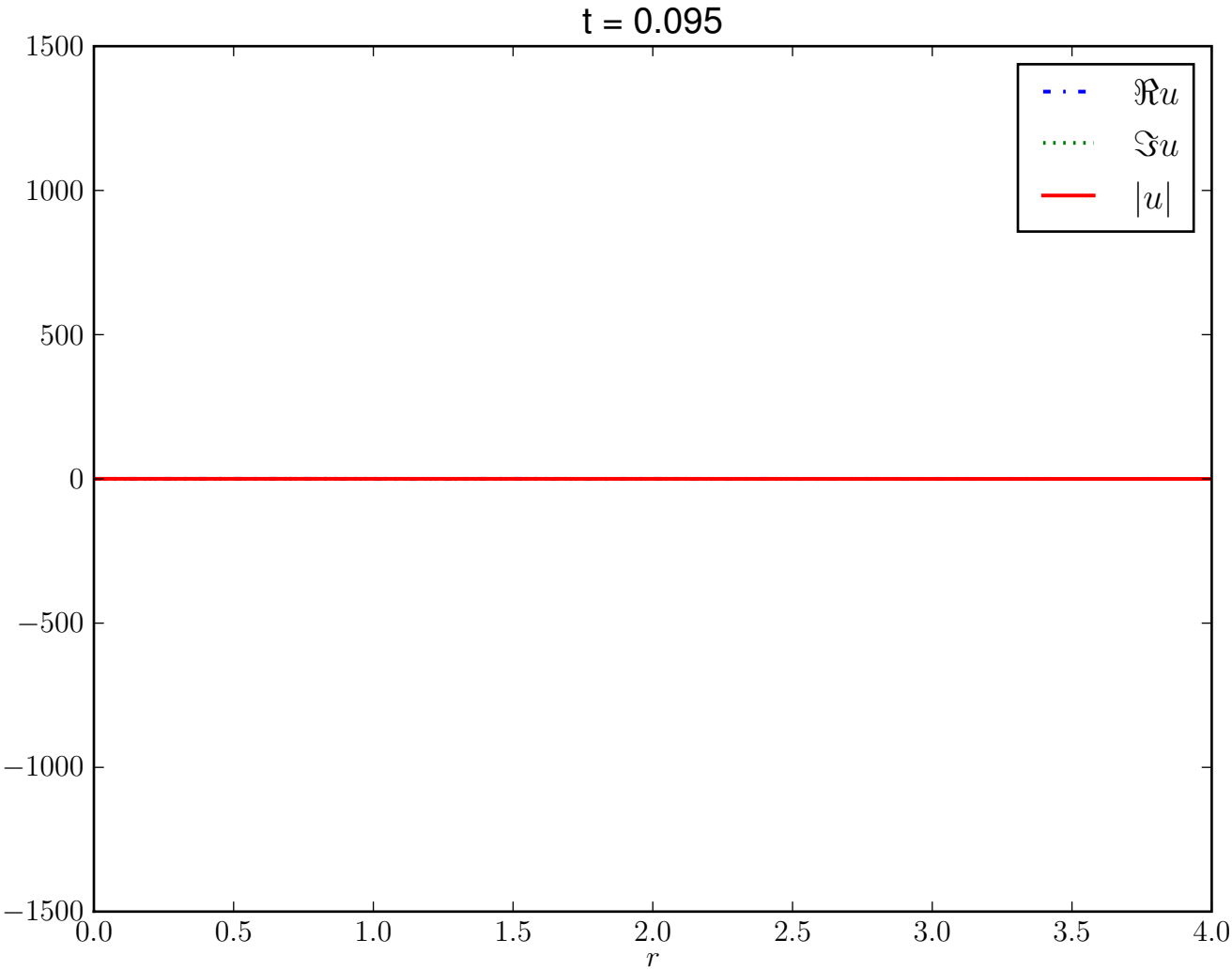

t = 0.07
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

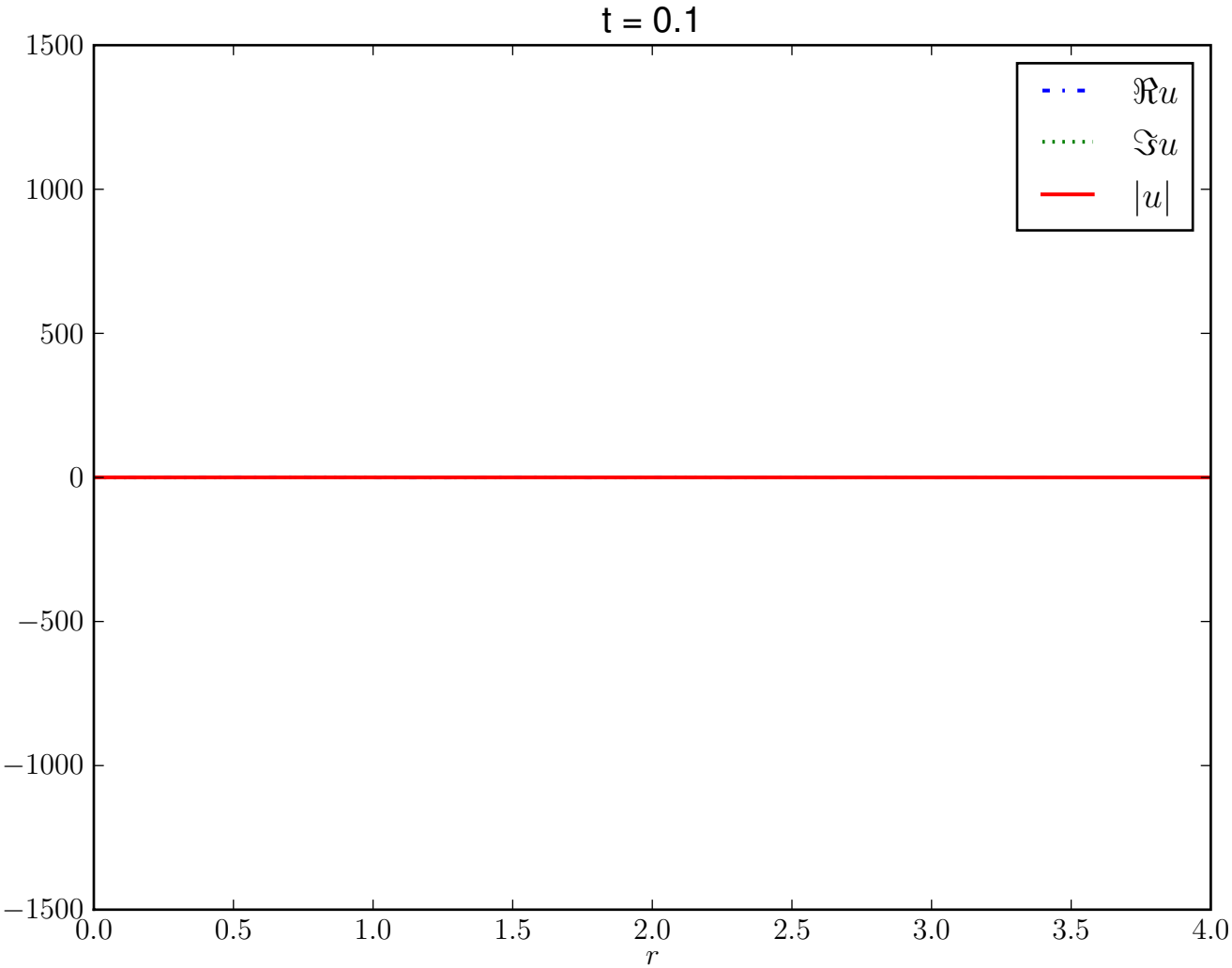

t = 0.075
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.08
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.085
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.09
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.095
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.1
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$