Maximal-in-time issues for nonlinear Schrödinger equations

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1. NLS Cauchy Problem

Nonlinear Schrödinger Initial Value Problem

We consider the initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^{p-1}u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_p^{\pm}(\mathbb{R}^d))$$

The $+$ case is called defocusing; $-$ is focusing.

- $NLS_3^{\pm}$ is ubiquitous in physics. $NLS_p^{\pm}$ introduced to explore interplay between dispersion and strength of nonlinearity.
- The **main question** about an evolution PDE: What is the ultimate fate of solutions? We want to understand the **maximal-in-time behavior** of the solutions.
- Conservation and invariance properties motivate the study of $NLS_p^{\pm}(\mathbb{R}^d)$ for low (and minimal) regularity initial data.

Time Invariant Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx \pm \frac{2}{p+1} |u(t)|^{p+1} dx.$$

- Mass is L^2; Momentum is close to $H^{1/2}$; Energy involves H^1.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Space/Frequency Localizations?

Dilation Invariance and Critical Regularity

One solution u generates parametrized family $\{u^\lambda\}_{\lambda>0}$ of solutions:

$$u : [0, T) \times \mathbb{R}^d_x \to \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)$$

$$\Updownarrow$$

$\boxed{\text{Scaling}}$

$$u^\lambda : [0, \lambda^2 T) \times \mathbb{R}^d_x \to \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)$$

where

$$u^\lambda(\tau, y) = \lambda^{-2/(p-1)} u(\lambda^{-2}\tau, \lambda^{-1}y).$$

Norms which are invariant under $u \longmapsto u_\lambda$ are *critical*.

Dilation Invariance and Critical Regularity

In the L^2-based Sobolev scale,

$$\|D^s u^\lambda(t)\|_{L^2} = \lambda^{-\frac{2}{p-1}-s+\frac{d}{2}}\|D^s u(t)\|_{L^2}.$$

The critical Sobolev index for $NLS_p^\pm(\mathbb{R}^d)$ is

$$s_c := \frac{d}{2} - \frac{2}{p-1}.$$

Scaling/Conservation Criticality

scaling	regime
$s_c < 0$	mass subcritical
$s = 0$	mass critical
$0 < s_c < 1$	mass super/energy subcritical
$s_c = 1$	energy critical
$1 < s_c < d/2$	energy supercritical

Optimal Local-in-time theory

Local-in-time theory for $NLS_p^{\pm}(\mathbb{R}^d)$ is essentially complete:

- Pioneering advances on spacetime dispersive estimates culminated in [Cazenave-Weissler 90] to prove local well-posedness for $s \geq s_{lwp} = \max(0, s_c)$.
  (discussed in more detail for $NLS_3(\mathbb{R}^2)$ soon.)
- Ill-posedness results for $s < s_{lwp}$ have been established.
  [Kenig-Ponce-Vega 01], [Christ-C-Tao 03], [Lebeau 01 05], [Burq-Gérard-Ibrahim], [Alazard-Carles 07].
- When $s_c < 0$, the Galilean symmetry obstructs well-posedness below $s = 0$.

Local-in-time theory for $NLS_3(\mathbb{R}^2)$

We pause to discuss the $L^2(\mathbb{R}^2)$-critical case.

- $\forall\ u_0 \in L^2(\mathbb{R}^2)\ \exists\ T_{lwp}(u_0)$ determined by
$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$
$\exists$ unique $u \in C([0,T_{lwp}];L^2) \cap L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)$ solving $NLS_3^+(\mathbb{R}^2)$.
- $\forall\ u_0 \in H^s(\mathbb{R}^2), s > 0,\ T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists: $u \in C([0,T_{lwp}];H^s(\mathbb{R}^2))$.
- Define the maximal forward existence time $T^*(u_0)$ by
$$\|u\|_{L^4_{tx}([0,T^*-\delta]\times\mathbb{R}^2)} < \infty$$
for all $\delta > 0$ but diverges to ∞ as $\delta \searrow 0$.
- $\exists$ small data scattering threshold $\mu_0 > 0$
$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

Global-in-time theory?

What is the ultimate fate of the local-in-time solutions?

L^2-critical Defocusing Scattering Conjecture:

$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Moreover, $\exists\ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \pm\infty} \|e^{\pm it\Delta} u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known [Tao-Visan-Zhang 06] for $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ for large $\boxed{\text{radial}}$ data, $d \geq 3$. Same for $d = 2$ [Killip-Tao-Visan 07].
- GWP for L^2 data $\iff$ Scattering for L^2 data. [Blue-C 06]

Critical Regularity Scattering Conjecture?

Consider defocusing case $NLS_p^+(\mathbb{R}^d)$ with critical Sobolev index

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

The critical (diagonal) Strichartz index is

$$q_c = \frac{(p-1)(2+d)}{2} \iff \frac{2}{q_c} + \frac{d}{q_c} = \frac{d}{2} - s_c.$$

<u>H^{s_c}**-critical defocusing scattering conjecture:**</u>

$H^{s_c}(\mathbb{R}^d) \ni u_0 \longmapsto u$ solving $NLS_p^+(\mathbb{R}^d)$ is global-in-time and

$$\|u\|_{L^{q_c}_{t,x}} < A(u_0) < \infty.$$

Critical Regularity Scattering Conjecture?

Present status of the defocusing scattering conjecture

criticality	general data	radial data	evidence
$s_c = 0$	???	[TVZ],[KTV]	GWP: $s_* < s < 1$
$0 < s_c < 1$	$\checkmark$: $s_c < s_* < s < 1$	$s = s_c$??	$\checkmark$: extra smooth
$s_c = 1$	[CKSTT],[RV],[V]	[B99], [T]	$\checkmark$: Resolved!
$1 < s_c < \frac{d}{2}$	[KV] if $\|u(t)\|_{H^{s_c}} < C$	????	Numerics [CSS]

Maiori

- Scattering for NLS_p^- under natural threshold.
- The existence (and value) of s_* depends upon p, d.
- Radial case with $s_c = \frac{1}{2}$ may be accessible using Morawetz??
- Induction-on-Mass + radial results $\to s_c = 0$ accessible???

2. Outline of Lectures

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I **Introduction: Outline of Course.**

II **I-method for Global Well-Posedness Below Energy.**

1. Abstract Scheme
2. Almost Conservation of $H[Iu]$
3. Multilinear Correction Terms
4. Resonant Decompositions

III **Low Regularity Theory for Focusing NLS.**

1. I-method for focusing NLS^- below ground state mass
2. Mass Concentration Properties of H^s Blowup Solutions
3. Mass Concentration Properties of L^2 Blowup Solutions

IV **The I-method with a Morawetz Bootstrap.**

1. Interaction Morawetz Estimates
2. $H[Iu]$ + Morawetz GWP & Scattering Results

V **A frequency cascading solution to $NLS_3^+(\mathbb{T}^2)$.**

"Weak Turbulence"

3. H^1 versus H^s Global Well-Posedness

3. H^1 versus H^s Global Well-Posedness

Consider $NLS_3^{\pm}(\mathbb{R}^2)$ with finite energy data $u_0 \in H^1$.
Classical H^1-GWP Scheme relies on three inputs:

1. LWP lifetime dependence on data norm: $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.
2. Energy controls data norm: $\|u(t)\|_{H^1}^2 \lesssim H[u(t)] + \|u(t)\|_{L^2}^2$.
3. Conservation: $H[u(t)] + \|u(t)\|_{L^2}^2 \leq C(Energy, Mass)$.

Fix arbitrary time interval $[0, T]$. Break $[0, T]$ into subintervals of uniform size $c(Energy, Mass)$ + LWP iteration $\implies$ GWP.

For $u_0 \in H^s$ with $0 < s < 1$, we may have infinite energy. Classical persistence of regularity from LWP/Duhamel only gives

$$\sup_{t \in [0, T_{lwp}]} \|u(t)\|_{H^s} \lesssim 2\|u_0\|_{H^s}$$

and LWP iteration fails due to (possible) doubling.

Abstract I-method Scheme for H^s-GWP

Let $H^s \ni u_0 \longmapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A smoothing operator $I = I_N : H^s \longmapsto H^1$. The NLS evolution $u_0 \longmapsto u$ induces a smooth reference evolution $H^1 \ni Iu_0 \longmapsto Iu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.
- A modified energy $\widetilde{E}[Iu]$ built using the reference evolution.

We postpone how we actually choose these objects.

Abstract I-method Scheme for H^s-GWP

We want I_N and $\widetilde{E}$ chosen to give a progressive H^s-GWP scheme:

1. Lifetime dependence on data norm: $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.✓
2. $\widetilde{E}$ controls data norm: $\exists\ t_g \in [\frac{1}{2}T_{lwp}, T_{lwp}]$ s.t. $\|u(t_g)\|_{H^s}^2 \lesssim \widetilde{E}[Iu(t_g)] + \|u(t_g)\|_{L^2}^2$.
3. Almost Conservation of Modifed Energy:

$$\sup_{t\in[0,T_{lwp}]} \widetilde{E}[Iu(t)] \le \widetilde{E}[Iu_0] + N^{-\alpha}.$$

The scheme advances over K uniform sized time steps of length $O(\widetilde{E}[u_0]^{-1/s})$ until the modified energy doubles

$$KN^{-\alpha} \sim \widetilde{E}[Iu_0].$$

This extends to solution for $t \in [0, N^{\alpha}\widetilde{E}[Iu_0]^{-\frac{1}{s}}]$ which contains $[0, T]$ for large enough N provided $s > s_\alpha$ with $s_\alpha < 1$.

First Version of the I-method: $\widetilde{E} = H[Iu]$

For $s < 1, N \gg 1$ define smooth monotone $m : \mathbb{R}^2_\xi \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N. \end{cases}$$

The associated Fourier multiplier operator, $\widehat{(Iu)}(\xi) = m(\xi)\widehat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Set $\widetilde{E}[Iu(t)] = H[Iu(t)]$. A detailed multilinear Fourier analysis establishes that $H[Iu]$ is almost conserved with $\alpha = \frac{3}{2}$ for $NLS_3^{\pm}(\mathbb{R}^2)$ and with $\alpha = 1$ for $NLS_3^{\pm}(\mathbb{R}^3)$. After some bookkeeping....

First Version of the I-method: $\widetilde{E} = H[Iu]$

Theorem (CKSTT:MRL02)

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

$NLS_3^+(\mathbb{R}^3)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{5}{6} < s < 1$. Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$ (both cases).

The same result applies for $NLS_3^-(\mathbb{R}^2)$ provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$ where Q is the ground state, the unique (up to translations) positive solution of $-Q + \Delta Q = Q^3$.

L^2-critical in Weighted L^2 spaces

High/Low Frequency Cut

Based on PC transformation & inspired by [Bourgain98], we have:

Theorem (Blue-C 06)

For $s \geq 0$, if $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ is GWP for $H^s(\mathbb{R}^d)$ initial data then $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ is GWP and scatters for data satisfying $\langle \cdot \rangle^s u_0(\cdot) \in L^2$. The same result applies to the focusing case provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

- Thus, GWP for L^2 data $\iff$ Scattering for L^2 data.
- H^s-GWP improvements imply weighted space improvements.
- PC transformation isometry in L^2-admissible Strichartz spaces.

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	**idea**	**reference**
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{1}{3}$	resonant cut & I-Morawetz	[C-Roy08]
$s > 0$?		Maiori

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

4. Low Regularity Theory for Focusing *NLS*

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Remark:

- The H^1-GWP scheme is relaxed to an H^s-GWP scheme by replacing the energy $H[u]$ by the modified energy $\widetilde{E}[Iu]$.
- The energy plays a basic role in other aspects of the NLS theory (e.g. soliton stability, properties of blowup).
- Natural idea: Explore whether existing H^1 theory may be systematically relaxed to H^s by replacing $H[u]$ by $\widetilde{E}[Iu]$.

Solitons and Log-log Blowup Stability

L^2 Critical Case: Blowup Solution Properties

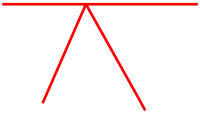
Explicit Blowup Solutions

- Arise as *pseudoconformal* image of $e^{it}Q(x)$:

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}.$$

- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$



All mass in S is conically concentrated into a point.

- Minimal mass H^1 blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

L^2 Critical Case: Blowup Solution Properties

Virial Identity $\implies$ $\exists$ **Many Blowup Solutions**

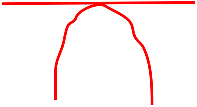
- Integration by parts and the equation yields

$$\partial_t^2 \int_{\mathbb{R}^2_x} |x|^2 |u(t,x)|^2 dx = 8H[u_0].$$

- $H[u_0] < 0, \int |x|^2 |u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

$\boxed{\text{What Happens?}}$

L^2 Critical Case: Mass Concentration



H^1 Theory of Mass Concentration

inessential

- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\liminf_{t \nearrow T^*} \int_{|x|<(T^*-t)^{1/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

  [Merle-Tsutsumi]
- H^1 blowups parabolically concentrate at least the ground state mass. Explicit blowups S concentrate mass much faster.
- Fantastic recent progress on the H^1 blowup theory. [Merle-Raphaël]

L^2 Critical Case: Mass Concentration

L^2 Theory of Mass Concentration

- $L^2 \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\limsup_{t \nearrow T^*} \sup_{cubes \ I, side(I) \leq (T^* - t)^{1/2}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

  [Bourgain98]

  L^2 blowups parabolically concentrate some mass.

- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
- For large L^2 data, do there exist tiny concentrations? ([TVZ], [KTV]: No, for radial data.)

Typical blowups leave an L^2 stain at time T^*

[Merle-Raphaël]:

$H^1 \cap \{\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*\} \ni u_0 \longmapsto u$ solving $NLS_3^-(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.
$\exists\ \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi\mathbb{Z})$ and u^* such that

$$u(t) - \lambda(t)^{-1} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\theta(t)} \to u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!

$\boxed{\text{Low Regularity Necessary!}}$

L^2 Critical Case: Conjectures/Questions

Consider focusing $NLS_3^-(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass.** ([KTV]:✓)

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies^{???} u_0 \longmapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$

  (Also, L^2 solutions of $NLS_3^+(\mathbb{R}^2)$ satisfy$^{???}$ $\|u\|_{L^4_{tx}} < \infty$.)
- **Minimal Mass Blowup Characterization.**

  $$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \longmapsto u, T^* < \infty \implies^{???} u = S,$$

  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in H^s for $s < 1$.
- **Concentrated mass amounts are quantized.**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.
- **Are there any general upper bounds?**

L^2 Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2 .$$

  H^s-blowup solutions concentrate ground state mass. [C-Raynor-C.Sulem-Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1$, $T^* < \infty \implies \exists\ t_n \nearrow T^*$ s.t. $u(t_n) \to Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence). For H^s blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an H^s analog of an H^1 result of [Weinstein] which preceded the minimal H^1 blowup solution characterization.
- Same results for $NLS^-_{\frac{4}{d}+1}(\mathbb{R}^d)$ in H^s, $s > \frac{d+8}{d+10}$. [Visan-Zhang]

L^2 Critical Case: Partial Results

[C-Roudenko 07]
Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

$$\limsup_{t\nearrow T^*} \sup_{cubes\ \ I, side(I)\leq(T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

This work refines the proof in [Bourgain 98].

5. Interaction Morawetz: Local Conservation

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Suppose $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{C}$ solves **generalized NLS**

$$(i\partial_t + \Delta)\phi = \mathcal{N}$$

for some $\mathcal{N} = \mathcal{N}(t, x, u) : [0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$. Assume ϕ is nice.

We introduce notation to compactly express mass and momentum (non)conservation for solutions of generalized NLS.

Write $\partial_{x_j}\phi = \partial_j\phi = \phi_j$.

Local mass/momentum (non)conservation

- mass density: $T_{00} = |\phi|^2$
- momentum density/mass current: $T_{0j} = T_{j0} = 2\Im(\overline{\phi}\phi_j)$
- (linear part of the) momentum current: $L_{jk} = L_{kj} = -\partial_j\partial_k|\phi|^2 + 4\Re(\overline{\phi_j}\phi_k)$
- mass bracket: $\{f, g\}_m = \Im(f\overline{g})$
- momentum bracket: $\{f, g\}_p^j = \Re(f\partial_j\overline{g} - g\partial_j\overline{f})$

Local mass (non)conservation identity:

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

Local momentum (non)conservation identity:

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j$$

Local mass/momentum (non)conservation

Consider $\mathcal{N} = F'(|\phi|^2)\phi$ for polynomial $F : \mathbb{R}^+ \to \mathbb{R}$.

- We calculate the mass bracket

$$\{F'(|\phi|^2)\phi, \phi\}_m = \Im(F'(|\phi|^2)\phi\overline{\phi}) = 0.$$

  Thus mass is conserved for these nonlinearities.

- We calculate the momentum bracket

$$\{F'(|\phi|^2)\phi, \phi\}_p^j = -\partial_j G(|\phi|^2)$$

  where $G(z) = zF'(z) - F(z) \sim F(z)$.
  Thus the momentum bracket contributes a divergence and momentum is conserved for these nonlinearities.

Generalized Virial Identity

Suppose $a : \mathbb{R}^d \to \mathbb{R}$. Form the **Morawetz Action**

$$M_a(t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the **generalized virial identity**

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) + 2a_j\{\mathcal{N}, \phi\}_p^j dx.$$

Idea of Morawetz Estimates: Cleverly choose the weight function a so that $\partial_t M_a \geq 0$ but $M_a \leq C(\phi_0)$ to obtain spacetime control on ϕ. This strategy imposes various constraints on a which suggest choosing $a(x) = |x|$.

Example: [Lin-Strauss 78] Morawetz identity

Consider $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. Choose $a(x) = |x|$. Observe that a is weakly convex, $\nabla a = \frac{x}{|x|}$ is bounded, and $-\Delta\Delta a = 4\pi\delta_0$. One gets the **Lin-Strauss Morawetz identity**

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^3} 4\pi\delta_0(x)|\phi(t,x)|^2 + (\geq 0) + 4\frac{G(|\phi|^2)}{|x|}dxdt$$

which implies the spacetime control estimate

$$(H[u_0])^{1/2}\|u_0\|_{L^2} \gtrsim \int_0^T \int_{\mathbb{R}^3} \frac{G(|\phi|^2)}{|x|}dxdt.$$

Tensor Product Idea

[CKSTT 04] (Hassell 04)

- Suppose ϕ_1, ϕ_2 are two solutions of $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. The "2-particle" wave function

$$\Psi(t, x_1, x_2) = \phi_1(t, x_1)\phi_2(t, x_2)$$

  satisfies an NLS-type equation on $\mathbb{R}^{1+6}$

$$(i\partial_t + \Delta_1 + \Delta_2)\Psi = [F'(|\phi_1|^2) + F'(|\phi_2|^2)]\Psi.$$

- Note that $[F'(|\phi_1|^2) + F'(|\phi_2|^2)] \geq 0$ so defocusing.
- Reparametrize $\mathbb{R}^6$ using center-of-mass coordinates $(\overline{x}, y)$ with $\overline{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3$. Note that $y = 0$ corresponds to the diagonal $x_1 = x_2 = \overline{x}$. Apply the generalized virial identity with the **choice** $a(x_1, x_2) = |y|$. Dismissing terms with favorable signs, one obtains...

Example: $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ Interaction Morawetz

$$\begin{aligned}
\|\nabla u\|_{L^\infty_{[0,T]}L^2_x}\|u_0\|^3_{L^2} &\geq \int_0^T \int_{\mathbb{R}^6} (-\Delta_6\Delta_6|y|)|\Psi(x_1,x_2)|^2 dx_1 dx_2 dt \\
&\geq c\int_0^T \int_{\mathbb{R}^6} \delta_{\{y=0\}}(x_1,x_2)|\phi_1(x_1)\phi_2(x_2)|^2 dx_1 dx_2 dt \\
&\geq c\int_0^T \int_{\mathbb{R}^3} |\phi_1(t,\overline{x})\phi_2(t,\overline{x})|^2 d\overline{x} dt.
\end{aligned}$$

Specializing to $\phi_1 = \phi_2$ gives the **interaction Morawetz estimate**

$$\int_0^T \int_{\mathbb{R}^3} |\phi(t,x)|^4 dx dt \leq C\|\nabla u\|_{L^\infty_{[0,T]}L^2_x}\|u_0\|^3_{L^2_x}$$

valid uniformly for all defocusing NLS equations on $\mathbb{R}^3$.

"The" Interaction Morawetz Estimate

Efforts to extend the $L^4(\mathbb{R}_t \times \mathbb{R}^3_x)$ interaction Morawetz to the $\mathbb{R}^2_x$ setting led to...

Theorem (C-Grillakis-Tzirakis 08)

Finite energy solutions of any defocusing $NLS^+(\mathbb{R}^d)$ satisfy

$$\|D^{\frac{3-d}{2}}|u|^2\|^2_{L^2_{t,x}} \lesssim \|u_0\|^3_{L^2_x}\|\nabla u\|_{L^\infty_t L^2_x}.$$

- Independently & simultaneously by [Planchon-Vega].
- Simplifies proof [Nakanishi] of H^1-scattering when $0 < s_c < 1$.
- Simplified proof extends to H^s for certain $s < 1$.
- Other applications?

6. A Cascading Solution to $NLS_3^+(\mathbb{T}^2)$.

6. A Cascading Solution to $NLS_3^+(\mathbb{T}^2)$.

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \text{where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x,t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$
$$\text{Energy} = E(u) = \int (\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4)\, dx = E(0)$$

We want to understand the shape of $|\hat{u}(t,\xi)|$. The conservation laws impose L^2-moment constraints on this object.

Past Results

- Bourgain: (late 90's)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove

  $$\|u(t)\|_{H^s}^2 \leq C_s|t|^{4s}.$$

  The idea is to improve the local estimate for $t \in [-1, 1]$

  $$\|u(t)\|_{H^s} \leq C_s\|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1$$

  ($\implies \|u(t)\|_{H^s} \lesssim C^{|t|}$ upper bounds) to obtain

  $$\|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s\|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,$$

  for some $\delta > 0$. This iterates to give

  $$\|u(t)\|_{H^s} \leq C_s|t|^{1/\delta}.$$

- Improvements: Staffilani, C-Delort-Kenig-Staffilani.

Past Results

- Bourgain: (late 90's)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x,t)$ to the modified wave equation
  $$(\partial_{tt} - \tilde{\Delta})u = u^p$$
  such that $\|u(t)\|_{H^s} \sim |t|^m$.
- Physics: Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ do not exhibit high Sobolev norm growth. $\exists$ solutions to dispersive equations on $\mathbb{T}^d$ with high Sobolev norm growth. In particular for $NLS(\mathbb{T}^2)$ there exists $u(t,x)$ s. t.

$$\|u(t)\|^2_{H^s} \to \infty \text{ as } t \to \infty.$$

Existence Result

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Theorem (C-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(t,x)$ and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$\|u(T)\|_{H^s}^2 \geq K.$$

Overview of Proof