Recent Progress on NLS-type Equations

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IHP: Ondes non-linéaires et dispersion
1. **Energy Supercritical NLS Simulations**
   - with G. Simpson and C. Sulem

2. **Global Well-Posedness of Cubic NLS on \( \mathbb{R}^2 \)**
   - with T. Roy
   - extends C-Grillakis-Tzirakis & C-Keel-Staffilani-Takaoka-Tao

3. **Elliptic-Elliptic Davey-Stewartson Blowup**
   - work of G. Richards

4. **Rough Blowup Solutions of Cubic NLS on \( \mathbb{R}^2 \)**
   - with P. Raphaël
   - builds on Merle-Raphaël and CKSTT ideas
1. Energy Supercritical NLS Simulations

Consider the defocusing monomial NLS initial value problem:

\[
\begin{cases}
(i \partial_t + \Delta) u = |u|^{p-1} u \\
u(0, x) = u_0(x).
\end{cases}
\]  \hspace{1cm} (NLS^+_p(\mathbb{R}^d))

**Dilation Invariance:**

\[
u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{C} \text{ solves } NLS^+_p(\mathbb{R}^d)
\]

\[
\forall \lambda > 0, \ u_\lambda : [0, \lambda^2 T] \times \mathbb{R}^d \rightarrow \mathbb{C} \text{ solves } NLS^+_p(\mathbb{R}^d)
\]

where

\[
u_\lambda(\tau, y) = \left(\frac{1}{\lambda}\right)^{p-1} u\left(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}\right).
\]
Critical Sobolev Regularity

A simple calculation shows that

\[ \| D^s u(\tau, \cdot) \|_{L^2} = \left( \frac{1}{\lambda} \right)^{\frac{2}{p-1} + \frac{s-d}{2}} \| D^s u(\tau) \|_{L^2}. \]

We encounter a Sobolev space with dilation invariant norm when

\[ s = s_c = \frac{d}{2} - \frac{2}{p-1}. \]

The space \( \dot{H}^{s_c}(\mathbb{R}^d) \) plays a basic role in theory for \( NLS_p(\mathbb{R}^d) \).

Global-in-time theory in the regime \( s_c > 1 \) is not understood.

Energy Supercritical Regime
Critical Norm Bounded \Rightarrow Scattering

- $NLW_p(\mathbb{R}^d)$: Radial + bounded $H^{s_c}$ norm \Rightarrow scattering. [Kenig-Merle] breakthrough, full supercritical range!
- $NLS^+_p(\mathbb{R}^d)$: Bounded $H^{s_c}$ norm \Rightarrow scattering. [Killip-Visan]

**Question**: What is the behavior of $\|u(t)\|_{H^{s_c}}$?

**Numerical simulations [CSS]**: $NLS^+_5(\mathbb{R}^5)$ has bounded $H^2$ norm.

Four simulations of radial data:
- Centered Gaussian
- Phased Centered Gaussian
- Phased Centered Gaussian (Linear flow diagnostic)
- Spherical Ring
PROBLEMS IN HAMILTONIAN PDE’S

J. Bourgain

1 Introduction

The purpose of this exposé is to describe a line of research and problems, which I believe, will not be by any means completed in the near future. As such, we certainly hope to encourage further investigations. The list of topics in this field is fairly extensive and only a few will be commented on here. Their choice was mainly dictated by personal research involvement. It should also be mentioned that the different groups of researchers may have very different styles and aims. As a science, claims and results range from pure experimentation to rigorous mathematical proofs. Although my primary interest is this last aspect, I have no doubt that numerics or heuristic argumentation may be equally interesting and important. The history of the Korteweg-de-Vries equation for instance is a striking example of how a problem may evolve through these different interacting stages to eventually create a beautiful theory. As a mathematician, I feel however that
Problem. Is there global scattering in the energy space for \( p = 2 + \frac{4}{d} \)?

(See also [C1,2] for other results on scattering).

(iv) We like to sketch the theoretical possibility for computer assisted proofs of global existence and scattering, for a given data \( \phi \). Consider for instance the 3D supercritical problem

\[
\begin{cases}
u_t + \Delta u - u|u|^6 = 0 \\
u(0) = \phi
\end{cases}
\] (3.22)

where \( \phi \) is a given smooth function. We do expect a global smooth solution + scattering. For this to hold, it is sufficient to show that for some time, \( 0 < T < \infty \),

(a) (3.22) has a smooth solution on \([0, T]\). Equivalently, \( T^* > T \), where \( T^* \) refers to Theorem 3.7

(b) The norm \( \|e^{i(t-T)\Delta}u(T)\|_{L^1_{t\geq T}L^1_x} < \delta \)

where \( \delta > 0 \) is some numerical constant (we do not explain the role of the \( L^1 \)-norm here). About step (a). If we fix a time \( T \), one may establish the result numerically. To do this, one first gathers sufficiently many discrete data and interpolates them with a (smooth) function \( v = v(x, t) \), \( t < T \). Assuming (3.22) has indeed a smooth solution, the function \( v \) will
Centered Gaussian Initial Data

\[
t = 0
\]

\[
\Re u, \Im u, |u|
\]
$t = 0.008$

Legend:
- $\Re u$
- $\Im u$
- $\bar{u}$
$t = 0.014$

- $\Re u$
- $\Im u$
- $|u|$
\[ t = 0.016 \]
$t = 0.018$

- $\Re u$
- $\Im u$
- $|u|$
\[ t = 0.022 \]

Diagram showing the function \( |u| \) at \( t = 0.022 \). The graph includes a red line representing \( |u| \), a blue dashed line for \( \Re u \), and a green dotted line for \( \Im u \). The x-axis represents \( r \) ranging from 0.0 to 4.0, and the y-axis represents a range from -10.0 to 10.0.
$t = 0.032$

\begin{align*}
\Re u & \\
\Im u & \\
|u| &
\end{align*}
$t = 0.034$

$\mathcal{R} u$

$\mathcal{I} u$

$|u|$
$t = 0.038$

$\Re u$

$\Im u$

$|u|$
$t = 0.04$
Centered Gaussian Fourier transform snapshots along nonlinear flow

- $|\hat{u}(k)| (t = 0)$
- $|\hat{u}(k)| (t = 0.008)$
- $|\hat{u}(k)| (t = 0.016)$
- $|\hat{u}(k)| (t = 0.024)$
- $|\hat{u}(k)| (t = 0.04)$
$H^2$ norm: Centered Gaussian along nonlinear flow
Sobolev vs. Besov: Centered Gaussian along nonlinear flow

\[ \| \Delta u \|_{L^2} \]

\[ \| u \|_{B^{2,\infty}} \]

\[ t \]

\[ 0 \]

\[ 0.005 \]

\[ 0.010 \]

\[ 0.015 \]

\[ 0.020 \]

\[ 0.025 \]

\[ 0.030 \]

\[ 0.035 \]

\[ 0.040 \]
Longer time $H^2$ norm: Centered Gaussian along nonlinear flow
Strichartz $L^{14}_x$ decay asymptotics
Phased Centered Gaussian Initial Data

Nonlinear Flow

t = 0

\[ \mathfrak{R}u \quad \mathfrak{I}u \quad |u| \]
$t = 0.005$

$\Re u$

$\Im u$

$|u|$
$t = 0.025$

- $\Re u$
- $\Im u$
- $|u|$
$t = 0.035$

Diagram showing the real and imaginary parts of $u$ as functions of $r$. The plot includes lines for $\Re u$, $\Im u$, and $|u|$. The graph ranges from $r = 0.0$ to $r = 4.0$, with a focus on the behavior of $u$ at $t = 0.035$. The real part of $u$ is represented by a blue dotted line, the imaginary part by a green dotted line, and the absolute value by a red solid line.
$t = 0.055$

- $\Re u$
- $\Im u$
- $|u|$
\[ t = 0.06 \]

The figure shows the behavior of \( |u| \) as a function of \( r \) for different values of \( t \). The graph includes lines labeled for different scenarios, such as \( \Re u \) and \( \Im u \), indicating real and imaginary parts of a complex function. The plot illustrates how these parts evolve with changes in the radius \( r \) at a specific time \( t \).
$t = 0.065$

The graph shows the evolution of $u(r)$ over a range of $r$ values. The plot includes different lines representing different components of $u(r)$.

- Blue dashed line: $\Re u$
- Green dotted line: $\Im u$
- Red line: $|u|$
$t = 0.08$

$|u|$

$\Re u$

$\Im u$

$|u|$

$r$
$t = 0.085$

- $\Re u$
- $\Im u$
- $|u|$

$r$ axis range: 0.0 to 4.0

$|u|$ axis range: -8.0 to 8.0
$t = 0.095$

The diagram shows plots of $\Re u$, $\Im u$, $\Re u$, and $|u|$ as functions of $r$. The plots appear to be constant for the given values of $r$. The axes are labeled with $t$ on the y-axis and $r$ on the x-axis.
Phased Centered Gaussian Fourier transform snapshots along nonlinear flow

- $|\hat{u}(k)| (t = 0)$
- $|\hat{u}(k)| (t = 0.01)$
- $|\hat{u}(k)| (t = 0.02)$
- $|\hat{u}(k)| (t = 0.04)$
- $|\hat{u}(k)| (t = 0.1)$
$H^2$ norm of Phased Centered Gaussian along nonlinear flow
Potential Energy Norm Decay: Phased Centered Gaussian along nonlinear flow
Phased Centered Gaussian Initial Data

Linear Flow

$t = 0$

$\Re u$

$\Im u$

$|u|$
$t = 0.02$
$t = 0.03$

\[ |u| \quad \Re u \quad \Im u \]

$r = 0.00 \quad 0.5 \quad 1.0 \quad 1.5 \quad 2.0 \quad 2.5 \quad 3.0 \quad 3.5 \quad 4.0$
$t = 0.055$

The graph shows the function $u(r, t)$ at $t = 0.055$, with contour lines for $\Re u$, $\Im u$, and $|u|$. The $r$-axis ranges from 0.0 to 4.0.
$t = 0.06$

- $\mathcal{R}u$
- $\mathcal{I}u$
- $|u|$
$t = 0.065$

Graph showing the real part ($\Re u$), imaginary part ($\Im u$), and absolute value ($|u|$) of some function $u$ as a function of $r$. The graph includes different line styles and markers to distinguish between the real and imaginary parts.
\( t = 0.07 \)

\[ \Re u \]
\[ \Im u \]
\[ |u| \]
$t = 0.075$
$t = 0.085$

The plot shows the real and imaginary parts of $u$ as functions of $r$, with $\Re u$ and $\Im u$ represented by dashed blue and dotted green lines, respectively. The magnitude of $u$ is represented by a solid red line.
$t = 0.095$

$\Re u$

$\Im u$

$|u|$
$t = 0.1$

$\mathbb{R}u$

$\mathbb{I}u$

$|u|$
Phased Centered Gaussian Fourier transform snapshots along linear flow

- $|\hat{u}(k)|(t = 0)$
- $|\hat{u}(k)|(t = 0.05)$
- $|\hat{u}(k)|(t = 0.1)$
Potential Energy Norm under linear flow
Phased Centered Gaussian under linear flow; bigger vertical axis

\[ \Re u, \Im u, |u| \]
\[ t = 0.03 \]

Diagram showing the behavior of \( u \) and \( \Re u \) as functions of \( r \) at different time points. The plot indicates the decay of \( u \) over time, with \( \Re u \) and \( \Im u \) being represented by different line styles for visual distinction.
$t = 0.035$

Graph showing the function $u$ at $t = 0.035$.
$t = 0.045$

$\Re u$

$\Im u$

$|u|$
$t = 0.05$

- $\Re u$
- $\Im u$
- $|u|$
$t = 0.055$

The graph shows the real ($\Re u$) and imaginary ($\Im u$) parts of $u$ along with the absolute value $|u|$ as functions of $r$. The plot is essentially flat, indicating that $u$ remains constant with respect to $r$ at $t = 0.055$. The x-axis represents $r$, and the y-axis represents the values of $u$. The graph includes a legend that distinguishes between the real, imaginary, and absolute value components.
Spherical Ring Initial Data

$\begin{align*}
t &= 0 \\
|u| \\
\Re u \\
\Im u
\end{align*}$
$t = 0.04$

$\mathbb{R} u$

$\mathbb{I} u$

$|u|$
$t = 0.1$

The graph shows the behavior of functions $\Re u$, $\Im u$, and $|u|$ as $r$ varies. The real part $\Re u$ is shown in blue, the imaginary part $\Im u$ in green, and the magnitude $|u|$ in red. The graph indicates a complex system, possibly related to a physical or mathematical model.
$t = 0.2$

The graph shows the behavior of $u(t, r)$ at $t = 0.2$. The real part $\Re u$ and imaginary part $\Im u$ are plotted, along with the magnitude $|u|$. The x-axis represents $r$, and the y-axis represents the values of $\Re u$, $\Im u$, and $|u|$. The graph indicates a complex function with oscillatory behavior in the radius $r$. The magnitude $|u|$ remains relatively constant, while the real and imaginary parts show more dynamic variation.
Spherical Ring Fourier transform snapshots along nonlinear flow

- $|\hat{u}(k)| (t = 0)$
- $|\hat{u}(k)| (t = 0.04)$
- $|\hat{u}(k)| (t = 0.08)$
- $|\hat{u}(k)| (t = 0.12)$
- $|\hat{u}(k)| (t = 0.2)$
$H^2$ norm of Spherical Ring along nonlinear flow
Sobolev vs. Besov: Spherical Ring along nonlinear flow

\[ \| \Delta u \|_{L^2} \]

\[ \| u \|_{B^2_{2,\infty}} \]
Potential Energy Norm Decay

\[ \|u\|_{L^6} \]

vs.

\[ t \]

The graph shows the decay of the potential energy norm \( \|u\|_{L^6} \) over time \( t \). As time progresses, the norm decreases smoothly, indicating an energy decay process.
We consider the initial value problems:

\[
\begin{cases}
(i\partial_t + \Delta)u = \pm |u|^2 u \\
u(0, x) = u_0(x).
\end{cases}
\tag{NLS^{\pm}_3(\mathbb{R}^2)}
\]

The + case is called **defocusing**; − is **focusing**. \(NLS^{\pm}_3\) is ubiquitous in physics. The solution has a dilation symmetry

\[u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1}y).
\]

which is invariant in \(L^2(\mathbb{R}^2)\). This problem is **\(L^2\)-critical**.
**Time Invariant Quantities**

\[
\text{Mass} = \int_{\mathbb{R}^d} |u(t, x)|^2 dx.
\]

\[
\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \bar{u}(t) \nabla u(t) dx.
\]

\[
\text{Energy} = H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx \pm \frac{1}{2} |u(t)|^4 dx.
\]

- Mass is \(L^2\); Momentum is close to \(H^{1/2}\); Energy involves \(H^1\).
- Dynamics on a sphere in \(L^2\); focusing/defocusing energy.
- Local conservation laws express how quantity is conserved: e.g., \(\partial_t |u|^2 = \nabla \cdot 2\Im(\bar{u}\nabla u)\). Frequency Localizations?
Local-in-time theory for $NLS^\pm_3(\mathbb{R}^2)$

- $\forall u_0 \in L^2(\mathbb{R}^2)$, $\exists T_{lwp}(u_0)$ determined by
  $$\|e^{it\Delta}u_0\|_{L^4_t([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}$$

  such that
  $\exists$ unique $u \in C([0, T_{lwp}]; L^2) \cap L^4_t([0, T_{lwp}] \times \mathbb{R}^2)$ solving
  $NLS^+_3(\mathbb{R}^2)$.

- $\forall u_0 \in H^s(\mathbb{R}^2), s > 0$, $T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists:
  $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.

- Define the **maximal forward existence time** $T^*(u_0)$ by
  $$\|u\|_{L^4_t([0, T^* - \delta] \times \mathbb{R}^2)} < \infty$$

  for all $\delta > 0$ but diverges to $\infty$ as $\delta \downarrow 0$.

- $\exists$ small data scattering threshold $\mu_0 > 0$
  $$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_t(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.$$
Robust, open set in $L^2$.

Asymptotically linear behavior.

Smallness brutally controls solution via fixed point argument.

What is the boundary of small data scattering portion of phase space $L^2$?
What is the ultimate fate of the local-in-time solutions?

**$L^2$-critical Scattering Conjecture:**

$L^2 \ni u_0 \mapsto u$ solving $\text{NLS}_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$ 

Moreover, $\exists \ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \pm \infty} \|e^{\pm it\Delta} u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$ 

Same statement for focusing $\text{NLS}_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known for large radial data [Killip-Tao-Visan 07].
**NLS$_3^\pm(\mathbb{R}^2)$: Present Status for General Data**

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<td>$s &gt; 0$?</td>
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- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?
Theorem (C-Roy) \( \text{NLS}_3^+(\mathbb{R}^2) \) GWP in \( H^s, s > 1/3 \).

Overview of Proof

Let \( u_0 \in H^s(\mathbb{R}^2), \ 0 \leq s \leq \frac{2}{5} \). Eventually, we require \( \frac{1}{3} < s \).

- **Task**: Construct \( u_0 \mapsto u(t) \ \forall \ t \in [0, T], \ T \) fixed large.
- **Equivalent Task**: Construct \( u_\lambda(\tau) \ \forall \ \tau \in [0, \lambda^2 T] \) where

\[
    u_\lambda(\tau, y) = \frac{1}{\lambda} u\left(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}\right).
\]

We reserve the right to choose \( \lambda > 0 \) later.
I-Method Setup

- Define a spatial smoothing operator $I_N : H^s \to H^1$ via
  \[ \hat{I_N} f(\xi) = m\left(\frac{\xi}{N}\right) \hat{f}(\xi) \]
  where the smooth monotone Fourier multiplier $m$ is defined
  
  \[ m(\xi) = \begin{cases} 
    1 & \text{for } |\xi| < 1 \\
    \xi^s - 1 & \text{for } |\xi| > 2.
  \end{cases} \]

- Rough solution induces finite energy reference evolution $I_N u$.
- Energy based control on $I_N u$ globalize $u$. 

**Modified Energy; Choice of \( \lambda \)**

\[
H[I_N u] = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} |I_N u|^4 \, dx.
\]

\[
\|u\|_{H^s}^2 \leq H[I_N u] + \|u_0\|_{L^2}^2.
\]

Scaling and parameter dependence of \( H[I_N u] \) gives

\[
H[I_N u] \leq C(u_0) \frac{N^{2(1-s)}}{\lambda^{2s}}.
\]

**Choose** \( \lambda = \lambda(N) = CN^{\frac{1-s}{s}} \implies \)

\[
H[I_N u_\lambda] \leq \frac{1}{100}.
\]

(Drop \( \lambda \) subscript; Recall target \([0, \lambda^2 T]\). Eventually \( N = N(T) \).)
We first construct $u(t) \forall t \in J_1 = [0, N^3] \subset [0, \lambda^2 T]$.

Decompose $J_1 = \bigcup_{i=1}^{\mathcal{I}} I_i$ where the $\{I_i\}$ satisfy $I_i \cap I_j = \emptyset$ and

$$\|u\|_{L^4_{t \in I_i, x}}^4 \sim \frac{1}{100}.$$ 

In principle, the number $\mathcal{I}$ of such intervals could be HUGE.

**Morawetz input**: On any $J$ where $u$ exists, [CGTz] proved

$$\|Iu\|_{L^4_{t \in J, x}}^4 \leq C_0 |J|^{1/3}.$$ 

Thus, taking $J = J_1$, we find (provided $u$ exists on all of $J_1$),

$$\mathcal{I} \lesssim N.$$
Using resonant decomposition, [CKSTT] constructed $\tilde{E}[u]$:

- Proximity to $H[Iu(t)]$ at each time $t$:
  \[ |H[Iu(t)] - \tilde{E}[u(t)]| \lesssim N^{-1+}(H[Iu(t)])^2. \]

- Almost Conservation Law:
  \[ \text{osc}_{l_1} \tilde{E}[u(\cdot)] := \sup_{l_1} \tilde{E}[u(\cdot)] - \inf_{l_1} \tilde{E}[u(\cdot)] \leq C_0 N^{-2+}. \]

(Corresponding estimate for $H[I_N u]$ had $N^{-3/2+}$.)
As $t$ traverses $I_1$:
- $H[lu(t)]$ stays with $N^{-1}$ of $\tilde{E}[u(t)]$;
- $\tilde{E}[u(t)]$ increments by at most $CN^{-2+}$.

As $t$ traverses $I_1 \cup I_2$:
- $H[lu(t)]$ stays with $N^{-1}$ of $\tilde{E}[u(t)]$;
- $\text{osc}_{I_1 \cup I_2} \tilde{E}[u(\cdot)] \leq 2CN^{-2+}$.

Morawetz control gives $I \lesssim N$ so
\[
\text{osc}_{J_1 = \bigcup_{i=1}^{I}} \tilde{E}[u(\cdot)] \leq C I N^{-2+} \lesssim N^{-1+}.
\]

Let $J_2 = [N^3, 2N^3] \subset [0, \lambda^2 T]$, $J_3 = [3N^3, 4N^3]$, $J_4$, ....
Process can be iterated $N^{-1}$ times before $\tilde{E}$ doubles.

We need $N^{4-} > \lambda^2 T \iff N^{-\frac{2}{s}+6} > T \implies s > \frac{1}{3}$ suffices.
This section describes work of **G. Richards** (Toronto student).

The Davey-Stewartson system is

\[
\begin{align*}
&i \partial_t u + \sigma u_{xx} + u_{yy} = \pm |u|^2 u + \phi u \\
&\alpha \phi_{xx} + \phi_{yy} + \gamma (|u|^2)_x = 0 \\
&u(0, x) = u_0(x). 
\end{align*}
\]

- DS arises in models of the ocean.
- Parameters $\sigma, \alpha \in \mathbb{R}$, $\gamma \geq 0$.
- When $\sigma > 0$ and $\alpha > 0$, system is called **elliptic-elliptic**.

Set $\sigma = \alpha = 1$. 
Using Fourier transform, elliptic equation for $\phi$ is reexpressed

$$\hat{\phi}_x = -\gamma \frac{\xi_1^2}{\xi_1^2 + \xi_2^2} |\hat{u}|^2(\xi) = -\gamma \hat{B} |\hat{u}|^2(\xi).$$

Substituting into the Schrödinger equation for $u$ yields

$$\begin{cases} i\partial_t u + u_{xx} + u_{yy} + \mathcal{L}(|u|^2)u = 0 \\ u(0, x) = u_0(x) \end{cases} \quad (DS_{e,e}^{\pm}(\mathbb{R}^2))$$

where $\mathcal{L} = \pm \mathbb{I} + \gamma B$. 
$DS_{e,e}^\pm$ IS SIMILAR TO $NLS_3^-$

- Conservation of mass, momentum, energy
- $L^2$ critical
- Pseudoconformal invariance
- Similar LWP theory
  - $L^4_{t,x}$ maximality criterion
  - Subcritical scaling of local existence time

(See [Ghidaglia-Saut])

In fact, the analogy between $DS_{e,e}^\pm$ and $NLS_3^-$ goes deeper.
**H¹ Theory for DS±_{e,e}**

- $E[u] = \int \frac{1}{2} |\nabla u|^2 - \frac{1}{4} \mathcal{L}(|u|^2)|u|^2 \, dx$ conserved.
- Weinstein Inequality [Papanicolau-Sulem-Sulem-Wang 94]:
  \[
  \int \mathcal{L}(|u|^2)|u|^2 \, dx \leq C_{opt} \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2
  \]
  where $C_{opt} = \frac{2}{\|R\|_{L^2}^2}$ for some $R \in H^1$, $R > 0$. **Uniqueness??**
- Soliton: $u(t, x) = e^{it} R(x)$ solves $DS^\pm_{e,e}$.
- PC(soliton) explicit blowup. (Not observed numerically.)
- Virial identity, energy criteria for blowup. [Ghidaglia-Saut 90]
- Log log blowups?? (Numerically and formally expected.)
**Theorem** (G. Richards) \( H^1 \) mass concentration

Let \( H^1 \ni u_0 \mapsto u \) solve \( DS_{e,e}^\pm \) which blows up as \( t \nearrow T^* < \infty \).

Fix any \( \lambda(t) > 0 \) such that \( \lambda(t) \| \nabla u(t) \|_{L^2} \to \infty \) as \( t \nearrow T^* \).

Then \( \exists x(t) \in \mathbb{R}^2 \) such that

\[
\liminf_{t \nearrow T^*} \int_{|x-x(t)|<\lambda(t)} |u(t,x)|^2 \, dx \geq \frac{2}{C_{opt}}.
\]

Remarks:
- Analog of [Merle-Tsutsumi], [Nawa] results for \( NLS_3^- \).
- Proof based on profile decomposition from [Hmidi-Keraani].
Theorem (G. Richards) $L^2$ mass concentration

Let $L^2 \ni u_0 \mapsto u$ solve $DS_{\pm,e}$ which blows up as $t \uparrow T^* < \infty$. Then

$$\limsup_{t \uparrow T^*} \sup_{\text{parabolic squares } Q} \int_Q |u(t, x)|^2 dx \geq \eta(\|u_0\|_{L^2}).$$

Remarks:

- Parabolic squares have sidelength $(Q) < (T^* - t)^{1/2}$.
- Analog of [Bourgain], [Merle-Vega] mass concentration result.
- Proof essentially the same; based on linear refinements.
4. Rough Blowup Solutions of $NLS_3^-(\mathbb{R}^2)$
Known Maximal-in-Time Solution Scenarios

1. Soliton solutions exist: \( u(t, x) = e^{it} R(x) \)
   - \( Q(x) \) ground state; also excited states.
   - non-scattering; Strichartz \( S^0 \) norms diverge global-in-time.
   - a priori \( H^1 \) control if \( \| u_0 \|_{L^2} < \| Q \|_{L^2} \). [Weinstein]

2. \{radial\} \( \cap L^2 \) \( \ni u_0 \mapsto u \) scatters if \( \| u_0 \|_{L^2} < \| Q \|_{L^2} \). [KTV]

3. \( PC \) transformation + solitons \( \implies \) explicit (fast) \( \frac{1}{t} \)-blowups.
   - \( PC \) is a Strichartz \( S^0 \) isometry.
   - There exists an enlarged class of \( \frac{1}{t} \)-blowups [Bourgain-Wang].
   - Stability?

4. Virial Blowup Solutions
   - Obstructive argument
   - Qualitative properties?

Open Question
Ground State

- $H^1$-GWP mass threshold for $\text{NLS}_3^-(\mathbb{R}^2)$ [Weinstein]:
  \[ \|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty, \]
  based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$
  \[ \|u\|_{L^4}^4 \leq \left[ \frac{2}{\|Q\|_{L^2}^2} \right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2. \]

- $Q$ is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $\text{NLS}_3^-(\mathbb{R}^2)$ is
  \[ u(t, x) = e^{it} Q(x). \]
**Pseudoconformal Symmetry**

- **Pseudoconformal transformation:**
  \[
  \mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u \left( -\frac{1}{\tau}, \frac{y}{\tau} \right),
  \]

- **$\mathcal{PC}$ is $L^2$-critical NLS solution symmetry:**
  Suppose $0 < t_1 < t_2 < \infty$. If
  \[
  u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS_{1+\frac{4}{d}}^\pm(\mathbb{R}^d)
  \]
  then
  \[
  \mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}
  \]
  solves
  \[
  i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v.
  \]

- **$\mathcal{PC}$ is an $L^2$-Strichartz isometry:**
  If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then
  \[
  \|\mathcal{PC}[u]\|_{L^q_t L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.\]
Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it} Q(x)$,

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + i},$$

is an explicit blowup solution.
- $S$ has minimal mass:

$$\| S(-1) \|_{L_x^2} = \| Q \|_{L^2}.$$

All mass in $S$ is conically concentrated into a point.
- Minimal mass $H^1$ blowup solution characterization:

$u_0 \in H^1, \| u_0 \|_{L^2} = \| Q \|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]
Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \rightarrow \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t, x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Re(\overline{\phi}\nabla \phi) dx.$$

Conservation identities lead to the generalized virial identity

$$\partial^2_t V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta \Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial^2_t \int_{\mathbb{R}^2} |x|^2|u(t, x)|^2 dx = 16H[u_0]$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.

- How do these solutions blow up?
**Question:** What are the dynamical properties of $NLS_{3^-}(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation
Numerical/Persuasive arguments [LPSS] led to:
- Prediction of blowups with log log speed:
  \[ \| u(t) \|_{H^1} \sim \sqrt{\frac{\log |\log (T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}. \]
- Prediction that such blowups are generic/stable/observed.
- Identification of certain mechanisms forecasting log log.
- \( NLS_5^{-}(\mathbb{R}^1) \) has log log blowup solutions. [Perelman]
- Detailed Description of log log regime in series by [MR].
Robust, open set in $H^1$.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomena in critical space ($L^2$ instability?).
- Conjectured quantization properties?
- Boundary of log log regime in phase space?
Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- **Small Excess Mass:** $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.
- **Negative Total Energy:** $H[u_0] < 0$.

The associated solution $u_0 \mapsto u$ explodes with $T^* < \infty$ and

- $\exists (\lambda(t), x(t), \gamma(t) \in \mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$ 

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.
- Sharp log log speed law holds:

$$\lambda(t) \sqrt{\frac{\log |\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$ 

- $u^* \not\in H^s$ for $s > 0$; $u^* \not\in L^p$ for $p > 2$. (Rough residual)
Theorem (Raphaël): $H^1$ Stability of log log

- **Fact:** $PC + \log\log$ for $E < 0 \implies \exists \log\log$ with $E > 0$.
- **$H^1$-Stability Theorem:** The set of data with $u_0 \in H^1$ with small excess mass blowing up in log log regime is open in $H^1$.
- Develops **bootstrap** approach to *constructing* log log.
- Further applications of Raphaël’s bootstrap/stability:
  - Domains: [Planchon-R:Ω]
  - Singular $S^1 \subset \mathbb{R}^2$: [R:Ring]
  - Singular $S^{d-1} \subset \mathbb{R}^d$: [R-Szeftel:Spheres]
  - Singular $S^1 \subset \mathbb{R}^3$: [Zwiers: Codimension Two Ring]
  - Higher Codimensional Singular Sets?
  - Rough Blowups
Theorem (C-Raphaël): $H^s$ Stability of log log

- Let $u_0 \in H^1$ evolve into the log log regime.
- $\forall \, s > 0 \, \exists \, \epsilon = \epsilon(s, u_0) > 0$ such that $\forall \, v_0 \in H^s(\mathbb{R}^2)$

\[
\|u_0 - v_0\|_{H^s} < \epsilon,
\]

$NLS_3^{-}(\mathbb{R}^2)$ solution $v_0 \mapsto v$ blows up in log log regime.

Thus, the $H^1$ log log blowup solutions constructed by [MR] are contained in an open superset of log log blowups in $H^s$, $\forall \, s > 0$. 
Remarks about the $H^s$ stability of $\log \log$

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of $H^s \log \log$ blowups.
- The condition $s > 0$ is expected to be optimal.
  Small $L^2$ (but huge $H^s$) perturbation destroys rough residual mass ($u^* \notin H^s$, $\forall \ s > 0$) leading to fast $\frac{1}{t}$-blowup?
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via $I$-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled $I$-method?
Aspects of the [MR] Analysis

- Geometrical description of log log blowup solutions.
  - Various profiles $Q, Q_b, \tilde{Q}_b, \tilde{Q}_b(t) + \zeta_b(t)$. (Obscure Notation)
  - Modulation parameters related to solution symmetries.
  - Three zones: blowup core, radiation, distant/decoupled.

- Virial/Coercivity constraints; Orthogonality conditions.

- A key role played by Energy conservation.
Near $T^*$, log log blowups satisfy **geometrical ansatz**

$$u(t, x) = \frac{1}{\lambda(t)}(Q_b(t) + \epsilon) \left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}.$$

- Parameters $(\lambda(t), x(t), \gamma(t), b(t))$ solve ODEs forced by $F(\epsilon)$.
- ODEs emerge from geometrical ansatz, taking inner products with equation, imposing orthogonality conditions. (These choices change across the [MR] works.)
Energy Conservation in [MR] Analysis

- Control of $\epsilon$:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$  

- Energy conservation and $\lambda \searrow 0 \implies$

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$  

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.

(Observation in [CRSW]; Led to [C-Raphaël] collaboration)