Maximal-in-time issues for nonlinear Schrödinger equations

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1 NLS Cauchy Problem

2 Outline of Lectures

3 The $I$-method

4 Low Regularity Theory for Focusing NLS

5 Interaction Morawetz

6 A Cascading Solution to $\text{NLS}_3^+ (\mathbb{T}^2)$. 
1. NLS Cauchy Problem
We consider the initial value problem:

\[
\begin{cases}
(i \partial_t + \Delta) u = \pm |u|^{p-1} u \\
u(0, x) = u_0(x).
\end{cases}
\]  
\((NLS_p^\pm(\mathbb{R}^d))\)

The + case is called defocusing; − is focusing.

- \(NLS_3^\pm\) is ubiquitous in physics. \(NLS_p^\pm\) introduced to explore interplay between dispersion and strength of nonlinearity.
- The **main question** about an evolution PDE: What is the ultimate fate of solutions? We want to understand the **maximal-in-time behavior** of the solutions.
- Conservation and invariance properties motivate the study of \(NLS_p^\pm(\mathbb{R}^d)\) for low (and minimal) regularity initial data.
Time Invariant Quantities

\[
\text{Mass} = \int_{\mathbb{R}^d} |u(t, x)|^2 dx.
\]
\[
\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.
\]
\[
\text{Energy} = H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx \pm \frac{2}{p + 1} |u(t)|^{p+1} dx.
\]

- Mass is \(L^2\); Momentum is close to \(H^{1/2}\); Energy involves \(H^1\).
- Dynamics on a sphere in \(L^2\); focusing/defocusing energy.
- Local conservation laws express how quantity is conserved: e.g., \(\partial_t |u|^2 = \nabla \cdot 2\Im (\overline{u} \nabla u)\). Space/Frequency Localizations?
Dilation Invariance and Critical Regularity

One solution \( u \) generates parametrized family \( \{ u^\lambda \}_{\lambda > 0} \) of solutions:

\[
u : [0, T) \times \mathbb{R}^d_x \rightarrow \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)\]

\[
\updownarrow
\]

\[
u^\lambda : [0, \lambda^2 T) \times \mathbb{R}^d_x \rightarrow \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)
\]

where

\[
u^\lambda(\tau, y) = \lambda^{-2/(p-1)} u(\lambda^{-2} \tau, \lambda^{-1} y).
\]

Norms which are invariant under \( u \mapsto u^\lambda \) are critical.
Dilation Invariance and Critical Regularity

In the $L^2$-based Sobolev scale,

$$\| D^s u^\lambda(t) \|_{L^2} = \lambda^{-\frac{2}{p-1} - s + \frac{d}{2}} \| D^s u(t) \|_{L^2}. $$

The critical Sobolev index for $NLS_p^\pm(\mathbb{R}^d)$ is

$$s_c := \frac{d}{2} - \frac{2}{p-1}. $$

Scaling/Conservation Criticality

<table>
<thead>
<tr>
<th>scaling</th>
<th>regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_c &lt; 0$</td>
<td>mass subcritical</td>
</tr>
<tr>
<td>$s = 0$</td>
<td>mass critical</td>
</tr>
<tr>
<td>$0 &lt; s_c &lt; 1$</td>
<td>mass super/energy subcritical</td>
</tr>
<tr>
<td>$s_c = 1$</td>
<td>energy critical</td>
</tr>
<tr>
<td>$1 &lt; s_c &lt; d/2$</td>
<td>energy supercritical</td>
</tr>
</tbody>
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Local-in-time theory for $NLS^\pm_p(\mathbb{R}^d)$ is essentially complete:

- Pioneering advances on spacetime dispersive estimates culminated in [Cazenave-Weissler 90] to prove local well-posedness for $s \geq s_{lwp} = \max(0, s_c)$. (discussed in more detail for $NLS_3(\mathbb{R}^2)$ soon.)

- Ill-posedness results for $s < s_{lwp}$ have been established. [Kenig-Ponce-Vega 01], [Christ-C-Tao 03], [Lebeau 01 05], [Burq-Gérard-Ibrahim], [Alazard-Carles 07].

- When $s_c < 0$, the Galilean symmetry obstructs well-posedness below $s = 0$. 
Local-in-time theory for \( NLS_3(\mathbb{R}^2) \)

We pause to discuss the \( L^2(\mathbb{R}^2) \)-critical case.

- \( \forall \ u_0 \in L^2(\mathbb{R}^2) \ \exists \ T_{lwp}(u_0) \) determined by
  \[
  \| e^{it\Delta} u_0 \|_{L^4_t([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100} \text{ such that}
  \]
  \[
  \exists \text{ unique } u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2) \text{ solving } NLS_3^+(\mathbb{R}^2).
  \]

- \( \forall \ u_0 \in H^s(\mathbb{R}^2), s > 0, T_{lwp} \sim \| u_0 \|_{H^s}^{-\frac{2}{s}} \) and regularity persists: \( u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2)) \).

- Define the maximal forward existence time \( T^*(u_0) \) by
  \[
  \| u \|_{L^4_t([0, T^* - \delta] \times \mathbb{R}^2)} < \infty
  \]
  for all \( \delta > 0 \) but diverges to \( \infty \) as \( \delta \downarrow 0 \).

- \( \exists \) small data scattering threshold \( \mu_0 > 0 \)
  \[
  \| u_0 \|_{L^2} < \mu_0 \implies \| u \|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.
  \]
**Global-in-time theory?**

What is the **ultimate fate** of the local-in-time solutions?

**$L^2$-critical Defocusing Scattering Conjecture:**

$L^2 \ni u_0 \mapsto u$ solving $NLS_3^+ (\mathbb{R}^2)$ is global-in-time and

$$\| u \|_{L_{t,x}^4} < A(u_0) < \infty.$$  

Moreover, $\exists \ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \pm \infty} \| e^{\pm it \Delta} u_\pm - u(t) \|_{L^2(\mathbb{R}^2)} = 0.$$  

**Remarks:**

- Known for small data $\| u_0 \|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known [Tao-Visan-Zhang 06] for $NLS_{1+\frac{4}{d}}^+ (\mathbb{R}^d)$ for large radial data, $d \geq 3$. Same for $d = 2$ [Killip-Tao-Visan 07].
- GWP for $L^2$ data $\iff$ Scattering for $L^2$ data. [Blue-C 06]
Consider defocusing case $NLS_p^+(\mathbb{R}^d)$ with critical Sobolev index

$$s_c = \frac{d}{2} - \frac{2}{p-1}. $$

The critical (diagonal) Strichartz index is

$$q_c = \frac{(p-1)(2+d)}{2} \iff \frac{2}{q_c} + \frac{d}{q_c} = \frac{d}{2} - s_c.$$ 

**$H^{s_c}$-critical defocusing scattering conjecture:**

$H^{s_c}(\mathbb{R}^d) \ni u_0 \mapsto u$ solving $NLS_p^+(\mathbb{R}^d)$ is global-in-time and

$$\|u\|_{L_{t,x}^{q_c}} < A(u_0) < \infty.$$
## Critical Regularity Scattering Conjecture?

### Present status of the defocusing scattering conjecture

<table>
<thead>
<tr>
<th>Criticality</th>
<th>General Data</th>
<th>Radial Data</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_c = 0$</td>
<td>$s_c &lt; s_* &lt; s &lt; 1$</td>
<td>[TVZ], [KTV]</td>
<td>GWP: $s_* &lt; s &lt; 1$</td>
</tr>
<tr>
<td>$0 &lt; s_c &lt; 1$</td>
<td>$s = s_c$</td>
<td>[CKSTT], [RV], [V]</td>
<td>$s = s_c$&lt;br&gt;$s = s_c$?</td>
</tr>
<tr>
<td>$s_c = 1$</td>
<td>$s = s_c$</td>
<td>[B99], [T]</td>
<td>$s = s_c$?</td>
</tr>
<tr>
<td>$1 &lt; s_c &lt; \frac{d}{2}$</td>
<td>$|u(t)|_{H^{s_c}} &lt; C$</td>
<td>$s = s_c$?</td>
<td>$s = s_c$?</td>
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</table>

- Scattering for $NLS_p^-$ under natural threshold.
- The existence (and value) of $s_*$ depends upon $p, d$.
- Radial case with $s_c = \frac{1}{2}$ may be accessible using Morawetz??
- Induction-on-Mass + radial results $\rightarrow s_c = 0$ accessible???
2. Outline of Lectures
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I Introduction: Outline of Course.

II $I$-method for Global Well-Posedness Below Energy.
   1 Abstract Scheme
   2 Almost Conservation of $H[Iu]$
   3 Multilinear Correction Terms
   4 Resonant Decompositions

III Low Regularity Theory for Focusing NLS.
   1 $I$-method for focusing $NLS^-$ below ground state mass
   2 Mass Concentration Properties of $H^s$ Blowup Solutions
   3 Mass Concentration Properties of $L^2$ Blowup Solutions

IV The $I$-method with a Morawetz Bootstrap.
   1 Interaction Morawetz Estimates
   2 $H[Iu] +$ Morawetz GWP & Scattering Results

V A frequency cascading solution to $NLS^+_3(\mathbb{T}^2)$.

"Weak Turbulence"
3. $H^1$ versus $H^s$ Global Well-Posedness
Consider $\mathcal{NLS}_3^\pm(\mathbb{R}^2)$ with finite energy data $u_0 \in H^1$. Classical $H^1$-GWP Scheme relies on three inputs:

1. **LWP lifetime dependence** on data norm: $T_{lwp} \sim \| u_0 \|_{H^s}^{-2/s}$.
2. **Energy controls data norm**: $\| u(t) \|_{H^1}^2 \lesssim H[u(t)] + \| u(t) \|_{L^2}^2$.
3. **Conservation**: $H[u(t)] + \| u(t) \|_{L^2}^2 \leq C(Energy, Mass)$.

Fix arbitrary time interval $[0, T]$. Break $[0, T]$ into subintervals of uniform size $c(\text{Energy, Mass}) + \text{LWP iteration} \implies \text{GWP}$.

For $u_0 \in H^s$ with $0 < s < 1$, we may have infinite energy. Classical persistence of regularity from LWP/Duhamel only gives

$$\sup_{t \in [0, T_{lwp}]} \| u(t) \|_{H^s} \lesssim 2 \| u_0 \|_{H^s}$$

and LWP iteration fails due to (possible) doubling.
Let $H^s \ni u_0 \mapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A **smoothing operator** $I = I_N : H^s \mapsto H^1$. The NLS evolution $u_0 \mapsto u$ induces a **smooth reference evolution** $H^1 \ni lu_0 \mapsto lu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.

- A **modified energy** $\tilde{E}[lu]$ built using the reference evolution.

We postpone how we actually choose these objects.
We want $I_N$ and $\tilde{E}$ chosen to give a progressive $H^s$-GWP scheme:

1. **Lifetime dependence on data norm:** $T_{lwp} \sim \|u_0\|^{-2/s}_{H^s}$. ✓
2. **$\tilde{E}$ controls data norm:** $\exists t_g \in \left[\frac{1}{2} T_{lwp}, T_{lwp}\right]$ s.t.
   $$\|u(t_g)\|^2_{H^s} \lesssim \tilde{E}[lu(t_g)] + \|u(t_g)\|^2_{L^2}.$$
3. **Almost Conservation of Modified Energy:**
   $$\sup_{t \in [0, T_{lwp}]} \tilde{E}[lu(t)] \leq \tilde{E}[lu_0] + N^{-\alpha}.$$

The scheme advances over $K$ uniform sized time steps of length $O(\tilde{E}[u_0]^{-1/s})$ until the modified energy doubles

$$KN^{-\alpha} \sim \tilde{E}[lu_0].$$

This extends to solution for $t \in [0, N^\alpha \tilde{E}[lu_0]^{-\frac{1}{s}}]$ which contains $[0, T]$ for large enough $N$ provided $s > s_\alpha$ with $s_\alpha < 1$. 
First Version of the $I$-method: $\tilde{E} = H[lu]$

For $s < 1$, $N \gg 1$ define smooth monotone $m : \mathbb{R}^2_\xi \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \nonumber \\ \left( \frac{|\xi|}{N} \right)^{s-1} & \text{for } |\xi| > 2N. \nonumber \end{cases}$$

The associated Fourier multiplier operator, $(\hat{lu})(\xi) = m(\xi) \hat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|lu\|_{H^1} \lesssim N^{1-s} \|u\|_{H^s}. \nonumber$$

Set $\tilde{E}[lu(t)] = H[lu(t)]$. A detailed multilinear Fourier analysis establishes that $H[lu]$ is almost conserved with $\alpha = \frac{3}{2}$ for $NLS^\pm_3(\mathbb{R}^2)$ and with $\alpha = 1$ for $NLS^\pm_3(\mathbb{R}^3)$. After some bookkeeping....
First Version of the I-method: $\tilde{E} = H[lu]$

**Theorem (CKSTT:MRL02)**

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

$NLS_3^+(\mathbb{R}^3)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{5}{6} < s < 1$.

Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$ (both cases).

The same result applies for $NLS_3^-(\mathbb{R}^2)$ provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$ where $Q$ is the ground state, the unique (up to translations) positive solution of $-Q + \Delta Q = Q^3$. 
**L^2-critical in Weighted L^2 spaces**

Based on PC transformation & inspired by [Bourgain98], we have:

**Theorem (Blue-C 06)**

For \( s \geq 0 \), if \( NLS^{+}_{1+\frac{4}{d}}(\mathbb{R}^d) \) is GWP for \( H^s(\mathbb{R}^d) \) initial data then \( NLS^{+}_{1+\frac{4}{d}}(\mathbb{R}^d) \) is GWP and scatters for data satisfying \( \langle \cdot \rangle^s u_0(\cdot) \in L^2 \). The same result applies to the focusing case provided \( \|u_0\|_{L^2} < \|Q\|_{L^2} \).

- Thus, GWP for \( L^2 \) data \( \iff \) Scattering for \( L^2 \) data.
- \( H^s \)-GWP improvements imply weighted space improvements.
- PC transformation isometry in \( L^2 \)-admissible Strichartz spaces.
### $NLS^\pm_3(\mathbb{R}^2)$: Present Status for General Data

<table>
<thead>
<tr>
<th>regularity</th>
<th>idea</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s &gt; \frac{2}{3}$</td>
<td>high/low frequency decomposition $H(Iu)$</td>
<td>[Bourgain98]</td>
</tr>
<tr>
<td>$s &gt; \frac{4}{7}$</td>
<td>resonant cut of 2nd energy $H(Iu)$</td>
<td>[CKSTT02]</td>
</tr>
<tr>
<td>$s &gt; \frac{1}{2}$</td>
<td>$H(Iu)$ &amp; Interaction Morawetz</td>
<td>[CKSTT07]</td>
</tr>
<tr>
<td>$s &gt; \frac{2}{5}$</td>
<td>$H(Iu)$ &amp; Interaction $I$-Morawetz</td>
<td>[Fang-Grillakis05]</td>
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<tr>
<td>$s &gt; \frac{1}{3}$</td>
<td>resonant cut &amp; $I$-Morawetz</td>
<td>[CGTz07]</td>
</tr>
<tr>
<td>$s &gt; 0?$</td>
<td></td>
<td>[C-Roy08]</td>
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</tbody>
</table>

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?
4. Low Regularity Theory for Focusing $\textit{NLS}$
Remark:

- The $H^1$-GWP scheme is relaxed to an $H^s$-GWP scheme by replacing the energy $H[u]$ by the modified energy $\tilde{E}[lu]$.
- The energy plays a basic role in other aspects of the NLS theory (e.g. soliton stability, properties of blowup).
- **Natural idea:** Explore whether existing $H^1$ theory may be systematically relaxed to $H^s$ by replacing $H[u]$ by $\tilde{E}[lu]$.
Explicit Blowup Solutions

- Arise as pseudoconformal image of $e^{it} Q(x)$:

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + i \frac{i}{t}}.$$

- $S$ has minimal mass:

$$\| S(-1) \|_{L^2_x} = \| Q \|_{L^2}.$$

All mass in $S$ is conically concentrated into a point.

- Minimal mass $H^1$ blowup solution characterization:

$u_0 \in H^1, \| u_0 \|_{L^2} = \| Q \|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]
Virial Identity $\implies \exists$ Many Blowup Solutions

Integration by parts and the equation yields

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 \, dx = 8H[u_0].$$

- $H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 \, dx < \infty$ blows up.
- How do these solutions blow up?
\textbf{\(L^2\) Critical Case: Mass Concentration}

\textbf{\(H^1\) Theory of Mass Concentration}

- \(H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty\) implies

\[
\liminf_{t \to T^*} \int_{|x| < (T^* - t)^{1/2}} |u(t, x)|^2 dx \geq \|Q\|_{L^2}^2.
\]

[Merle-Tsutsumi]

- \(H^1\) blowups \textit{parabolically} concentrate at least the ground state mass. Explicit blowups \(S\) concentrate mass much faster.

- Fantastic recent progress on the \(H^1\) blowup theory.

[Merle-Raphaël]
\( L^2 \) Critical Case: Mass Concentration

\( L^2 \) Theory of Mass Concentration

\( L^2 \ni u_0 \rightarrow u, T^* < \infty \) implies

\[
\limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.
\]

[Bourgain98]

\( L^2 \) blowups parabolically concentrate some mass.

Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].

For large \( L^2 \) data, do there exist tiny concentrations? ([TVZ], [KTV]: No, for radial data.)
Typical blowups leave an $L^2$ stain at time $T^*$

[Merle-Raphaël]:

$$H^1 \cap \{ \| Q \|_{L^2} < \| u_0 \|_{L^2} < \| Q \|_{L^2} + \alpha^* \} \ni u_0 \mapsto u$$

solving $\text{NLS}_3^-(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.

$\exists \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi \mathbb{Z})$ and $u^*$ such that

$$u(t) - \lambda(t)^{-1} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\theta(t)} \rightarrow u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!

\[\text{Low Regularity Necessary!}\]
Consider focusing $NLS_3^-(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass.** ([KTV]: ✓)
  \[ \| u_0 \|_{L^2} < \| Q \|_{L^2} \implies \text{??? } u_0 \mapsto u \text{ with } \| u \|_{L^4_{tx}} < \infty. \]

  (Also, $L^2$ solutions of $NLS_3^+(\mathbb{R}^2)$ satisfy ???. $\| u \|_{L^4_{tx}} < \infty$.)

- **Minimal Mass Blowup Characterization.**
  \[ \| u_0 \|_{L^2} = \| Q \|_{L^2}, \quad u_0 \mapsto u, \quad T^* < \infty \implies \text{??? } u = S, \]
  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$.

- **Concentrated mass amounts are quantized.**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.

- **Are there any general upper bounds?**
\[ L^2 \text{ Critical Case: Partial Results} \]

- For \( 0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1 \), \( H^s \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty \implies \)

\[
\limsup_{t \nearrow T^*} \int_{|x| < (T^*-t)^{s/2}} |u(t,x)|^2 \, dx \geq \|Q\|_{L^2}^2.
\]

\( H^s \)-blowup solutions concentrate ground state mass. [C-Raynor-C.Sulem-Wright]

- \( \|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies \exists t_n \nearrow T^* \text{ s.t. } u(t_n) \to Q \text{ in } H^{\tilde{s}(s)} \text{ (mod symmetry sequence).} \)

For \( H^s \) blowups with \( \|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \in H^1 \) (mod symmetry sequence). [Hmidi-Keraani] This is an \( H^s \) analog of an \( H^1 \) result of [Weinstein] which preceded the minimal \( H^1 \) blowup solution characterization.

- Same results for \( NLS^-_{\frac{4}{d+1}}(\mathbb{R}^d) \) in \( H^s, s > \frac{d+8}{d+10} \). [Visan-Zhang]
\textbf{L}^2 \textbf{C}ritical \textbf{C}ase: \textbf{P}artial \textbf{R}esults

\[ \| u \|_{L^4_t([0,t] \times \mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta} \]

is linked with mass concentration rate

\[ \limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1 + \frac{\beta}{2}}} \int_I |u(t, x)|^2 dx \geq \| u_0 \|_{L^2}^{-M}. \]

This work refines the proof in [Bourgain 98].
5. Interaction Morawetz: Local Conservation
5. Interaction Morawetz: Local Conservation

Suppose $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{C}$ solves generalized NLS

$$(i\partial_t + \Delta)\phi = \mathcal{N}$$

for some $\mathcal{N} = \mathcal{N}(t, x, u) : [0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$. Assume $\phi$ is nice.

We introduce notation to compactly express mass and momentum (non)conservation for solutions of generalized NLS.

Write $\partial_{x_j} \phi = \partial_j \phi = \phi_j$. 
Local mass/momentum (non)conservation

- mass density: $T_{00} = |\phi|^2$
- momentum density/mass current:
  $T_{0j} = T_{j0} = 2\Im(\overline{\phi}\phi_j)$
- (linear part of the) momentum current:
  $L_{jk} = L_{kj} = -\partial_j\partial_k|\phi|^2 + 4\Re(\overline{\phi_j}\phi_k)$
- mass bracket: $\{f, g\}_m = \Im(f\overline{g})$
- momentum bracket: $\{f, g\}_p^j = \Re(f\partial_j\overline{g} - g\partial_j\overline{f})$

Local mass (non)conservation identity:

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

Local momentum (non)conservation identity:

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j$$
Consider \( \mathcal{N} = F'(|\phi|^2)\phi \) for polynomial \( F : \mathbb{R}^+ \rightarrow \mathbb{R} \).

- We calculate the mass bracket

\[
\{ F'(|\phi|^2)\phi, \phi \}_m = \Im(F'(|\phi|^2)\phi\bar{\phi}) = 0.
\]

Thus mass is conserved for these nonlinearities.

- We calculate the momentum bracket

\[
\{ F'(|\phi|^2)\phi, \phi \}_p = -\partial_j G(|\phi|^2)
\]

where \( G(z) = zF'(z) - F(z) \sim F(z) \).

Thus the momentum bracket contributes a divergence and momentum is conserved for these nonlinearities.
**Generalized Virial Identity**

Suppose $a : \mathbb{R}^d \to \mathbb{R}$. Form the **Morawetz Action**

$$M_a(t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\Im(\bar{\phi}\nabla \phi) \, dx.$$  

Conservation identities lead to the **generalized virial identity**

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta \Delta a) |\phi|^2 + 4a_{jk} \Re(\bar{\phi}_j \phi_k) + 2a_j \{\mathcal{N}, \phi\}_p \, dx.$$  

**Idea of Morawetz Estimates:** Cleverly choose the weight function $a$ so that $\partial_t M_a \geq 0$ but $M_a \leq C(\phi_0)$ to obtain spacetime control on $\phi$. This strategy imposes various constraints on $a$ which suggest choosing $a(x) = |x|$. 
Consider \((i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi\) with \(F' \geq 0\) and \(x \in \mathbb{R}^3\). Choose \(a(x) = |x|\). Observe that \(a\) is weakly convex, \(\nabla a = \frac{x}{|x|}\) is bounded, and \(-\Delta \Delta a = 4\pi \delta_0\). One gets the \textbf{Lin-Strauss Morawetz identity}

\[
M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^3} 4\pi \delta_0(x)|\phi(t, x)|^2 + (\geq 0) + 4 \frac{G(|\phi|^2)}{|x|} \, dxdt
\]

which implies the spacetime control estimate

\[
(H[u_0])^{1/2} \| u_0 \|_{L^2} \gtrsim \int_0^T \int_{\mathbb{R}^3} \frac{G(|\phi|^2)}{|x|} \, dxdt.
\]
Tensor Product Idea

[CKSTT 04] (Hassell 04)

- Suppose $\phi_1, \phi_2$ are two solutions of $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$
  with $F' \geq 0$ and $x \in \mathbb{R}^3$. The “2-particle” wave function

$$\Psi(t, x_1, x_2) = \phi_1(t, x_1)\phi_2(t, x_2)$$

satisfies an NLS-type equation on $\mathbb{R}^{1+6}$

$$(i\partial_t + \Delta_1 + \Delta_2)\Psi = [F'(|\phi_1|^2) + F'(|\phi_2|^2)]\Psi.$$

- Note that $[F'(|\phi_1|^2) + F'(|\phi_2|^2)] \geq 0$ so defocusing.

- Reparametrize $\mathbb{R}^6$ using center-of-mass coordinates $(\bar{x}, y)$
  with $\bar{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3$. Note that $y = 0$ corresponds to
  the diagonal $x_1 = x_2 = \bar{x}$. Apply the generalized virial identity
  with the choice $a(x_1, x_2) = |y|$. Dismissing terms with
  favorable signs, one obtains...
**Example: \( L^4(\mathbb{R}_t \times \mathbb{R}^3) \) Interaction Morawetz**

\[
\| \nabla u \|_{L^\infty_{[0,T]} L^2_x} \| u_0 \|^3_{L^2_x} \geq \int_0^T \int_{\mathbb{R}^6} (-\Delta_6 \Delta_6 |y|) |\psi(x_1, x_2)|^2 \, dx_1 \, dx_2 \, dt \\
\geq c \int_0^T \int_{\mathbb{R}^6} \delta_{\{y=0\}}(x_1, x_2) |\phi_1(x_1) \phi_2(x_2)|^2 \, dx_1 \, dx_2 \, dt \\
\geq c \int_0^T \int_{\mathbb{R}^3} |\phi_1(t, \bar{x}) \phi_2(t, \bar{x})|^2 \, d\bar{x} \, dt.
\]

Specializing to \( \phi_1 = \phi_2 \) gives the **interaction Morawetz estimate**

\[
\int_0^T \int_{\mathbb{R}^3} |\phi(t, x)|^4 \, dx \, dt \leq C \| \nabla u \|_{L^\infty_{[0,T]} L^2_x} \| u_0 \|^3_{L^2_x}
\]

valid uniformly for all defocusing NLS equations on \( \mathbb{R}^3 \).
Efforts to extend the $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ interaction Morawetz to the $\mathbb{R}_x^2$ setting led to...

**Theorem (C-Grillakis-Tzirakis 08)**

*Finite energy solutions of any defocusing $\text{NLS}^+(\mathbb{R}^d)$ satisfy*

\[ \| D^{3-d/2} |u|^2 \|^2_{L^2_{t,x}} \lesssim \| u_0 \|^3_{L^2_x} \| \nabla u \|_{L^\infty_t L^2_x}. \]

- Independently & simultaneously by [Planchon-Vega].
- Simplifies proof [Nakanishi] of $H^1$-scattering when $0 < s_c < 1$.
- Simplified proof extends to $H^s$ for certain $s < 1$.
- Other applications?
6. A Cascading Solution to $NLS_3^+(\mathbb{T}^2)$. 
6. A Cascading Solution to $NLS^+_3(\mathbb{T}^2)$.

We consider the defocusing initial value problem:

$$\begin{cases} 
(-i\partial_t + \Delta)u = |u|^2u \\
u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2.
\end{cases} \quad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x, t)$ exists globally and

- **Mass**: $M(u) = \|u(t)\|^2 = M(0)$
- **Energy**: $E(u) = \int \left( \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(x, t)|^4 \right) dx = E(0)$

We want to understand the shape of $|\hat{u}(t, \xi)|$. The conservation laws impose $L^2$-moment constraints on this object.
**Past Results**

- **Bourgain**: (late 90’s)
  For the periodic IVP \( NLS(\mathbb{T}^2) \) one can prove

  \[
  \|u(t)\|_{H^s}^2 \leq C_s |t|^{4s}.
  \]

  The idea is to improve the local estimate for \( t \in [-1, 1] \)

  \[
  \|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1
  \]

  \( \implies \|u(t)\|_{H^s} \lesssim C|t| \text{ upper bounds} \) to obtain

  \[
  \|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,
  \]

  for some \( \delta > 0 \). This iterates to give

  \[
  \|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}.
  \]

- **Improvements**: Staffilani, C-Delort-Kenig-Staffilani.
**Past Results**

- **Bourgain:** (late 90’s)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x, t)$ to the modified wave equation
  \[(\partial_{tt} - \tilde{\Delta})u = u^p\]
  such that $\|u(t)\|_{H^s} \sim |t|^m$.

- **Physics:** Weak turbulence theory: Hasselmann & Zakharov.
  **Numerics (d=1):** Majda-McLaughlin-Tabak; Zakharov et. al.

**Conjecture**

*Solutions to dispersive equations on $\mathbb{R}^d$ do not exhibit high Sobolev norm growth.* ∃ solutions to dispersive equations on $\mathbb{T}^d$ with high Sobolev norm growth. In particular for \(\text{NLS}(\mathbb{T}^2)\) there exists $u(t, x)$ s. t.

\[\|u(t)\|_{H^s}^2 \to \infty \text{ as } t \to \infty.\]
Existence Result

We consider the defocusing initial value problem:

\[
\begin{cases}
(-i\partial_t + \Delta)u = |u|^2u \\
u(0, x) = u_0(x), \quad x \in \mathbb{T}^2.
\end{cases}
\]

(NLS(\mathbb{T}^2))

Theorem (C-Keel-Staffilani-Takaoka-Tao)

Let \( s > 1, K \gg 1 \) and \( 0 < \sigma < 1 \) be given. Then there exists a global smooth solution \( u(t, x) \) and \( T > 0 \) such that

\[
\|u_0\|_{H^s} \leq \sigma
\]

and

\[
\|u(T)\|_{H^s}^2 \geq K.
\]
Overview of Proof

- NLS (Gauge; Mass)
- FNLS → Projection → RFNLS
- RNFLS_L
- Toy Model
- Approximation
- Choice of L
- Arnold Diffusion