Resonant decompositions and the I-method for the cubic NLS on R^2

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1. Cubic NLS Initial Value Problem on $\mathbb{R}^2$

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We consider the initial value problems:

$$\begin{cases} (i\partial_t + \Delta)u = \pm |u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^{\pm}(\mathbb{R}^2))$$

The $+$ case is called defocusing; $-$ is focusing. $NLS_3^{\pm}$ is ubiquitous in physics. The solution has a dilation symmetry

$$u^{\lambda}(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y). \qquad \boxed{\text{Scaling}}$$

which is invariant in $L^2(\mathbb{R}^2)$. This problem is L^2-critical.

Time Invariant Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx \pm \frac{1}{2} |u(t)|^4 dx.$$

- Mass is L^2; Momentum is close to $H^{1/2}$; Energy involves H^1.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Frequency Localizations?

Local-in-time theory for $NLS_3^{\pm}(\mathbb{R}^2)$

- $\forall\ u_0 \in L^2(\mathbb{R}^2)\ \exists\ T_{lwp}(u_0)$ determined by

Strichartz

$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$

  $\exists$ unique $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$ solving $NLS_3^{+}(\mathbb{R}^2)$.

- $\forall\ u_0 \in H^s(\mathbb{R}^2), s > 0,\ T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists: $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.
- Define the maximal forward existence time $T^*(u_0)$ by

$$\|u\|_{L^4_{tx}([0,T^*-\delta]\times\mathbb{R}^2)} < \infty$$

  for all $\delta > 0$ but diverges to ∞ as $\delta \searrow 0$.

- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

Global-in-time theory?

What is the ultimate fate of the local-in-time solutions?

L^2-critical Scattering Conjecture:
$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Asymptotic Representation

Moreover, $\exists\, u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \|e^{\pm it\Delta}u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Same statement for focusing $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known for large radial data [Killip-Tao-Visan 07].

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{1}{3}$	resonant cut & I-Morawetz	[C-Roy08]
$s > 0$?		

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

H^1 Global Well-Posedness Scheme

Consider $NLS_3^{\pm}(\mathbb{R}^2)$ with finite energy data $u_0 \in H^1$.
Classical H^1-GWP Scheme relies on three inputs:

1. LWP lifetime dependence on data norm: $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.
2. Energy controls data norm: $\|u(t)\|_{H^1}^2 \lesssim H[u(t)] + \|u(t)\|_{L^2}^2$.
3. Conservation: $H[u(t)] + \|u(t)\|_{L^2}^2 \leq C(Energy, Mass)$.

Fix arbitrary time interval $[0, T]$. Break $[0, T]$ into subintervals of uniform size $c(Energy, Mass)$ + LWP iteration $\implies$ GWP.

For $u_0 \in H^s$ with $0 < s < 1$, we may have infinite energy. Classical persistence of regularity from LWP/Duhamel only gives

$$\sup_{t \in [0, T_{lwp}]} \|u(t)\|_{H^s} \lesssim 2\|u_0\|_{H^s}$$

and LWP iteration fails due to (possible) doubling. [Bourgain98]

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Refinements of Strichartz' Inequality and Applications to 2D-NLS with Critical Nonlinearity

J. Bourgain

Summary

Consider the 2D IVP

$$\begin{cases} iu_t + \Delta u + \lambda |u|^2 u = 0 \\ u(0) = \varphi \in L^2(\mathbb{R}^2). \end{cases} \tag{†}$$

The theory on the Cauchy problem asserts a unique maximal solution

$$u \in \mathcal{C}(]-T_*, T^*[, L^2(\mathbb{R}^2)) \cap L^4(]-T_*, T^*[; L^4(\mathbb{R}^2)),$$

where $T_*, T^* > 0$. Assume, for instance, $T^* < \infty$. It is shown then that

2. Abstract I-method Scheme for H^s-GWP

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Let $H^s \ni u_0 \longmapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A smoothing operator $I = I_N : H^s \longmapsto H^1$. The NLS evolution $u_0 \longmapsto u$ induces a smooth reference evolution $H^1 \ni Iu_0 \longmapsto Iu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.
- A modified energy $\widetilde{E}[Iu]$ built using the reference evolution.

We postpone how we actually choose these objects.

First Version of the I-method: $\widetilde{E} = H[Iu]$

For $s < 1, N \gg 1$ define smooth monotone $m : \mathbb{R}^2_\xi \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N. \end{cases}$$

The associated Fourier multiplier operator, $\widehat{(Iu)}(\xi) = m(\xi)\widehat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Set $\widetilde{E}[Iu(t)] = H[Iu(t)]$. Other choices of $\widetilde{E}$ are considered later.

AC Law Decay and Sobolev GWP index

1. **Modified LWP.** Initial v_0 s.t. $\|\nabla I v_0\|_{L^2} \sim 1$ has $T_{lwp} \sim 1$.
2. **Goal.** $\forall\ u_0 \in H^s$, $\forall\ T > 0$, construct $u : [0, T] \times \mathbb{R}^2 \to \mathbb{C}$.
3. $\iff$ **Dilated Goal.** Construct $u^\lambda : [0, \lambda^2 T] \times \mathbb{R}^2 \to \mathbb{C}$.
4. **Rescale Data.** $\|I\nabla u_0^\lambda\|_{L^2} \lesssim N^{1-s}\lambda^{-s}\|u_0\|_{H^s} \sim 1$ provided we choose $\lambda = \lambda(N) \sim N^{\frac{1-s}{s}} \iff N^{1-s}\lambda^{-s} \sim 1$.
5. **Almost Conservation Law.** $\|I\nabla u(t)\|_{L^2} \lesssim H[Iu(t)]$ and
$$\sup_{t\in[0,T_{lwp}]} H[Iu(t)] \leq H[Iu(0)] + N^{-\alpha}.$$
6. **Delay of Data Doubling.** Iterate modified LWP N^α steps with $T_{lwp} \sim 1$. We obtain rescaled solution for $t \in [0, N^\alpha]$.
$$\lambda^2(N)T < N^\alpha \iff T < N^{\alpha + \frac{2(s-1)}{s}} \text{ so } s > \frac{2}{2+\alpha} \text{ suffices.}$$

First Version of the I-method: $\widetilde{E} = H[Iu]$

A Fourier analysis established the almost conservation property of $\widetilde{E} = H[Iu]$ with $\alpha = \frac{3}{2}$ which led to...

Theorem (CKSTT 02)

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$.

- Same result for $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Here Q is the ground state (unique positive solution of $-Q + \Delta Q = -Q^3$).
- Fourier analysis leading to $\alpha = \frac{3}{2}$ in fact gives $\alpha = 2$ for most frequency interactions.

Almost Conservation Law for $H[Iu]$

Proposition

Given $s > \frac{4}{7}$, $N \gg 1$, and initial data $\phi_0 \in C_0^\infty(\mathbb{R}^2)$ with $E(I_N u_0) \leq 1$, then there exists a $T_{lwp} \sim 1$ so that the solution

$$u(t,x) \in C([0, T_{lwp}], H^s(\mathbb{R}^2))$$

of $NLS_3^+(\mathbb{R}^2)$ satisfies

$$E(I_N u)(t) = E(I_N u)(0) + O(N^{-\frac{3}{2}+}),$$

for all $t \in [0, T_{lwp}]$.

Ideas in the Proof of Almost Conservation

- Standard Energy Conservation Calculation:

$$\partial_t H(u) = \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u)dx$$
$$= \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u - iu_t)dx = 0.$$

- For the smoothed reference evolution, we imitate.... $\boxed{i}$

$$\partial_t H(Iu) = \Re \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - \Delta Iu - Iu_t)dx$$
$$= \Re \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - I(|u|^2 u))dx \neq 0.$$

- The increment in modified energy involves a commutator,

$$H(Iu)(t) - H(Iu)(0) = \Re \int_0^t \int_{\mathbb{R}^2} \overline{Iu_t}(|Iu|^2 Iu - I(|u|^2 u))dxdt.$$

- Littlewood-Paley, Case-by-Case, (Bi)linear Strichartz, $X_{s,b}$....

Remarks

- The almost conservation property

$$\sup_{t\in[0,T_{lwp}]} \widetilde{E}[Iu(t)] \leq \widetilde{E}[Iu_0] + N^{-\alpha}$$

  leads to GWP for

$$\boxed{s > s_\alpha = \frac{2}{2+\alpha}.}$$

- The I-method is a *subcritical method*. To prove the Scattering Conjecture at $s = 0$ via the I-method would require $\alpha = +\infty$.
- The I-method *localizes the conserved density in frequency*. Similar ideas appear in recent critical scattering results.
- There is a *multilinear corrections algorithm* for defining new choices of $\widetilde{E}$ which should have a better AC property.

Focusing Case Below the Ground State Mass

- Modified LWP lifetime is controlled by $\|I\nabla u_0\|_{L^2}$.
- The GWP scheme progresses if $\|I\nabla u(t)\|_{L^2}^2 \lesssim H[Iu(t)]$.
- Weinstein's optimal Gagliardo-Nirenberg Inequality:

$$\|w\|_{L^4}^4 \leq \frac{2}{\|Q\|_{L^2}^2}\|w\|_{L^2}^2\|\nabla w\|_{L^2}^2.$$

- I has symbol m satisfying $|m| \leq 1$ so $\|If\|_{L^2} \leq \|f\|_{L^2}$. Thus,

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies \|Iu_0\|_{L^2} < \|Q\|_{L^2}.$$

- The required control then follows:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies \|I\nabla u(t)\|_{L^2}^2 \lesssim H[Iu(t)].$$

3. Multilinear Correction Terms

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(Inspired by [Coifman-Meyer]; following [CKSTT:KdV])

1. For $k \in \mathbb{N}$, define the *convolution hypersurface*

$$\Sigma_k := \{(\xi_1, \ldots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \ldots + \xi_k = 0\} \subset (\mathbb{R}^2)^k.$$

2. For $M : \Sigma_k \to \mathbb{C}$ and $u_1, \ldots, u_k$ nice, define *k-linear functional*

$$\Lambda_k(M; u_1, \ldots, u_k) := c_k \, \Re \int_{\Sigma_k} M(\xi_1, \ldots, \xi_k) \widehat{u_1}(\xi_1) \ldots \widehat{u_k}(\xi_k).$$

3. For $k \in 2\mathbb{N}$ abbreviate $\Lambda_k(M; u) = \Lambda_k(M; u, \overline{u}, \ldots, \overline{u})$.
4. $\Lambda_k(M; u)$ invariant under interchange of even/odd arguments,

$$M(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) \mapsto \overline{M}(\xi_2, \xi_1, \ldots, \xi_k, \xi_{k-1}).$$

5. We can define a symmetrization rule via group orbit.

Examples

- $$\int_x u\bar{u}u\bar{u}dx = \int \Big(\int e^{ix\cdot\xi_1}\widehat{u}(\xi_1)d\xi_1\Big)\dots\Big(\int e^{ix\cdot\xi_4}\widehat{\overline{u}}(\xi_4)d\xi_4\Big)dx$$

  $$= \int_{\xi_1,\dots,\xi_4} \left[\int_x e^{ix\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}dx\right] \widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4)d\xi_{1,\dots,4}$$

  $$= \int_{\Sigma_4} \widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4) = \Lambda_4(1;u).$$

- $$\Lambda_2(-\xi_1\cdot\xi_2;u) = \|\nabla u\|_{L^2}^2.$$

Thus, $H[u] = \Lambda_2(-\xi_1\cdot\xi_2;u) \pm \Lambda_4(\frac{1}{2};u)$.

Time Dependence of Multilinear Forms

Suppose u nicely solves $NLS_3^+(\mathbb{R}^2)$; M is time independent, symmetric. Calculations produce the *time differentiation formula*

$$\begin{aligned}\partial_t \Lambda_k(M; u(t)) &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}(ikX(M); u(t)) \\ &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}([ikX(M)]_{sym}; u(t)).\end{aligned}$$

Here

$$\alpha_k(\xi_1, \dots, \xi_k) := -|\xi_1|^2 + |\xi_2|^2 - \dots - |\xi_{k-1}|^2 + |\xi_k|^2$$

(so $\alpha_2 = 0$ on Σ_2) and

$$X(M)(\xi_1, \dots, \xi_{k+2}) := M(\xi_{123}, \xi_4, \dots, \xi_{k+2}).$$

We use the notation $\xi_{ab} := \xi_a + \xi_b$, $\xi_{abc} := \xi_a + \xi_b + \xi_c$, etc.

Exercise: Conservation of Energy

Verify, using the time differentiation formula,

$$\begin{aligned}\partial_t \Lambda_k(M; u(t)) &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}(ikX(M); u(t)) \\ &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}([ikX(M)]_{sym}; u(t)).\end{aligned}$$

that the Hamiltonian

$$H[u] = \Lambda_2(-\xi_1 \cdot \xi_2; u) \pm \Lambda_4(\frac{1}{2}; u)$$

is conserved.

AC Quantities via Multilinear Corrections

- Abbreviate $m(\xi_j)$ as m_j. Define σ_2 s.t. $\|I\nabla u\|_{L^2}^2 = \Lambda_2(\sigma_2; u)$:

$$\sigma_2(\xi_1, \xi_2) := -\frac{1}{2}\xi_1 m_1 \cdot \xi_2 m_2 = \frac{1}{2}|\xi_1|^2 m_1^2$$

- With $\tilde{\sigma}_4$ (symmetric, time independent) to be determined, set

$$\widetilde{E} := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u).$$

- Using the time differentiation formula, we calculate

$$\partial_t \widetilde{E} = \Lambda_4(\{i\tilde{\sigma}_4\alpha_4 - i2[X(\sigma_2)]_{sym}\}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}; u).$$

We'd like to $\boxed{\text{define } \tilde{\sigma}_4}$ to cancel away the Λ_4 contribution.

Small Divisor Problem

Resonant interactions obstruct the natural choice:

$$\tilde{\sigma}_4 \stackrel{?}{=} \frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4}.$$

impossible!

On Σ_4, we can reexpress $\alpha_4 = -|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 + |\xi_4|^2$ as

$$\alpha_4 = -2\xi_{12} \cdot \xi_{14} = -2|\xi_{12}||\xi_{14}| \cos \angle(\xi_{12}, \xi_{14}),$$

and

$$[2iX(\sigma_2)]_{sym} = \frac{1}{4}(-m_1^2|\xi_1|^2 + m_2^2|\xi_2|^2 - m_3^2|\xi_3|^2 + m_4^2|\xi_4|^2).$$

When all the $m_j = 1$ (so $\max_j |\xi_j| < N$), $\tilde{\sigma}_4$ is well-defined. However, α_4 can also vanish when ξ_{12} and ξ_{14} are orthogonal.

Remark: Integrable Systems Connection?

For $NLS_3^+(\mathbb{R})$, the resonant obstruction disappears. Thus,

$$\widetilde{E}^1 = \Lambda_2(\sigma_2) + \Lambda_4(\tilde{\sigma}_4);$$

$$\partial_t \widetilde{E}^1 = -\Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}).$$

We can then define, with $\tilde{\sigma}_6$ to be determined,

$$\widetilde{E}^2 = \widetilde{E}^1 + \Lambda_6(\tilde{\sigma}_6);$$

$$\partial_t \widetilde{E}^2 = \Lambda_6(\{i\tilde{\sigma}_6\alpha_6 - [i4X(\tilde{\sigma}_4)]_{sym}\}) + \Lambda_8([i6X(\tilde{\sigma}_6)]_{sym}).$$

Let's define

$$\tilde{\sigma}_6 = \frac{[i4X(\tilde{\sigma}_4)]_{sym}}{i\alpha_6}.$$

Remark: Integrable Systems Connection?

Thus, we formally obtain a continued-fraction-like algorithm.

$$\tilde{\sigma}_6 = \frac{\left[i4X\left(\frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4} \right) \right]_{sym}}{i\alpha_6},$$

$$\tilde{\sigma}_8 = \frac{\left[i6X\left(\frac{\left[i4X\left(\frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4} \right) \right]_{sym}}{i\alpha_6} \right) \right]_{sym}}{i\alpha_8}, \ldots.$$

Each step gains two derivatives but costs two more factors.
This is a big gain!

Conjecture: The multipliers $\tilde{\sigma}_6, \tilde{\sigma}_8, \ldots$ are well defined and lead to better AC properties. Same for other integrable systems?

4. Resonant Decomposition

4. Resonant Decomposition

We return to $NLS_3^+(\mathbb{R}^2)$.
Since the natural choice is not well-defined, we choose

$$\tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4}\ \chi_{\Omega_{nr}}$$

where the non-resonant set $\Omega_{nr} \subset \Sigma_4$ such that

$$\Omega_{nr} := \{\max_{1\leq j\leq 4} |\xi_j| \leq N\} \cup \{|\cos\angle(\xi_{12}, \xi_{14})| \geq \theta_0\}.$$

Eventually, we choose θ_0 to balance the 4-linear and 6-linear contributions to the modified energy increment. We have

$$\partial_t \widetilde{E} = \Lambda_4(\{i\tilde{\sigma}_4\alpha_4 - i2[X(\sigma_2)]_{sym}\}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}; u).$$

The 4-linear contribution is constrained to the resonant set Ω_{nr}^C.

Improved Almost Conservation Property

Lemma

If $\|u_0\|_{L^2_x(\mathbb{R}^2)} \leq A$*;* $E(Iu_0) \leq 1$*;* u *is a nice solution of* $NLS_3^+(\mathbb{R}^2)$ *on a time interval* $[0, t_0]$*, then if* $t_0 = t_0(A)$ *is small enough,*

$$\left|\int_0^{t_0} \Lambda_4([-2iX(\sigma_2)]_{sym} + i\tilde{\sigma}_4\alpha_4; u(t))\ dt\right| + \left|\int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{sym}; u(t))\ dt\right| \lesssim C(A)[N^{-2+} + \theta_0^{1/2} N^{-3/2+} + \theta_0^{-1} N^{-3+}].$$

The choice $\theta_0 = N^{-1}$ produces the AC property with $\alpha = 2.$

N^{-2} AC law!

Overview and Delicate Case of Proof

- The 4-linear contribution has multiplier

$$([-2iX(\sigma_2)]_{sym} + i\tilde{\sigma}_4\alpha_4)(\xi) = [-2iX(\sigma_2)]_{sym}\chi_{\Omega_r}$$

where the *resonant set* $\Omega_r = \Omega_{nr}^C \subset \Sigma_4$,

$$\Omega_r := \{\max(|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|) > N; |\cos\angle(\xi_{12}, \xi_{14})| < \theta_0\}.$$

- We wish to bound the associated energy incremement

$$\int_0^{T_{lwp}} \Lambda_4([-2iX(\sigma_2)]_{sym}\chi_{\Omega_r}; u)dt.$$

- The 4 factors u are dyadically decomposed. The integral is studied case-by-case based on dyadic frequency sizes.
- On Σ_4, the two largest frequencies are comparable.

Overview and Delicate Case of Proof

- Let $|\xi_j| \sim N_j \in 2^{\mathbb{Z}}$. Symmetry properties and the Ω_r constraint allow to assume

$$N_1 \sim N_2 \gtrsim N, N_2 \gtrsim N_3 \gtrsim N_4 \gtrsim 1.$$

- For most cases, suffices to use (enhanced) [CKSTT 02] and

Lemma

$\forall\ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4,$

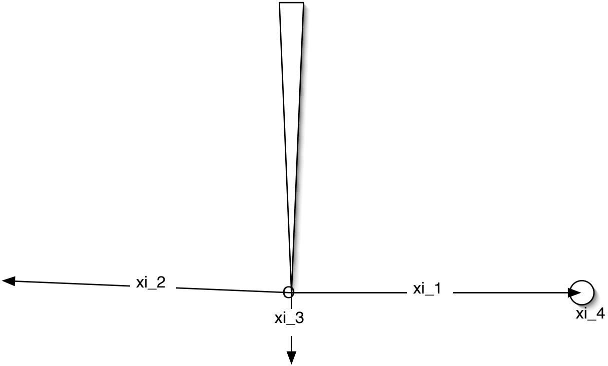
$$|[2iX(\sigma_2)]_{sym}| \lesssim \min(m_1, m_2, m_3, m_4)^2 |\xi_{12}||\xi_{14}|.$$

This follows from the mean value theorem.

Overview and Delicate Case of Proof

The most delicate case occurs in Ω_r and when

$$N_1 \sim N_2 \gg N, N_3 \gg N_4 \gtrsim 1.$$



A REMARK ON NORMAL FORMS AND THE "*I*-METHOD" FOR PERIODIC NLS

By

JEAN BOURGAIN

0 Introduction

In this paper, we explore the combination of two ideas in establishing global wellposedness results (GWP) for Hamiltonian PDE's with rough data. The first is a "normal forms" reduction by symplectic transformations that in some sense reduces the nonlinearity to its "essential" part. This construction was exploited in [Bo1] in the slightly different context of estimating the growth in time of higher Sobolev norms of smooth solutions. The second idea is the so-called "*I*-method," different from) that to establish LWP results in $H^s(\mathbb{T})$, $s > 0$ (the key ingredient is again Strichartz' inequality). For self containedness sake, this analysis is repeated here in §7 (Appendix).

In principle, once the appropriate normal form is obtained, the estimates are direct and purely spatial. The main point is that now the remaining monomials in the nonlinearity satisfy certain specific frequency configurations, which allow us to improve on the I-method approach. This is exactly how we proceed to establish GWP in $H^s, s > \frac{1}{2}$. To go beyond this, we rely on a refined bilinear (or trilinear) Strichartz inequality with different frequency ranges for the factors; see §4. We only prove a qualitative statement, and it would be worth extracting a precise formulation. The actual proof of GWP for $s < \frac{1}{2}$ proceeds in two stages. We first carry out the preceding analysis, upgraded with the improved Strichartz inequality, for a truncated equation $|n| \leq N_1$ (§5). The full equation is then treated perturbatively in §6 along a scheme similar to that in [Bo4] but replacing the usual Hamiltonian by the almost conserved quantities studied in §5. Possibly, this part of the analysis could be avoided by modifying the framework of §5 to avoid the need of restricting Fourier modes. Rather than trying this, we return in §6 to space-time estimates, following a scheme that, while perhaps not technically obvious, is conceptually rather familiar (cf. [Bo4], [St]).

Overview and Delicate Case of Proof

Angle constraint in Ω_r gives better estimates based on two effects:

- Cancellation with $[X(\sigma_2)]_{sym}$,
- Angular refinement of Bilinear Strichartz.

We use a refinement exploiting spherical symmetry of m.

Lemma

Let $N_1, \ldots, N_4$ be in the delicate case with $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r$. Then

$$|[X(\sigma_2)]_{sym}| \lesssim m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2.$$

Angular Refinement of Bilinear Strichartz

Lemma (Angle Refined Bilinear Strichartz)

Let $0 < N_1 \leq N_2$ *and* $0 < \theta < \frac{1}{50}$. *Then for any* $v_1, v_2 \in X^{0,1/2+}$ *with spatial frequencies* N_1, N_2 *respectively, the spacetime function*

$$F(t,x) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(t(\tau_1+\tau_2)+x\cdot(\xi_1+\xi_2))} \times \chi_{\{|\cos\angle(\xi_1,\xi_2)|\leq\theta\}} \tilde{v}_1(\tau_1,\xi_1)\tilde{v}_2(\tau_2,\xi_2)\, d\xi_1 d\xi_2$$

obeys the bound

$$\|F\|_{L^2_{t,x}} \lesssim \theta^{1/2}\|v_1\|_{X^{0,1/2+}}\|v_2\|_{X^{0,1/2+}}.$$

as in P. Gérard's Lectures