Recent Progress on NLS-type Equations

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IHP: *Ondes non-linéaires et dispersion*

1. **Energy Supercritical NLS Simulations**
   - with G. Simpson and C. Sulem
2. **Global Well-Posedness of Cubic NLS on $\mathbb{R}^2$**
   - with T. Roy
   - extends C-Grillakis-Tzirakis & C-Keel-Staffilani-Takaoka-Tao
3. **Elliptic-Elliptic Davey-Stewartson Blowup**
   - work of G. Richards
4. **Rough Blowup Solutions of Cubic NLS on $\mathbb{R}^2$**
   - with P. Raphaël
   - builds on Merle-Raphaël and CKSTT ideas

1. Energy Supercritical NLS Simulations

Consider the defocusing monomial NLS initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = |u|^{p-1}u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_p^+(\mathbb{R}^d))$$

Dilation Invariance:

$$u : [0, T] \times \mathbb{R}^d \longmapsto \mathbb{C} \text{ solves } NLS_p^+(\mathbb{R}^d)$$

$$\Updownarrow$$

$$\forall\ \lambda > 0,\ \ u_\lambda : [0, \lambda^2 T] \times \mathbb{R}^d] \longmapsto \mathbb{C} \text{ solves } NLS_p^+(\mathbb{R}^d)$$

where

$$u_\lambda(\tau, y) = (\frac{1}{\lambda})^{\frac{2}{p-1}} u(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}).$$

Critical Sobolev Regularity

A simple calculation shows that

$$\|D^s u_\lambda(\tau,\cdot)\|_{L^2} = (\frac{1}{\lambda})^{\frac{2}{p-1}+s-\frac{d}{2}} \|D^s u(\tau)\|_{L^2}.$$

We encounter a Sobolev space with dilation invariant norm when

$$s = s_c = \frac{d}{2} - \frac{2}{p-1}.$$

The space $\dot{H}^{s_c}(\mathbb{R}^d)$ plays a basic role in theory for $NLS_p(\mathbb{R}^d)$.

Global-in-time theory in the regime $s_c > 1$ is not understood.
Energy Supercritical Regime

Critical Norm Bounded $\implies$ Scattering

- $NLW_p(\mathbb{R}^d)$: Radial + bounded H^{s_c} norm $\implies$ scattering. [Kenig-Merle] breakthrough, full supercritical range!
- $NLS_p^+(\mathbb{R}^d)$: Bounded H^{s_c} norm $\implies$ scattering. [Killip-Visan]

Question: What is the behavior of $\|u(t)\|_{H^{s_c}}$?

Numerical simulations [CSS]: $NLS_5^+(\mathbb{R}^5)$ has bounded H^2 norm.

Four simulations of radial data:

- Centered Gaussian
- Phased Centered Gaussian
- Phased Centered Gaussian (Linear flow diagnostic)
- Spherical Ring

GAFA, Geom. funct. anal.
Special Volume – GAFA2000, 32 – 56
1016-443X/00/S1032-25 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

PROBLEMS IN HAMILTONIAN PDE'S

J. Bourgain

1 Introduction

The purpose of this exposé is to describe a line of research and problems, which I believe, will not be by any means completed in the near future. As such, we certainly hope to encourage further investigations. The list of topics in this field is fairly extensive and only a few will be commented on here. Their choice was mainly dictated by personal research involvement. It should also be mentioned that the different groups of researchers may have very different styles and aims. As a science, claims and results range from pure experimentation to rigorous mathematical proofs. Although my primary interest is this last aspect, I have no doubt that numerics or heuristic argumentation may be equally interesting and important. The history of the Korteweg-de-Vries equation for instance is a striking example of how a problem may evolve through these different interacting stages to eventually create a beautiful theory. As a mathematician, I feel however that

PROBLEM. *Is there global scattering in the energy space for $p = 2 + \frac{4}{d}$?*

(See also [C1,2] for other results on scattering).

(iv) We like to sketch the theoretical possibility for computer assisted proofs of global existence and scattering, for a given data ϕ. Consider for instance the 3D supercritical problem

$$\begin{cases} iu_t + \Delta u - u|u|^6 = 0 \\ u(0) = \phi \end{cases} \tag{3.22}$$

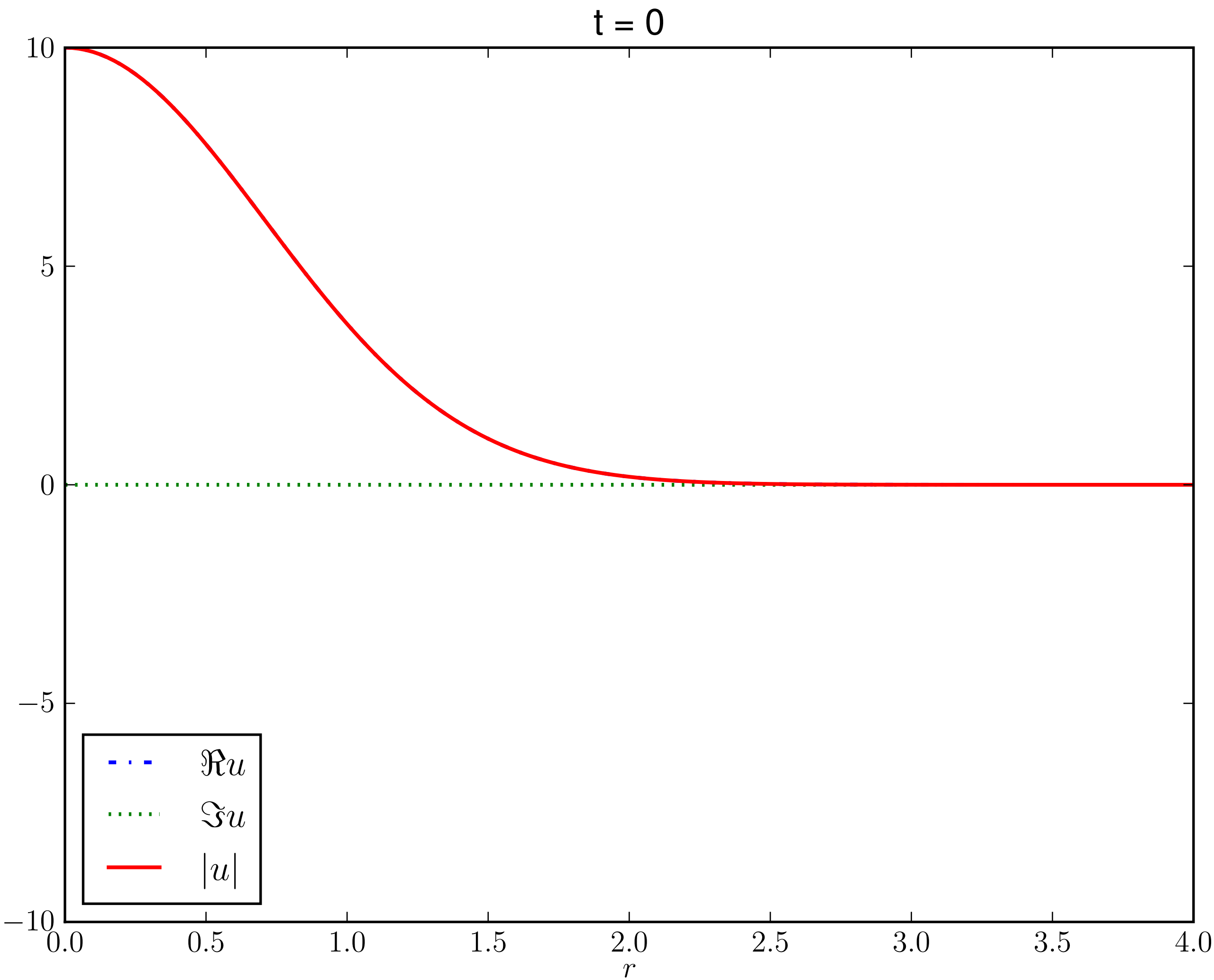
where ϕ is a given smooth function. We do expect a global smooth solution + scattering. For this to hold, it is sufficient to show that for some time, $0 < T < \infty$,

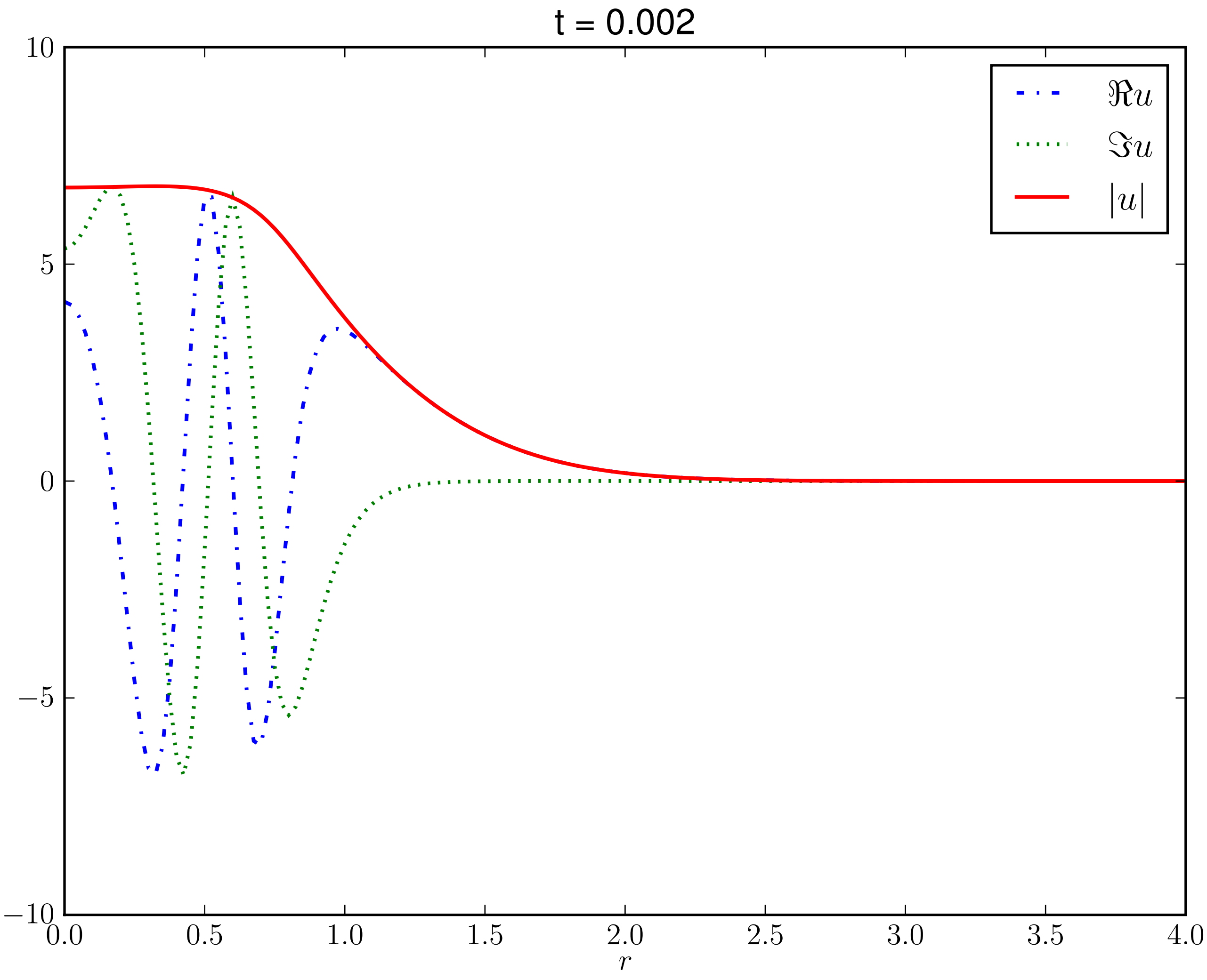
(a) (3.22) has a smooth solution on [0, T]. Equivalently, $T^* > T$, where T^* refers to Theorem 3.7

(b) The norm $\|e^{i(t-T)\Delta}u(T)\|_{L^{15}_{t\geq T} L^{15}_x} < \delta$

where $\delta > 0$ is some numerical constant (we do not explain the role of the L^{15}-norm here). About step (a). If we fix a time T, one may establish the result numerically. To do this, one first gathers sufficiently many discrete data and interpolates them with a (smooth) function $v = v(x,t)$, $t < T$. Assuming (3.22) has indeed a smooth solution, the function v will

Centered Gaussian Initial Data

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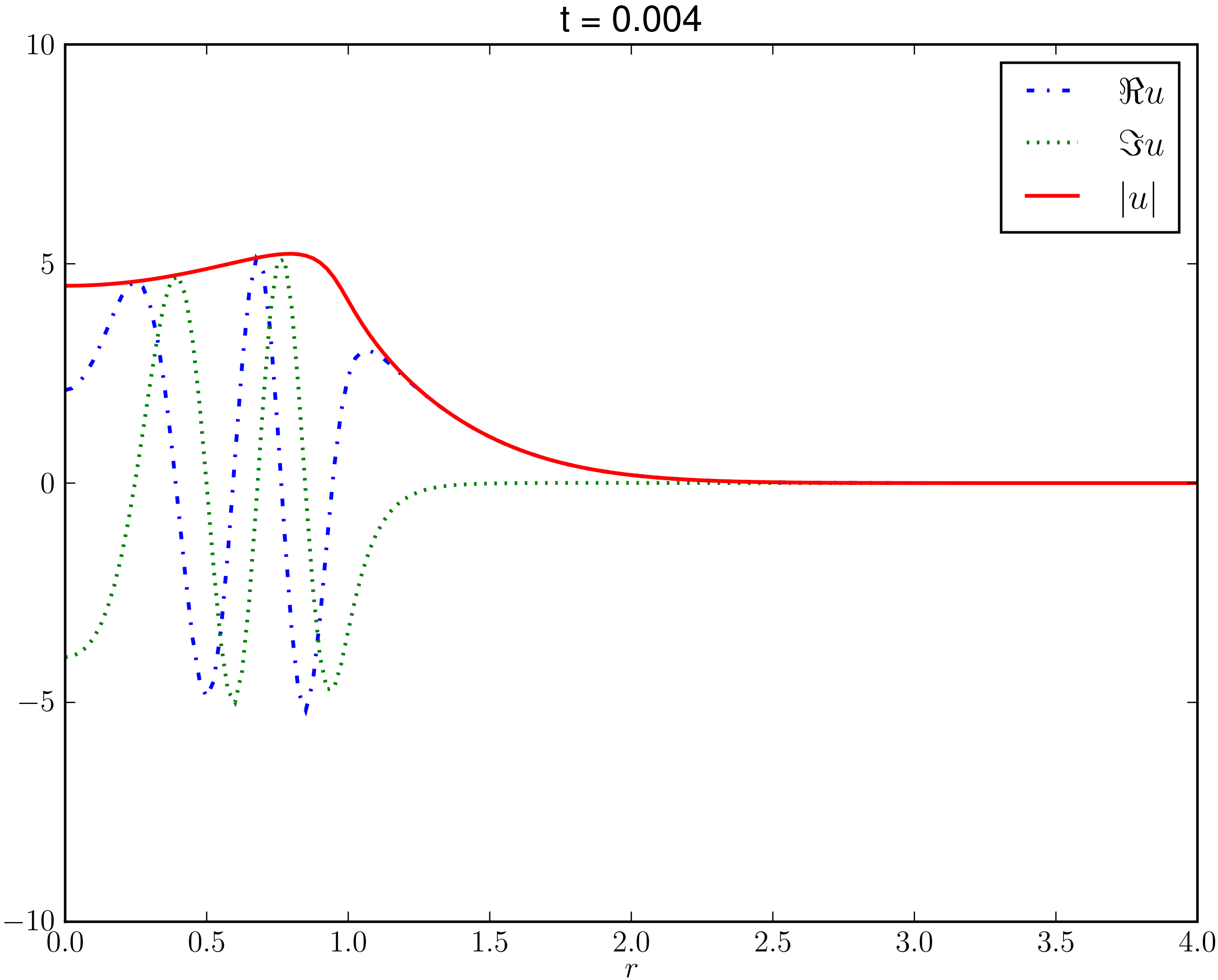
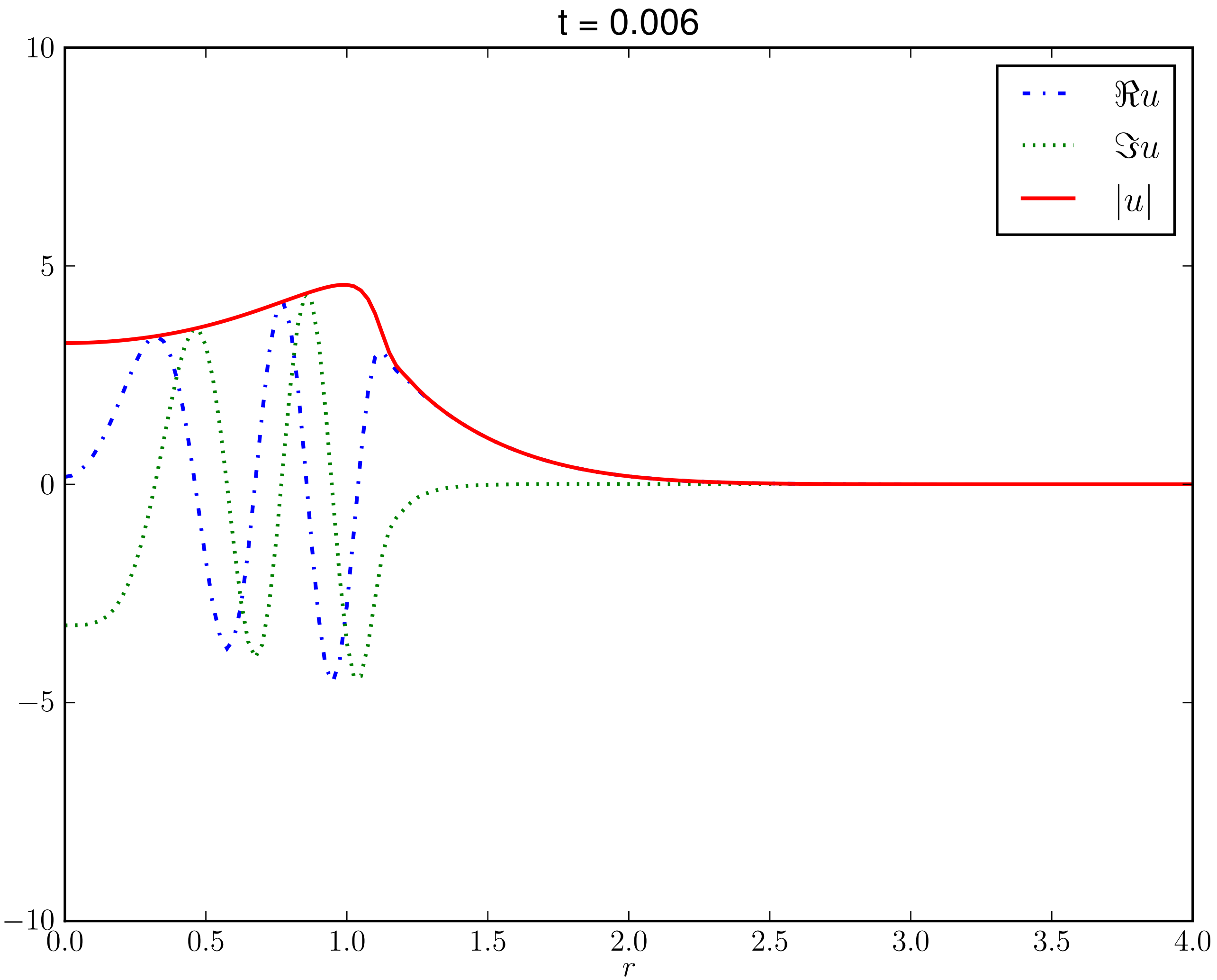
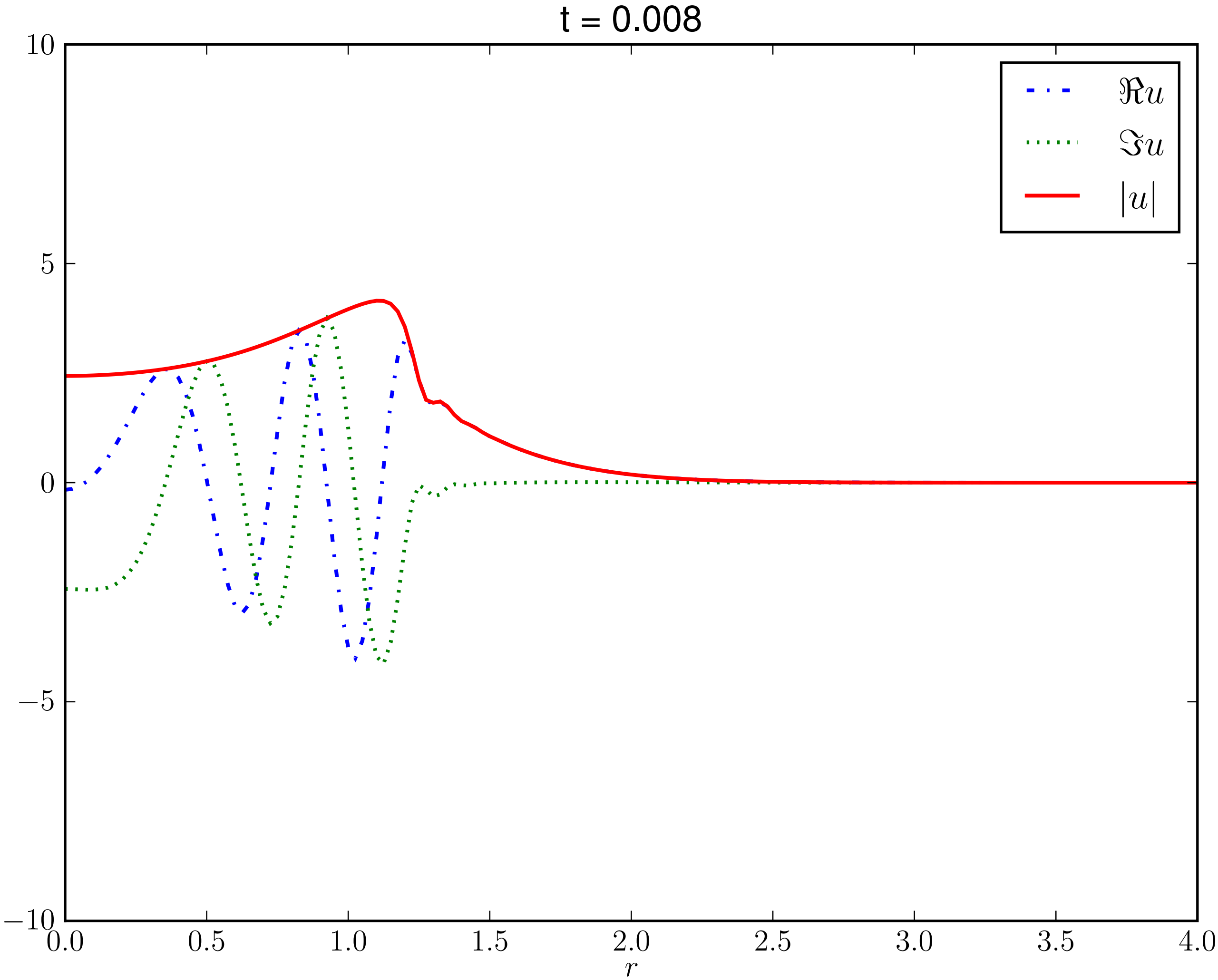
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

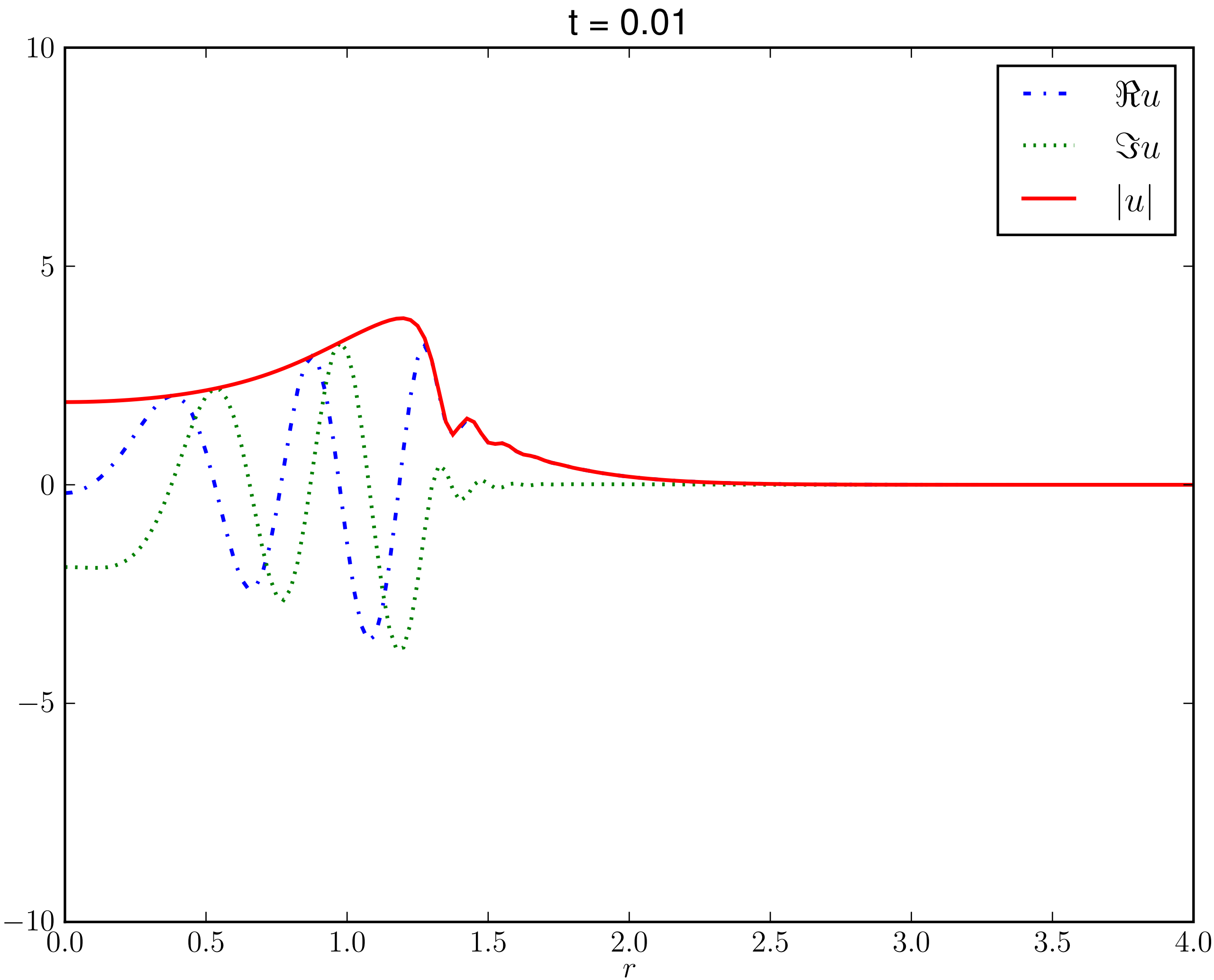


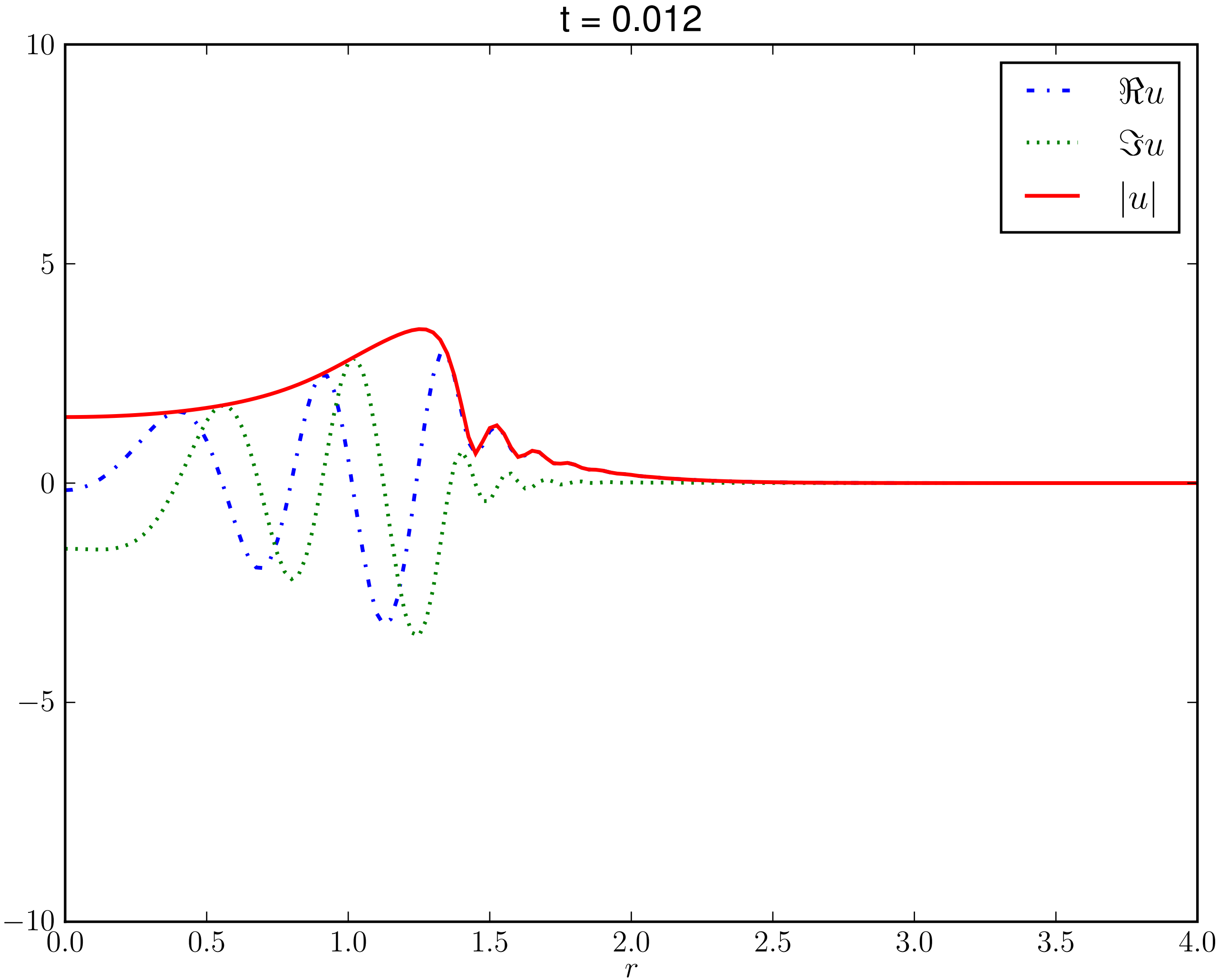
t = 0.004
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



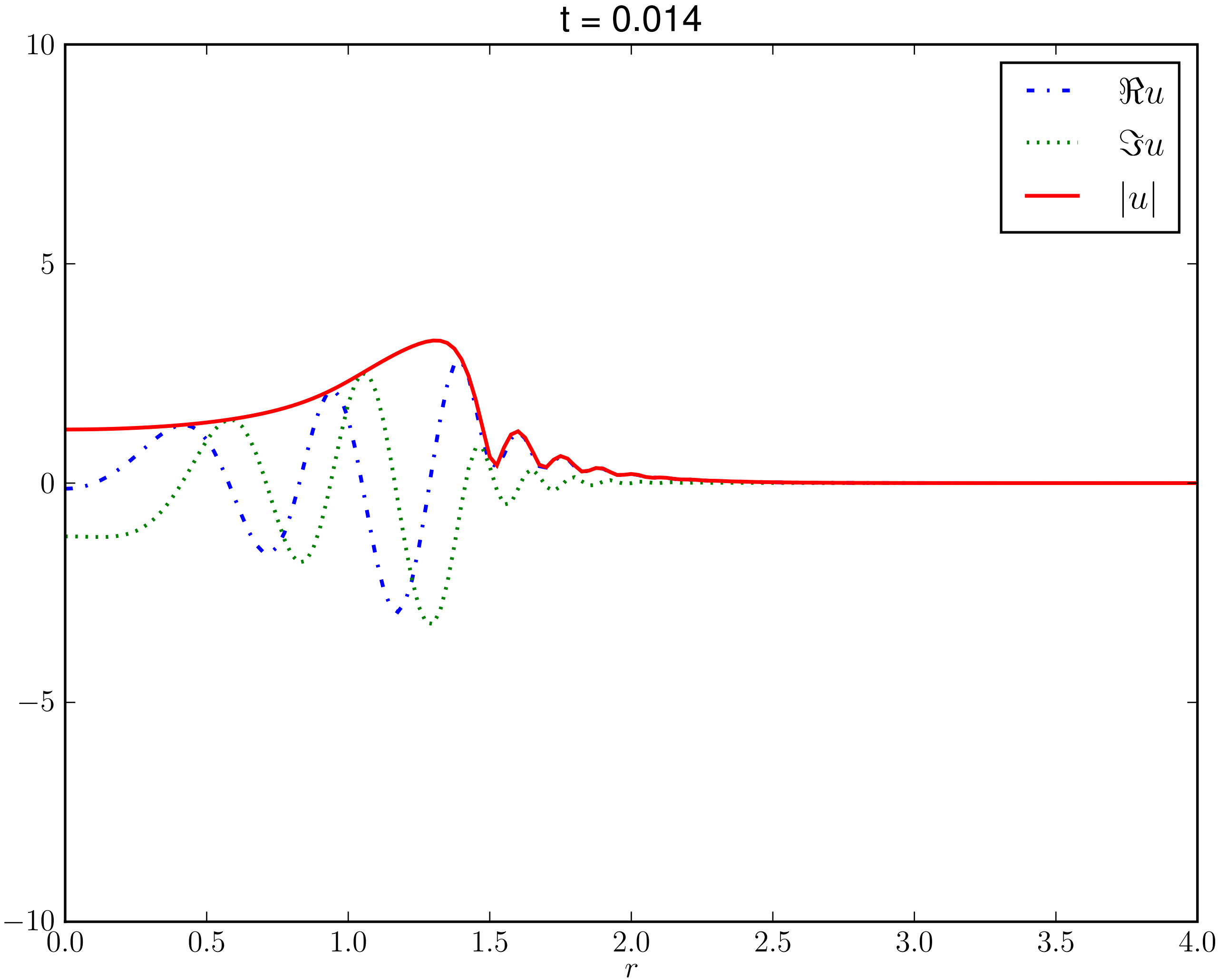
t = 0.006
10
5
0
-5
-10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.008
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

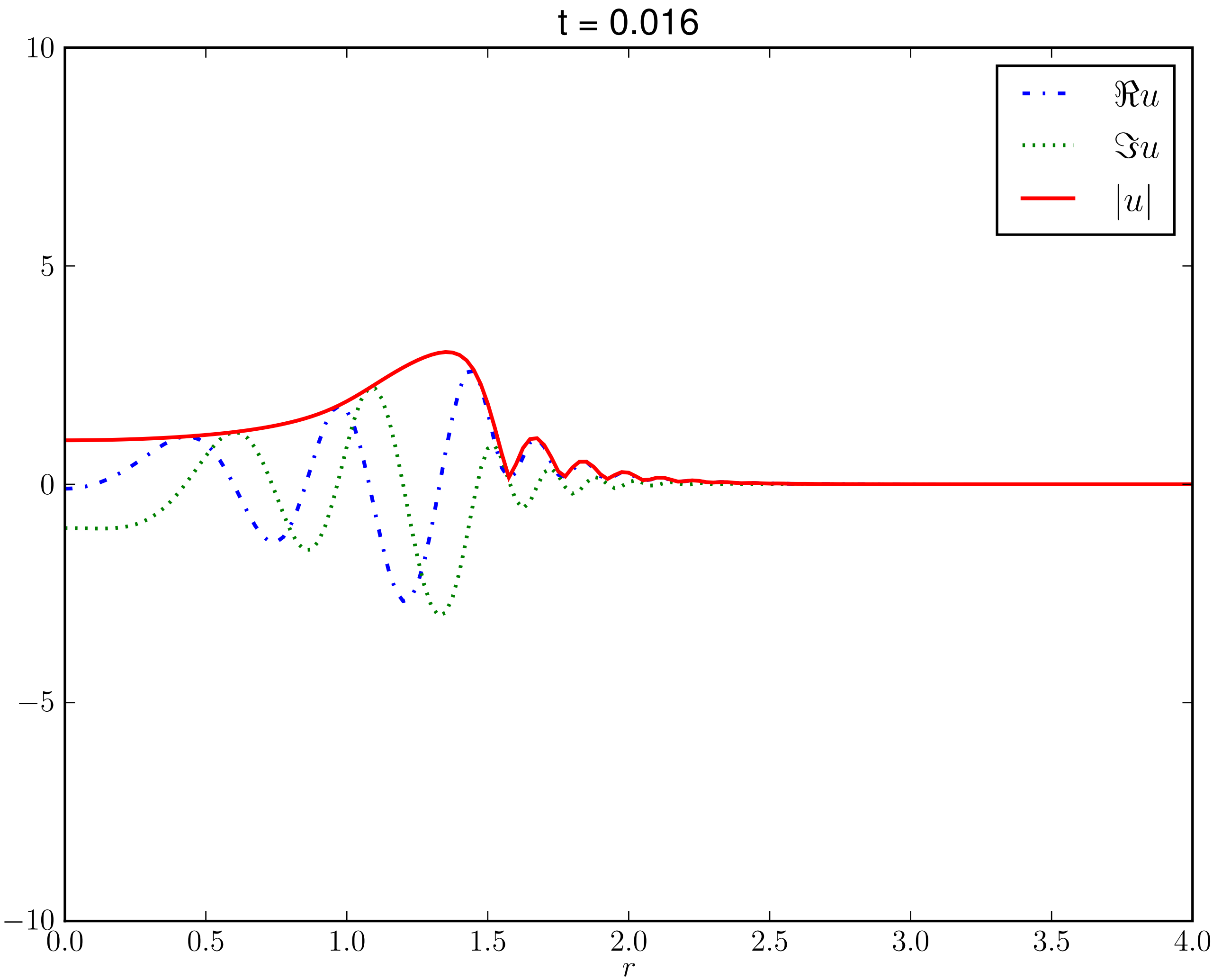

t = 0.01
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

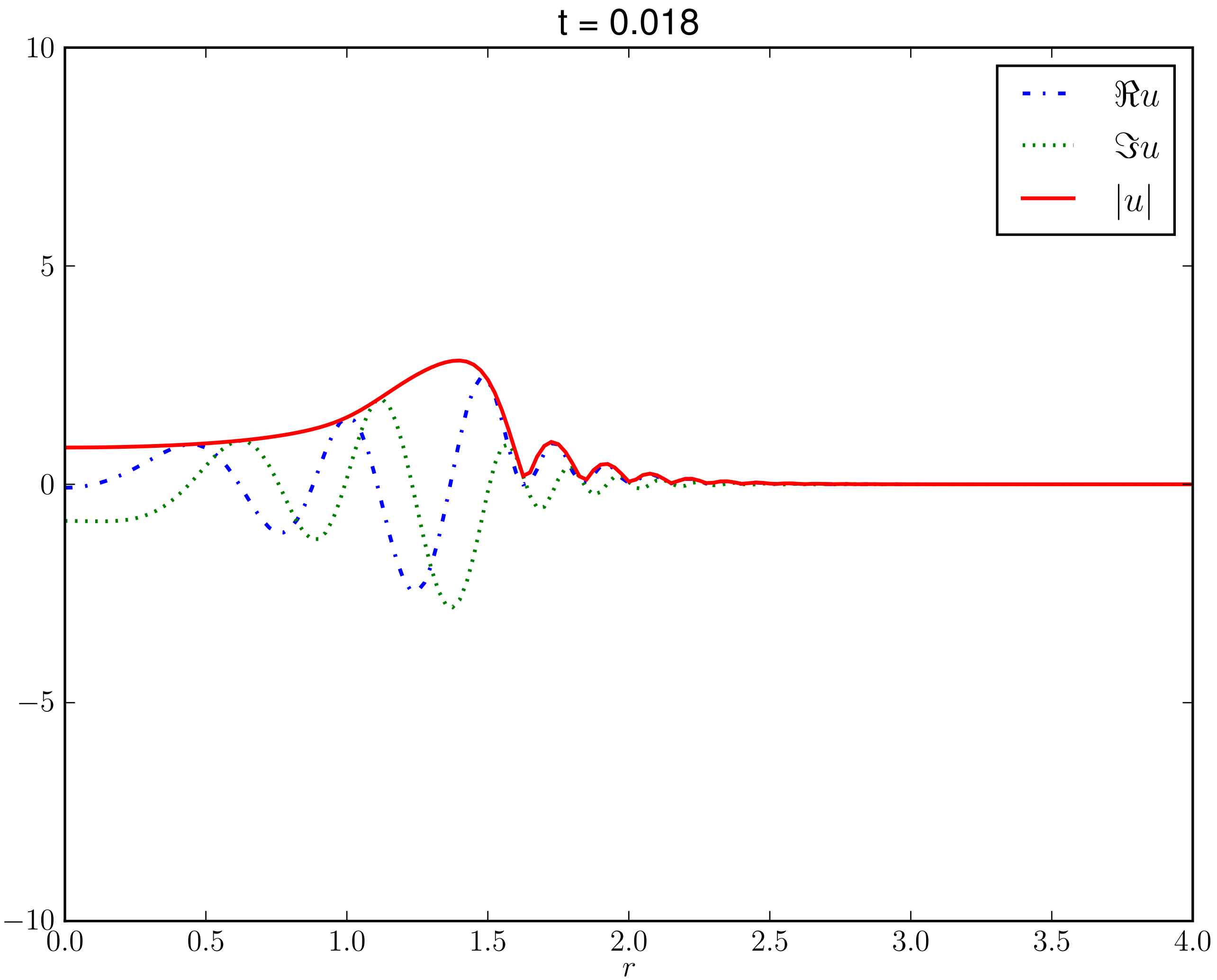


t = 0.012
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

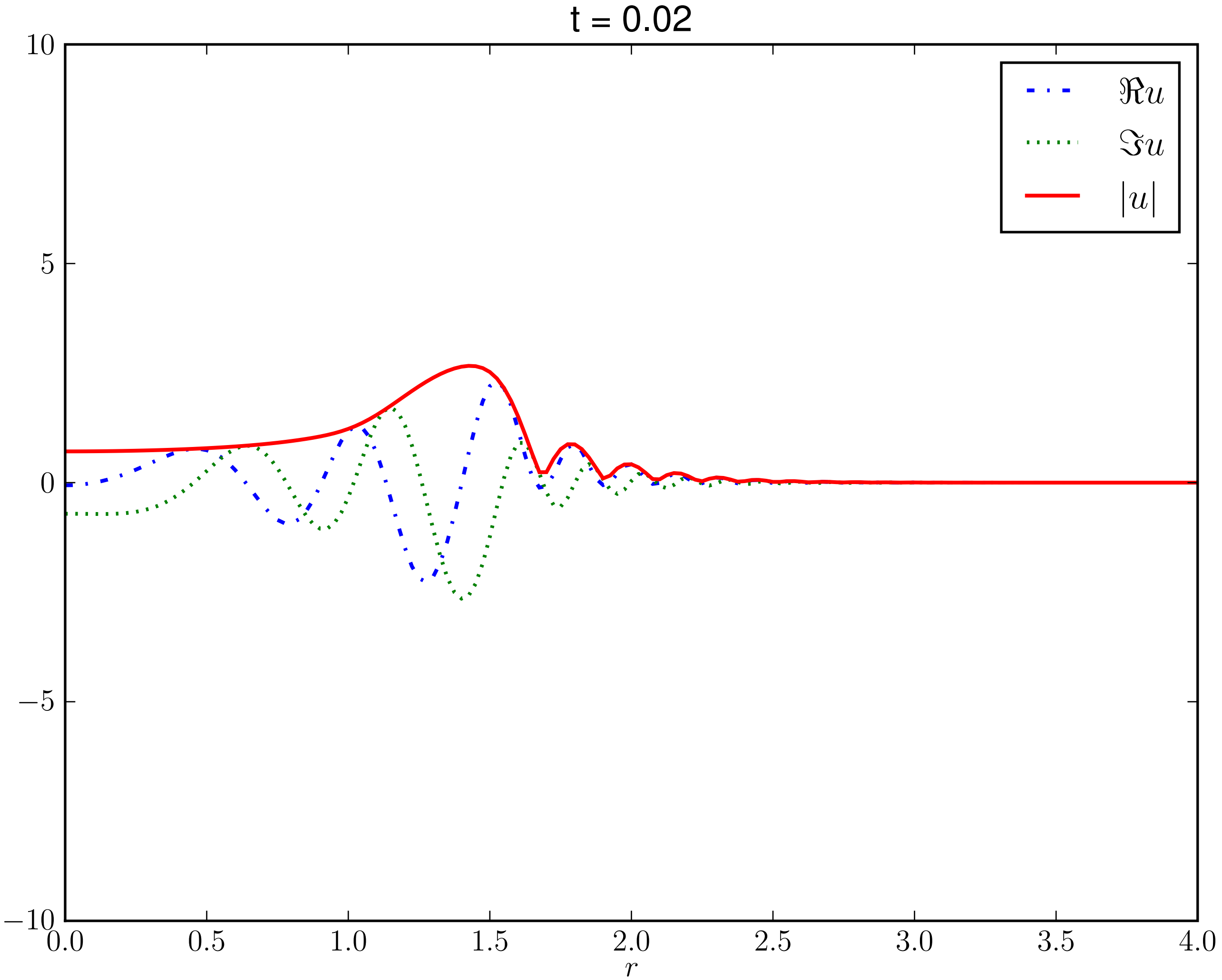


t = 0.014
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.016
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



t = 0.018
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|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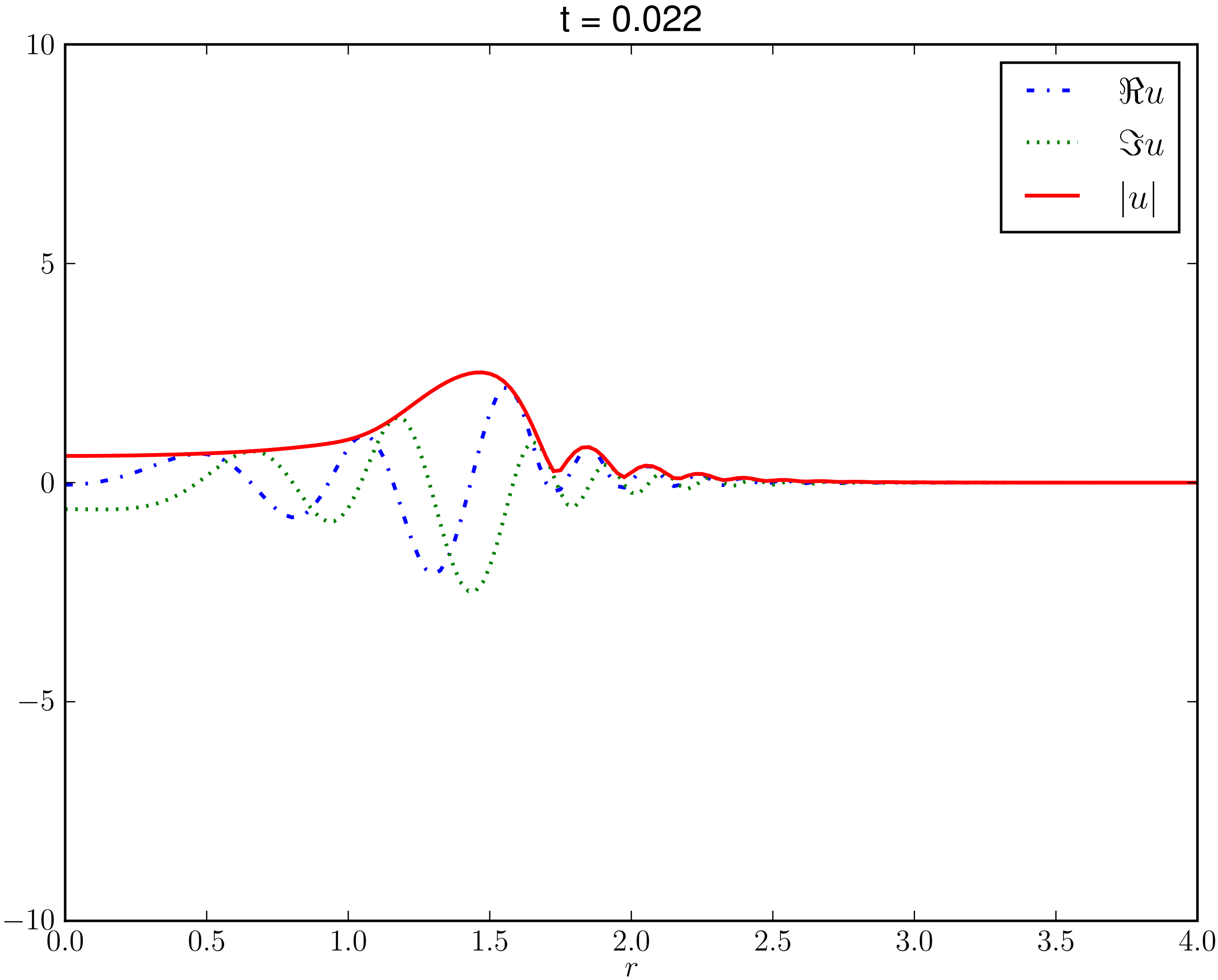
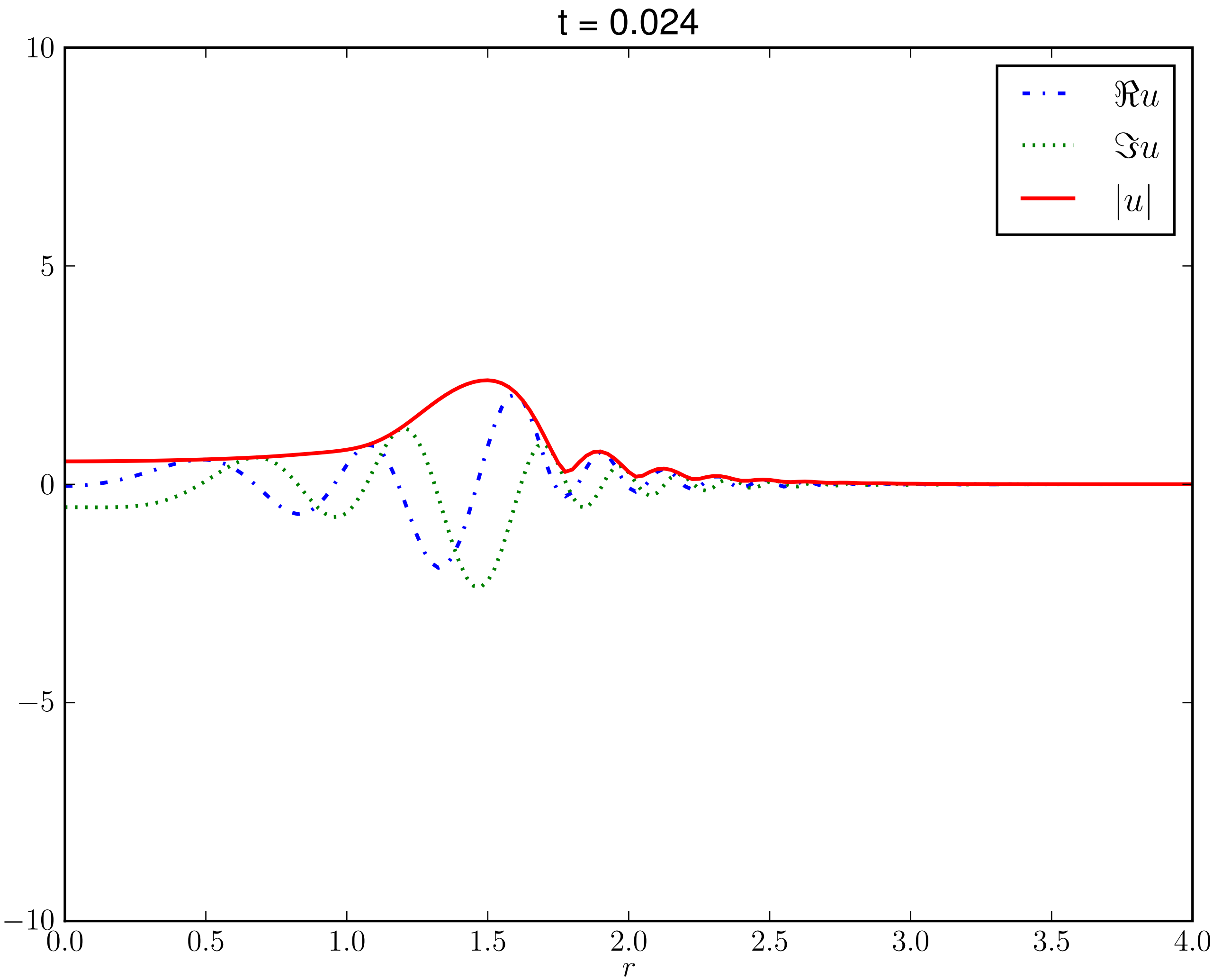
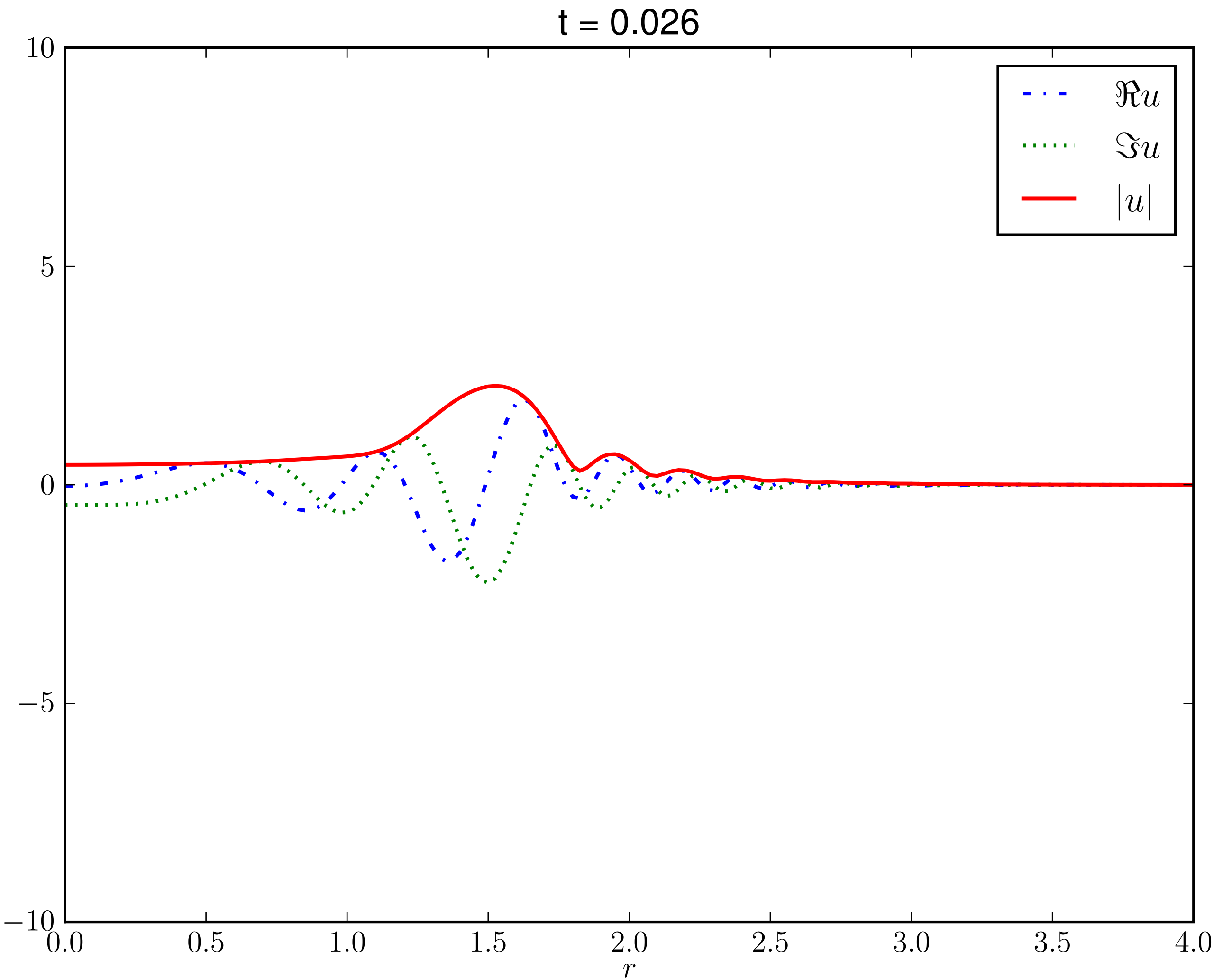
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



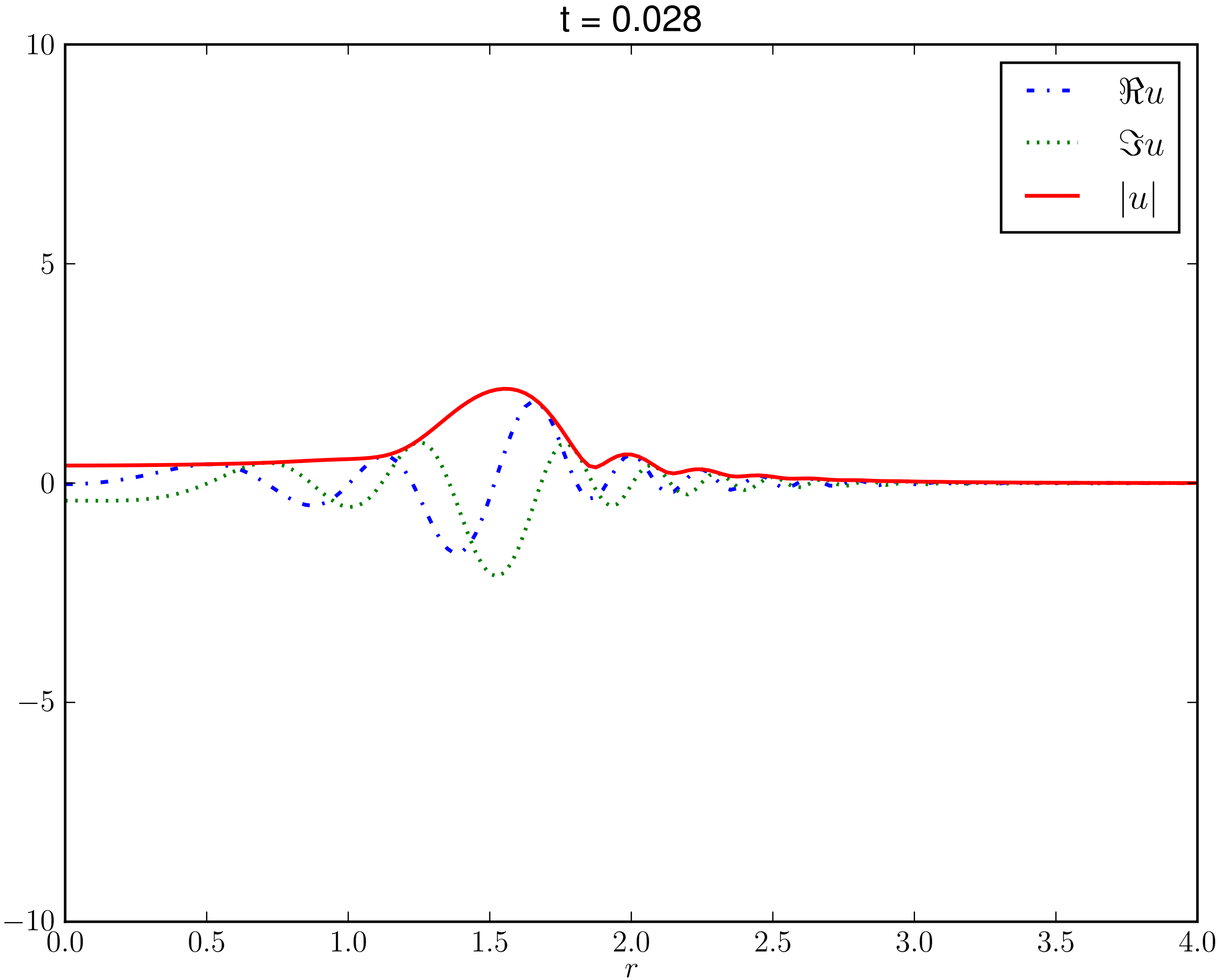
t = 0.022
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

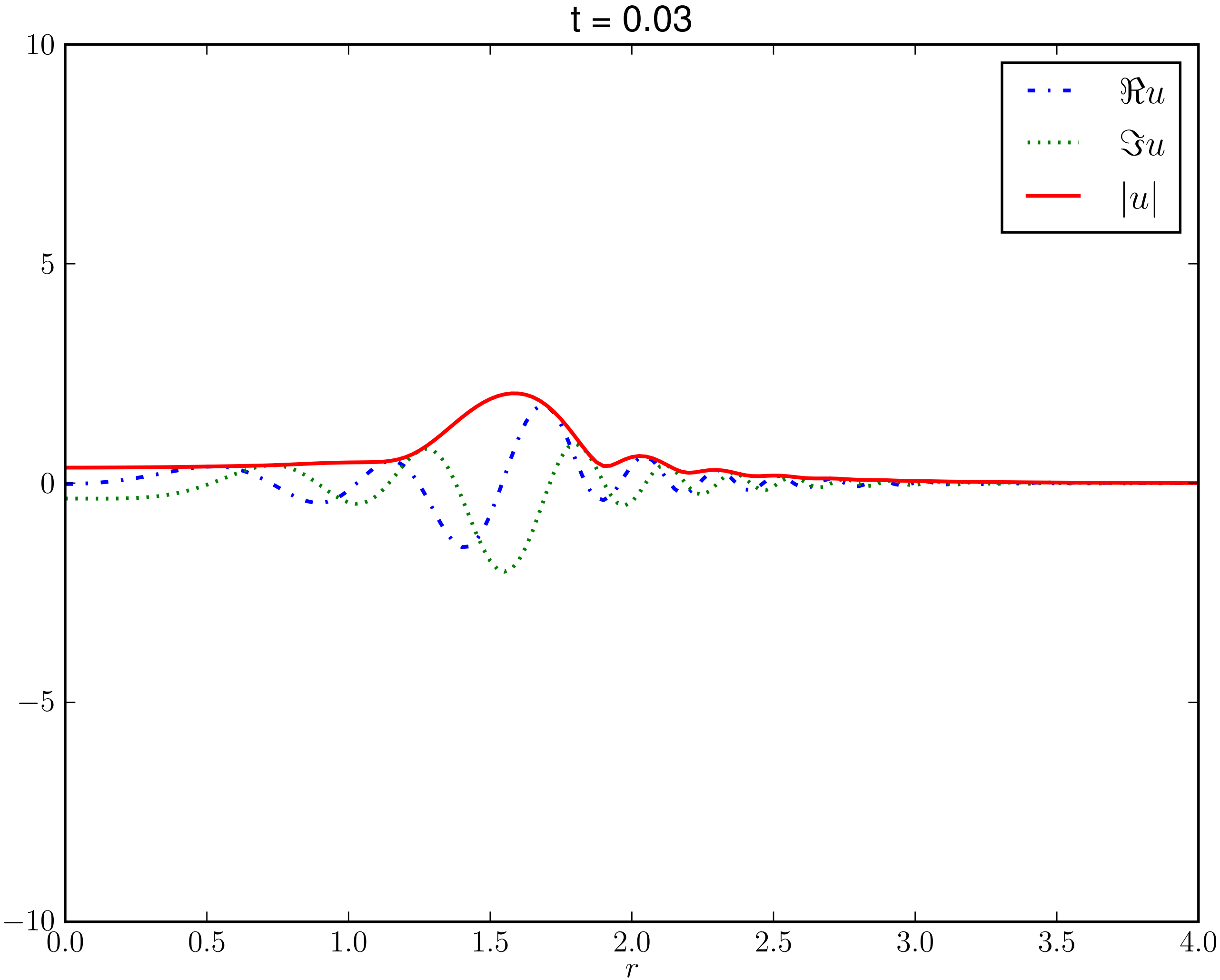


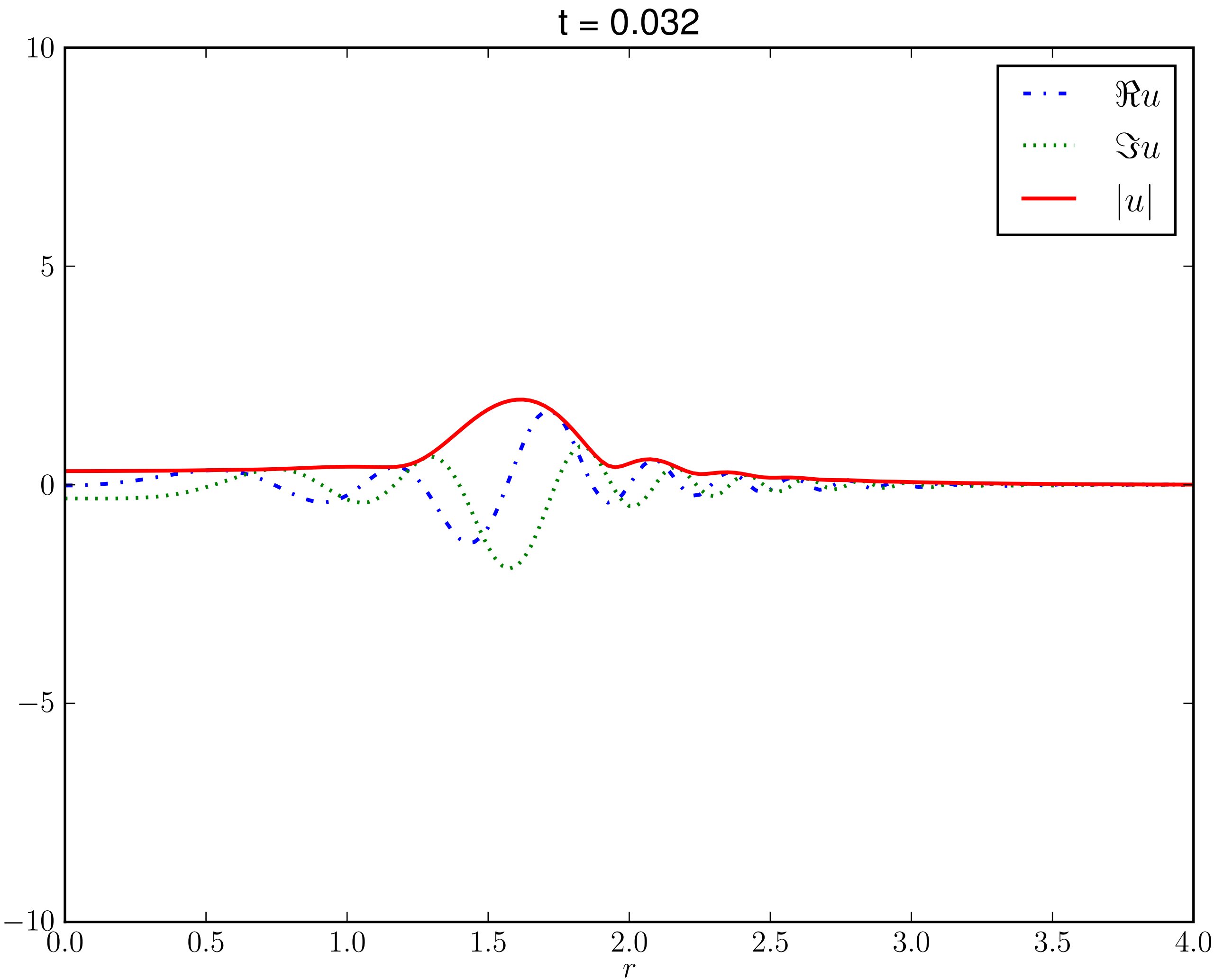
t = 0.024
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

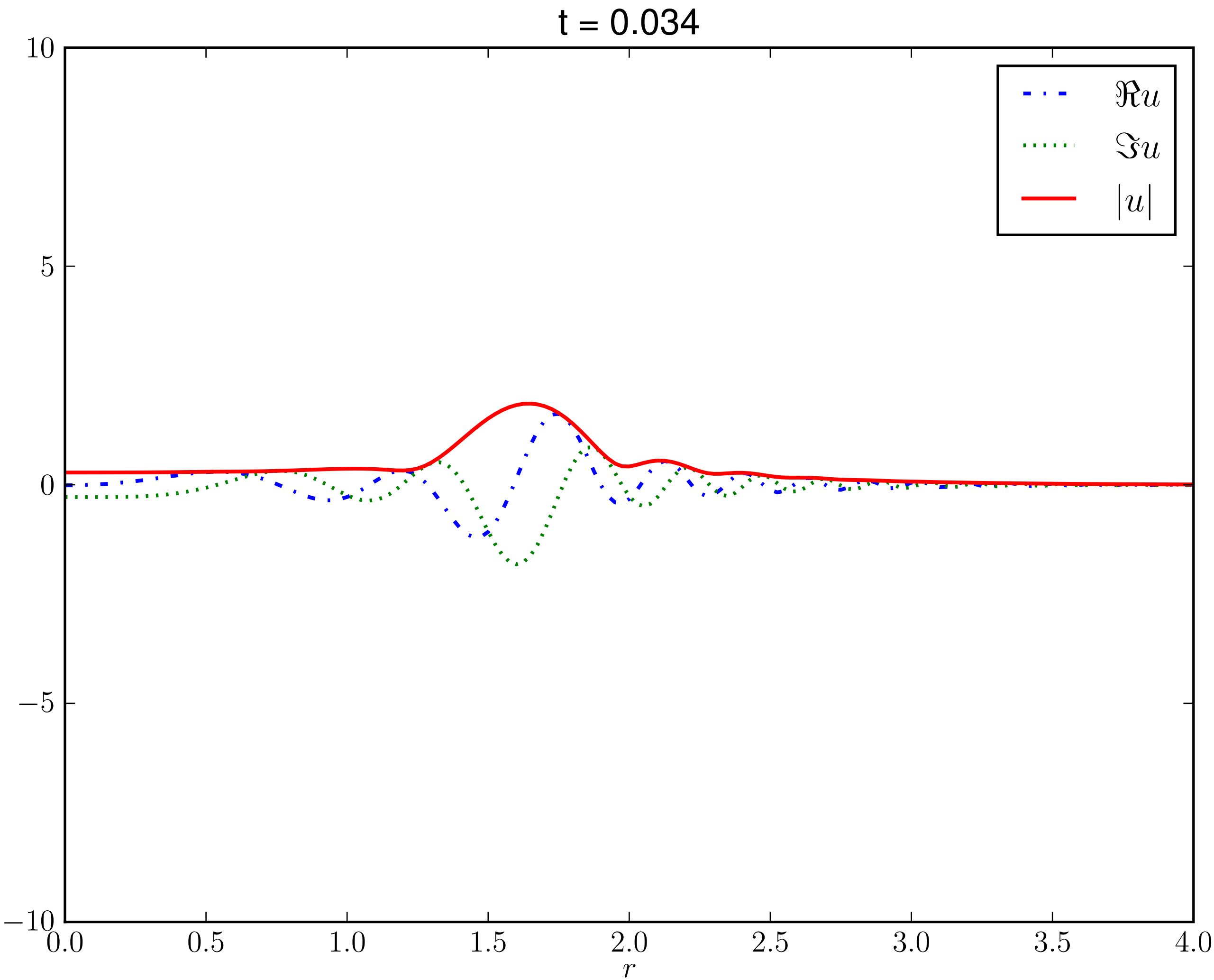


t = 0.026
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

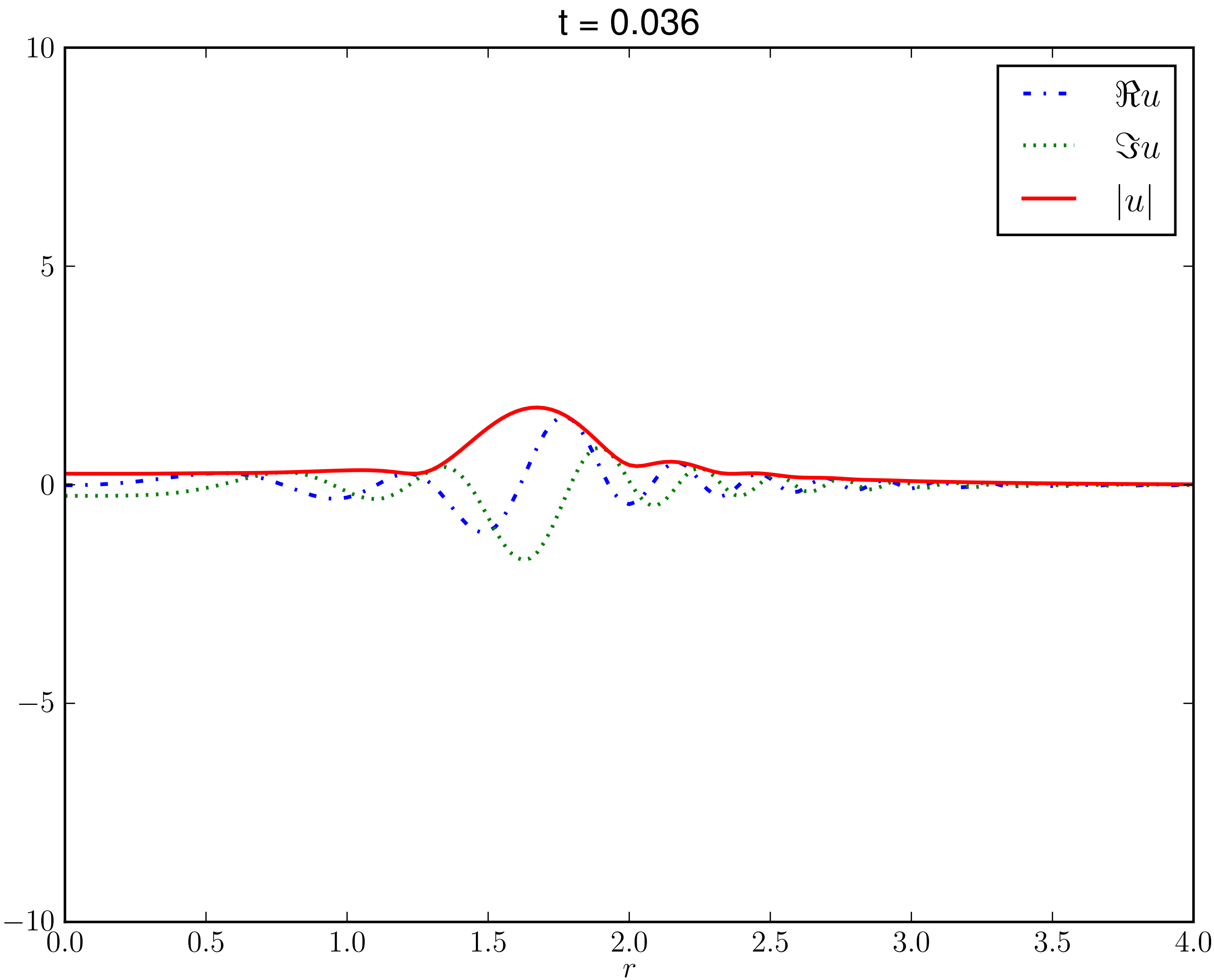

t = 0.028
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.03
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

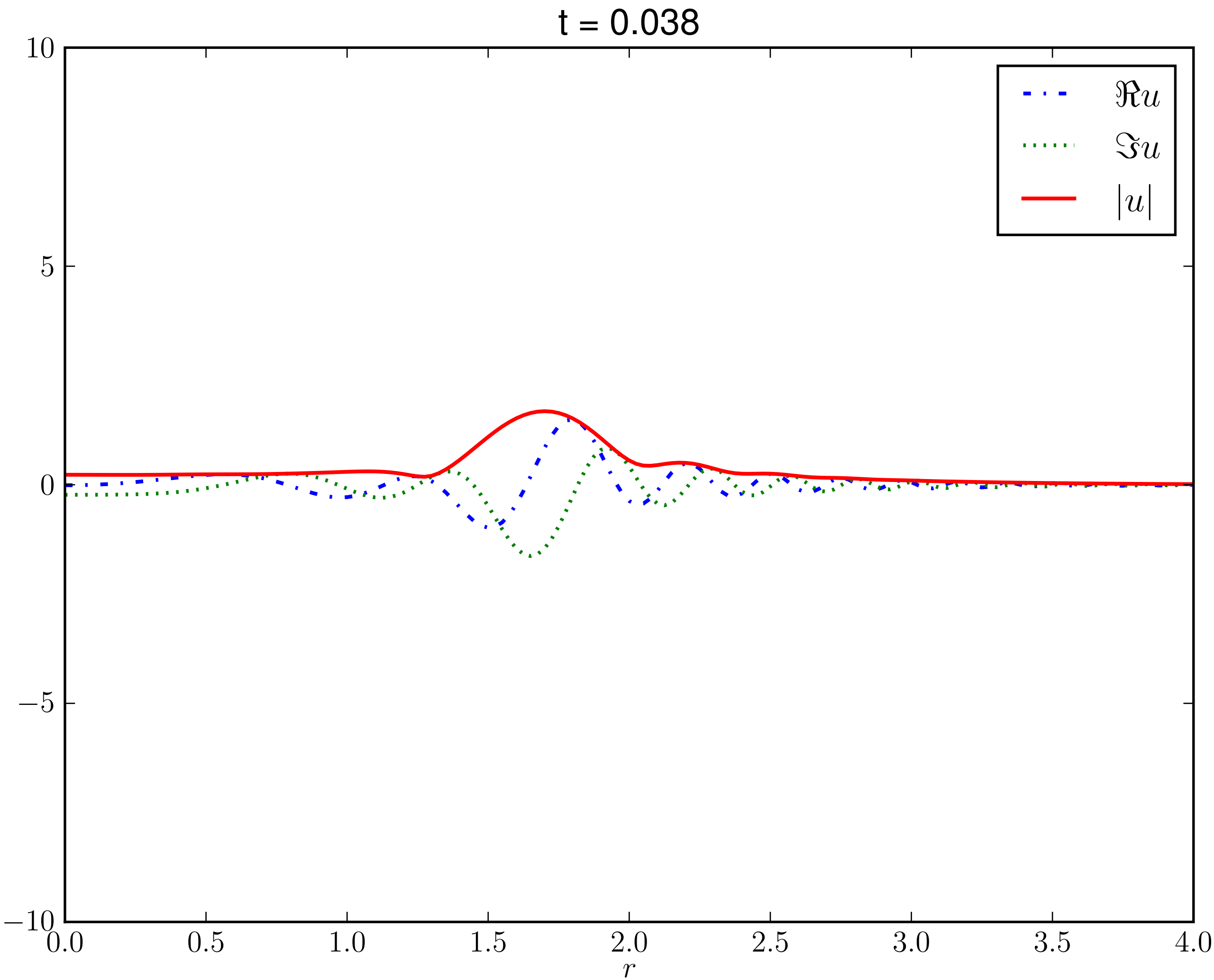

t = 0.032
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

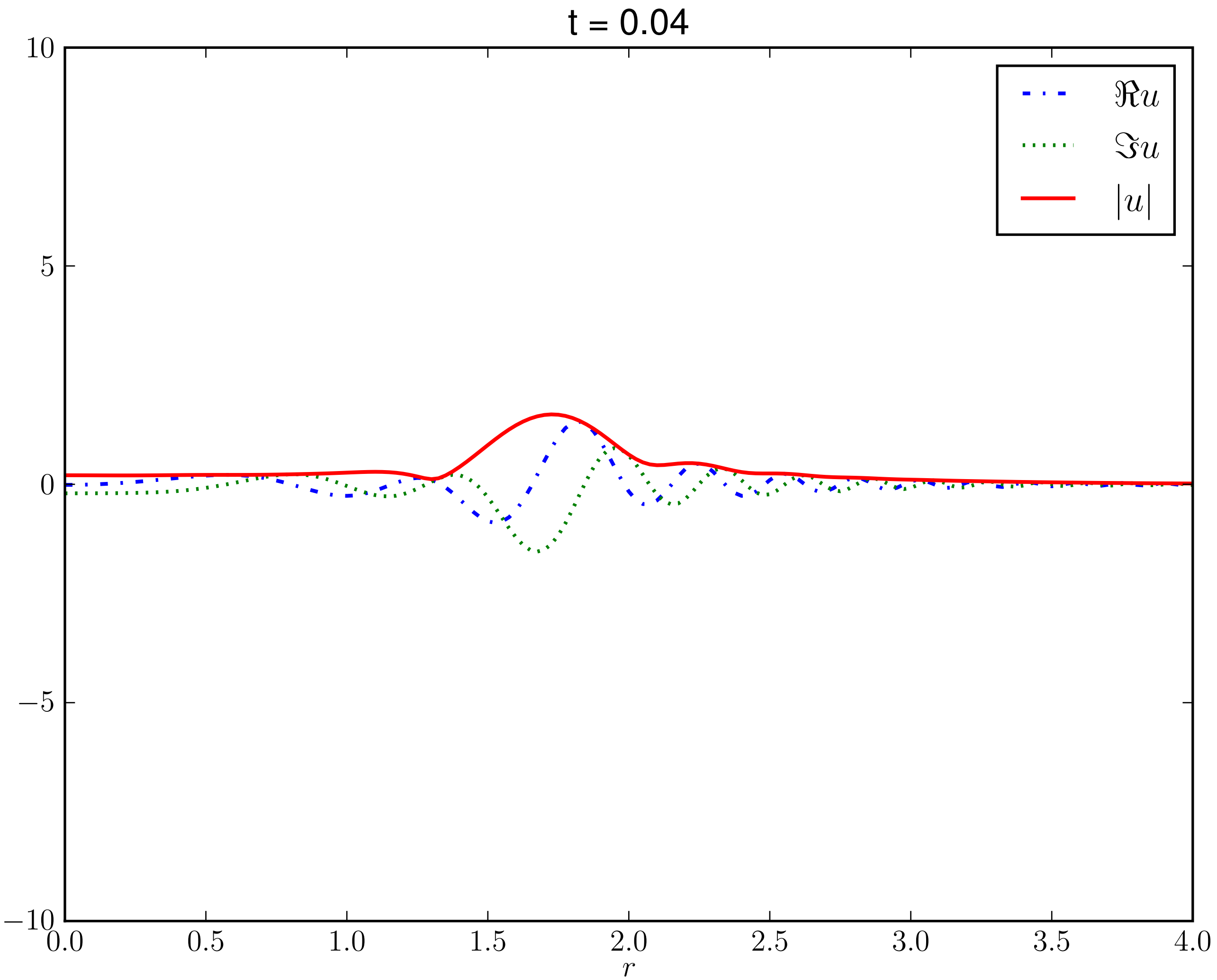


t = 0.034
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

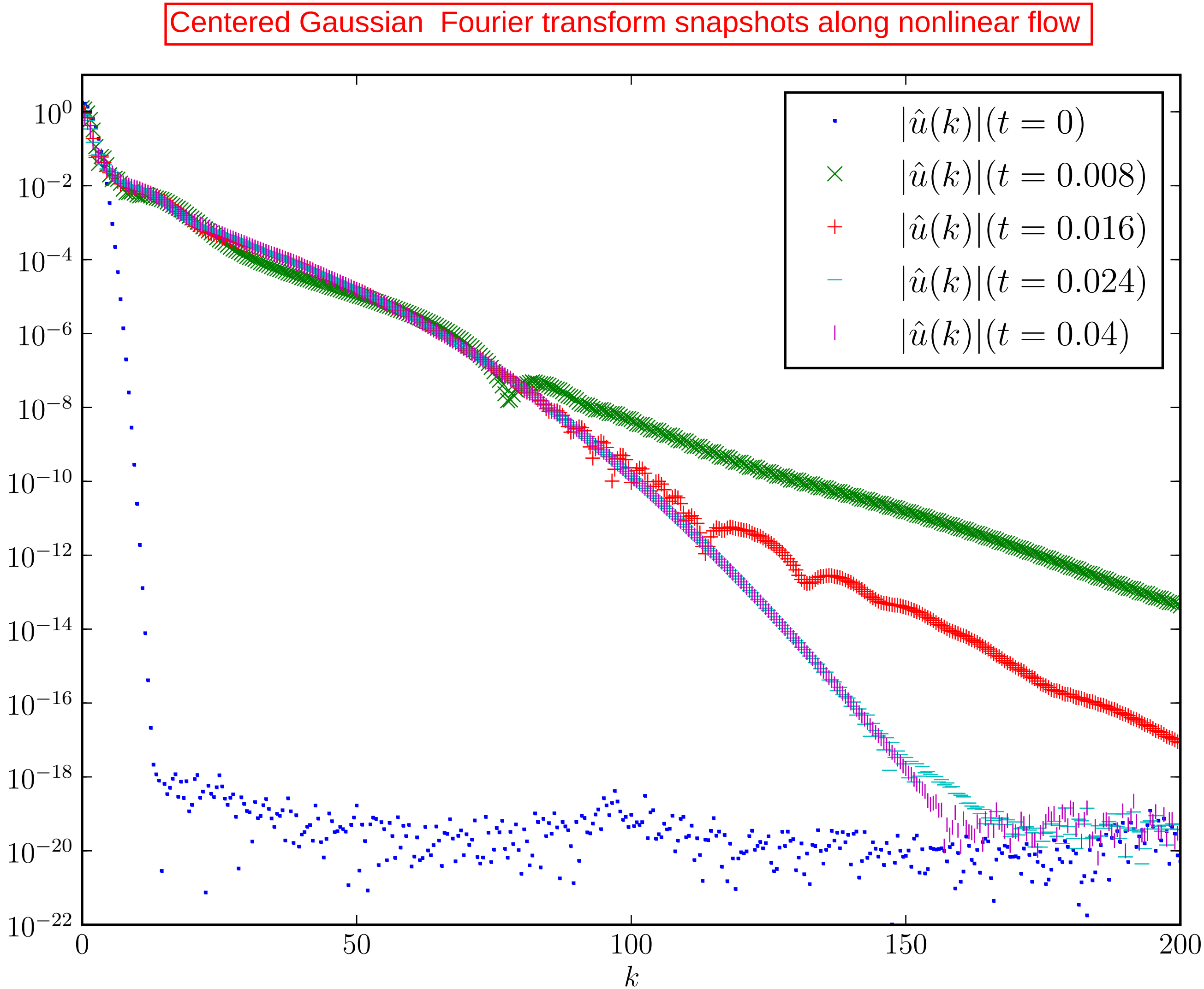


t = 0.036
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

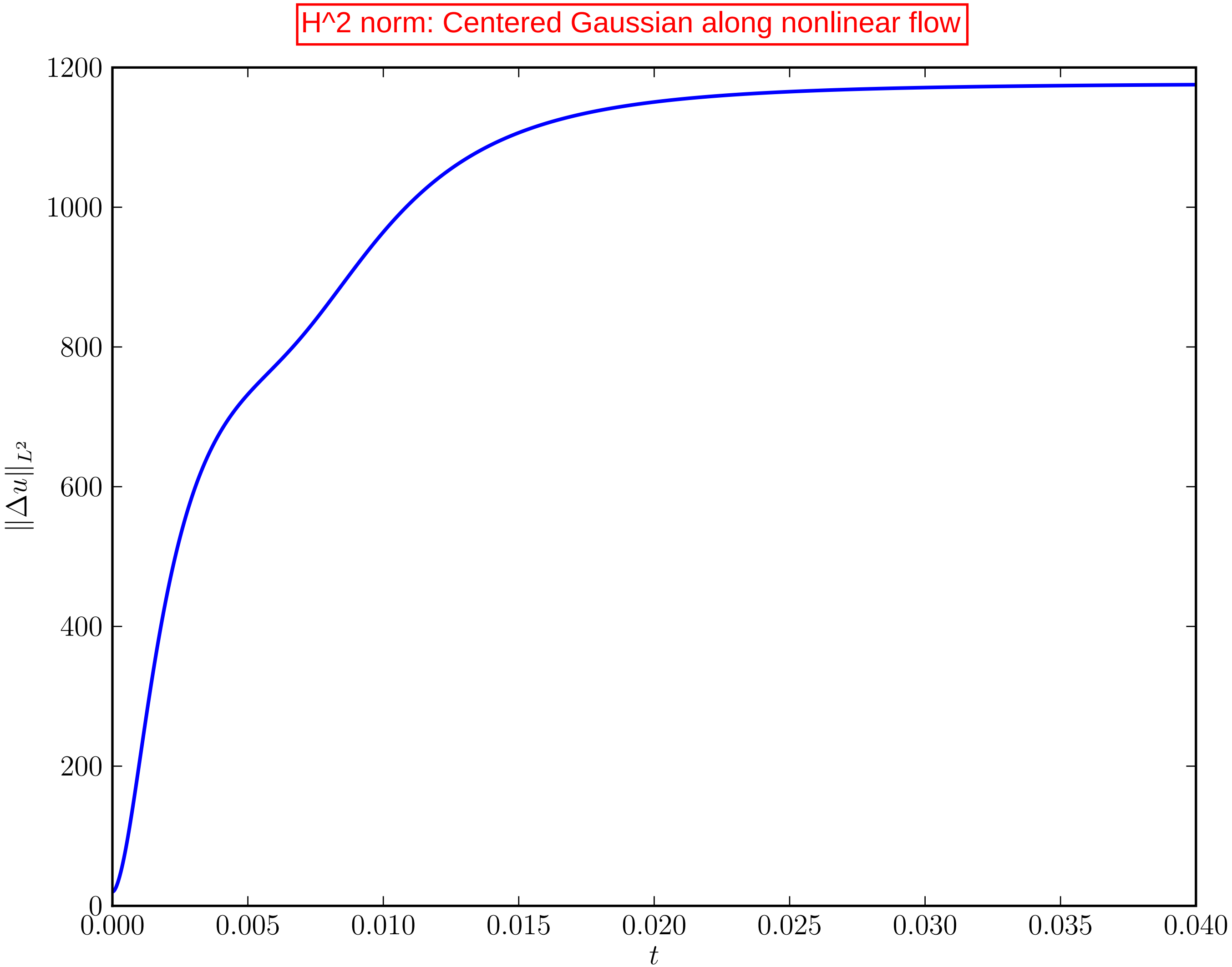

t = 0.038
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

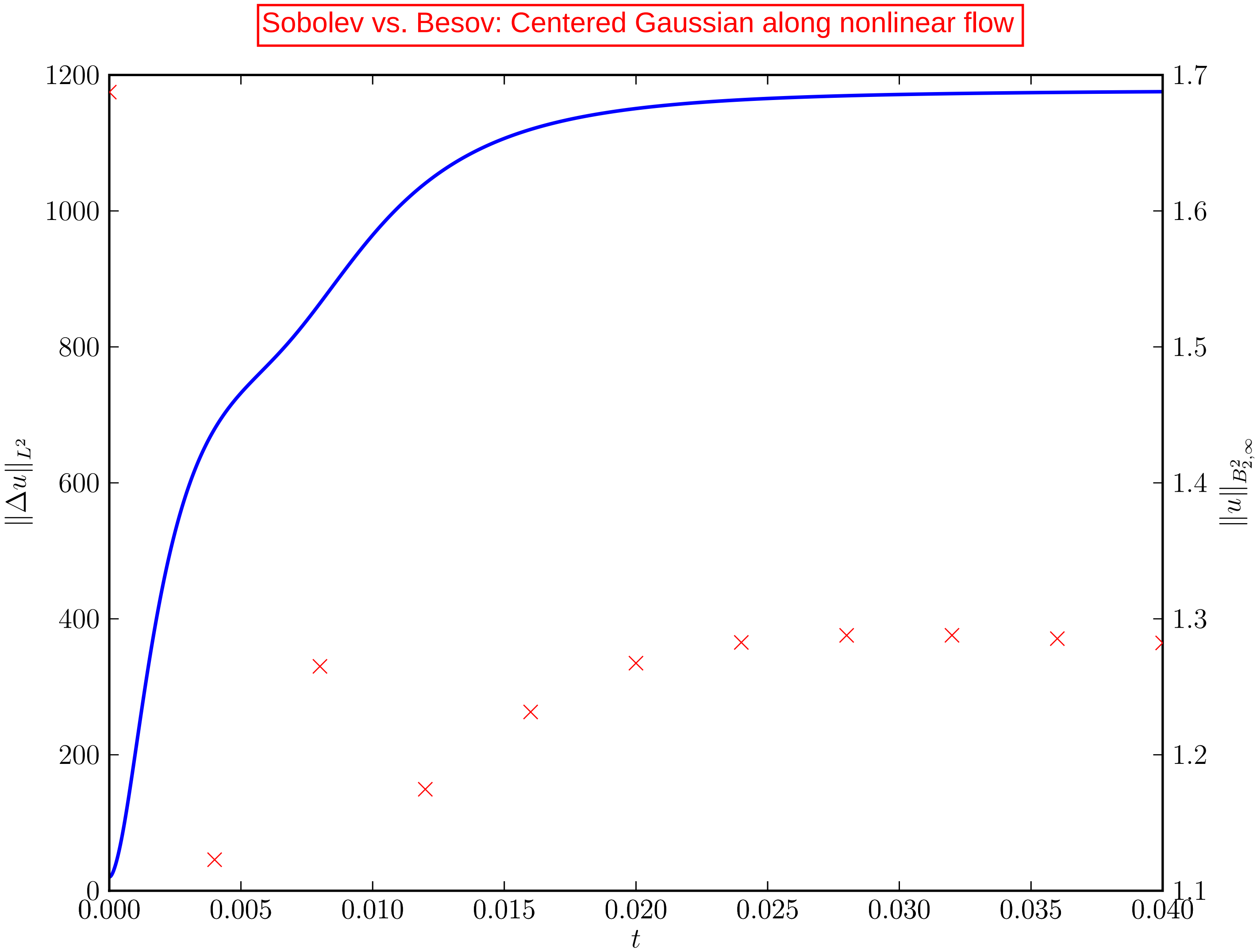


t = 0.04
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

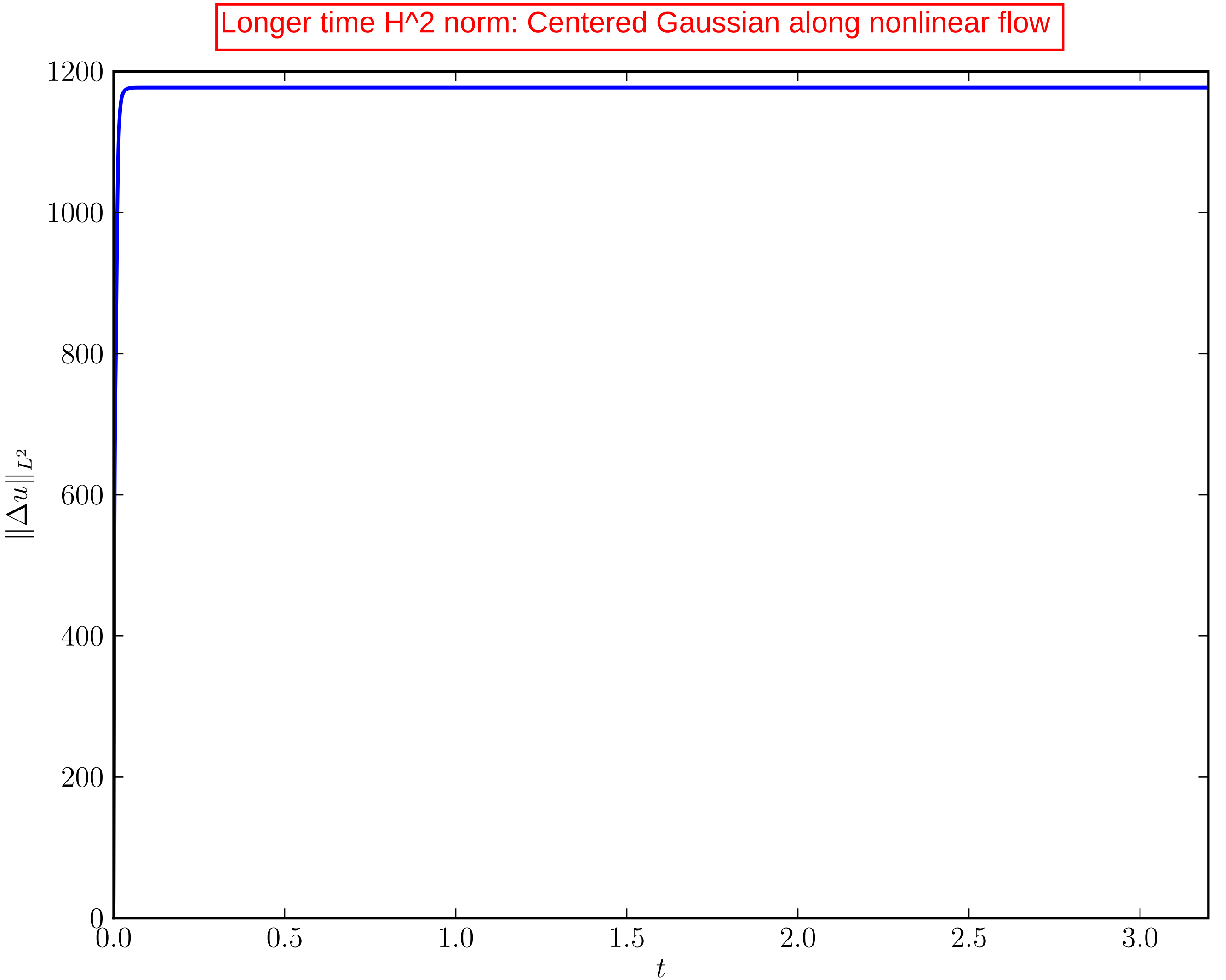


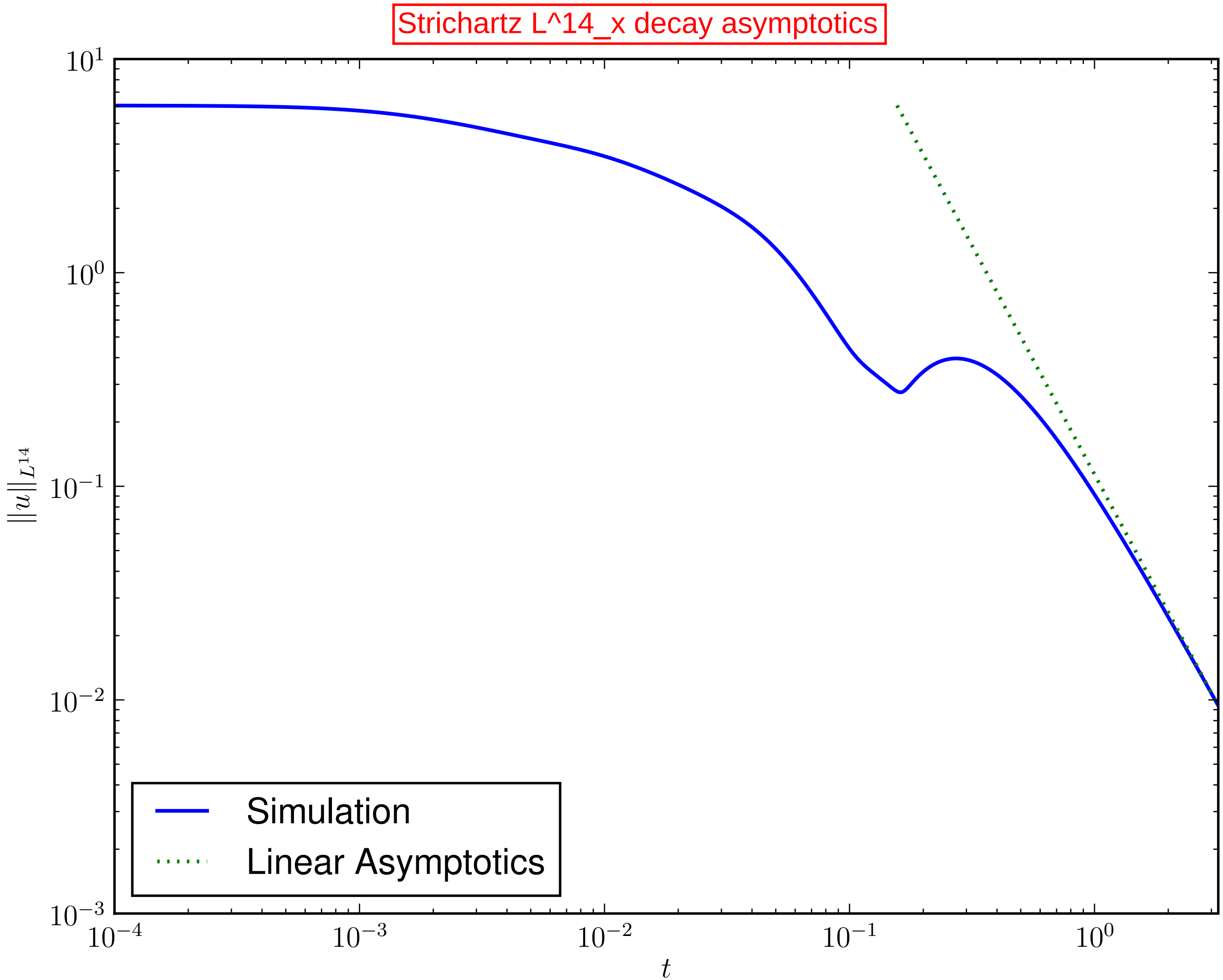
Centered Gaussian Fourier transform snapshots along nonlinear flow
$|\hat{u}(k)|(t = 0)$
$|\hat{u}(k)|(t = 0.008)$
$|\hat{u}(k)|(t = 0.016)$
$|\hat{u}(k)|(t = 0.024)$
$|\hat{u}(k)|(t = 0.04)$
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k

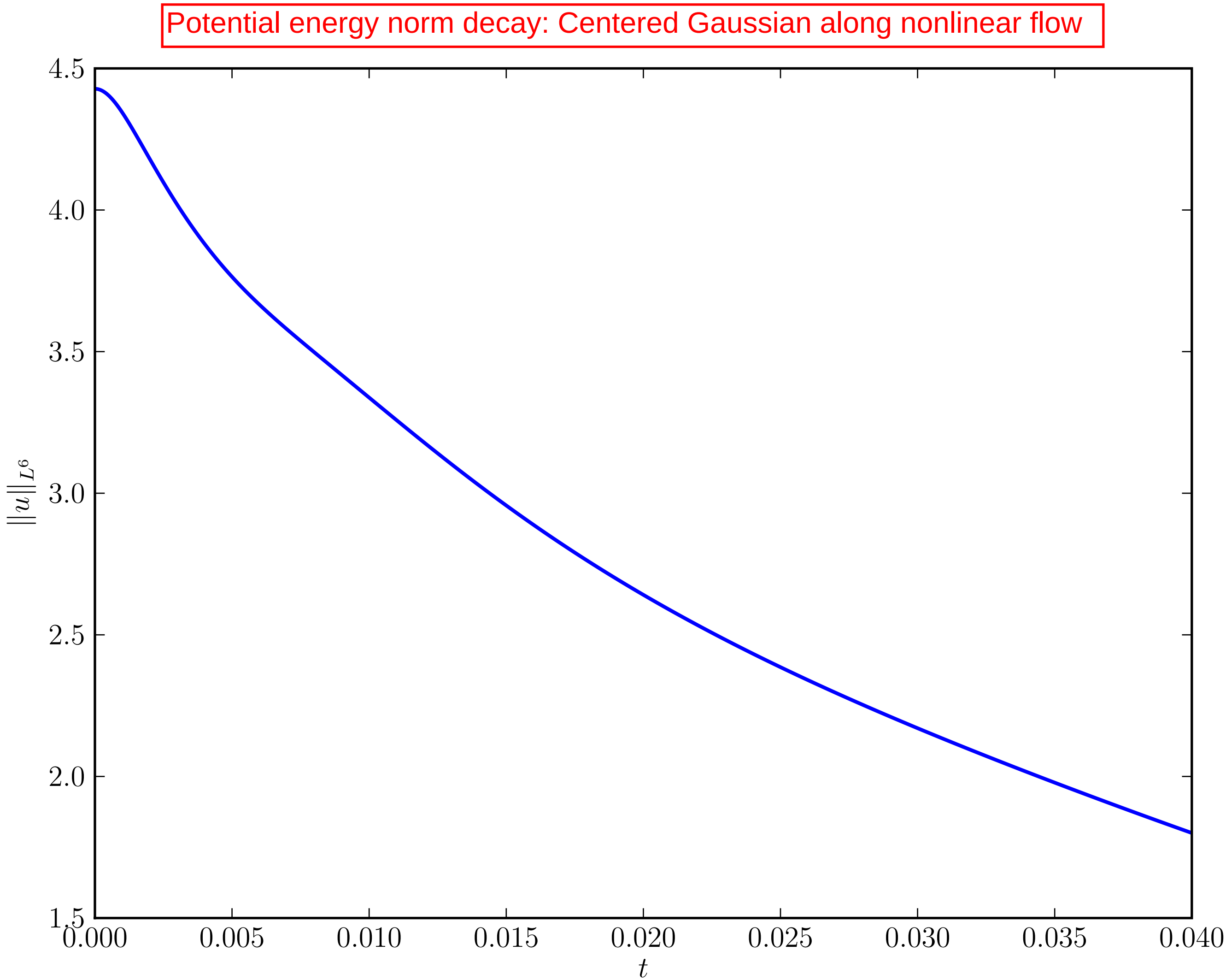

H^2 norm: Centered Gaussian along nonlinear flow
1200
1000
800
600
400
200
0
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t
$\|\Delta u\|_{L^2}$

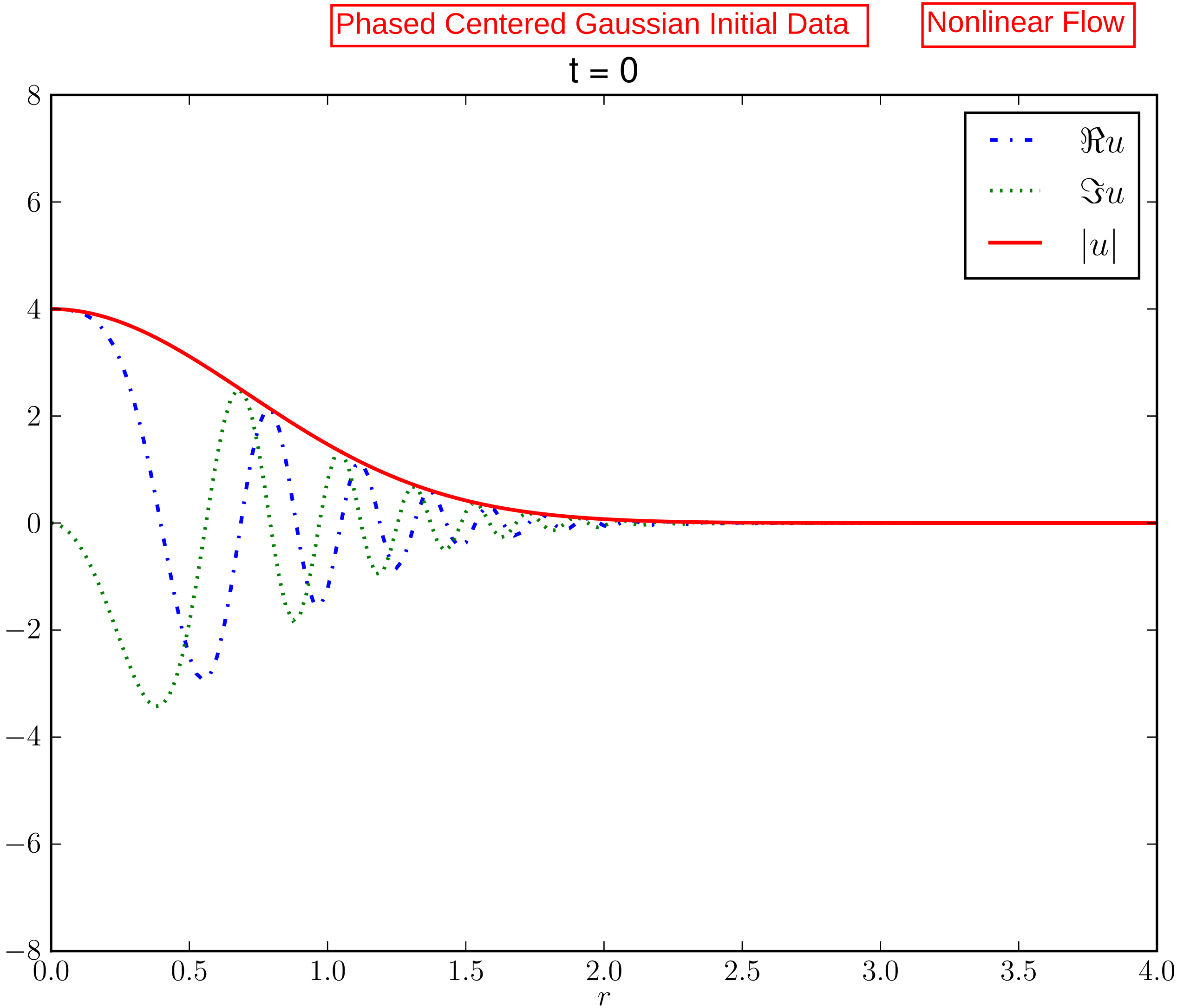


Sobolev vs. Besov: Centered Gaussian along nonlinear flow
1200
1000
800
600
400
200
0
1.7
1.6
1.5
1.4
1.3
1.2
1.1
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t
$\|\Delta u\|_{L^2}$
$\|u\|_{B^2_{2,\infty}}$

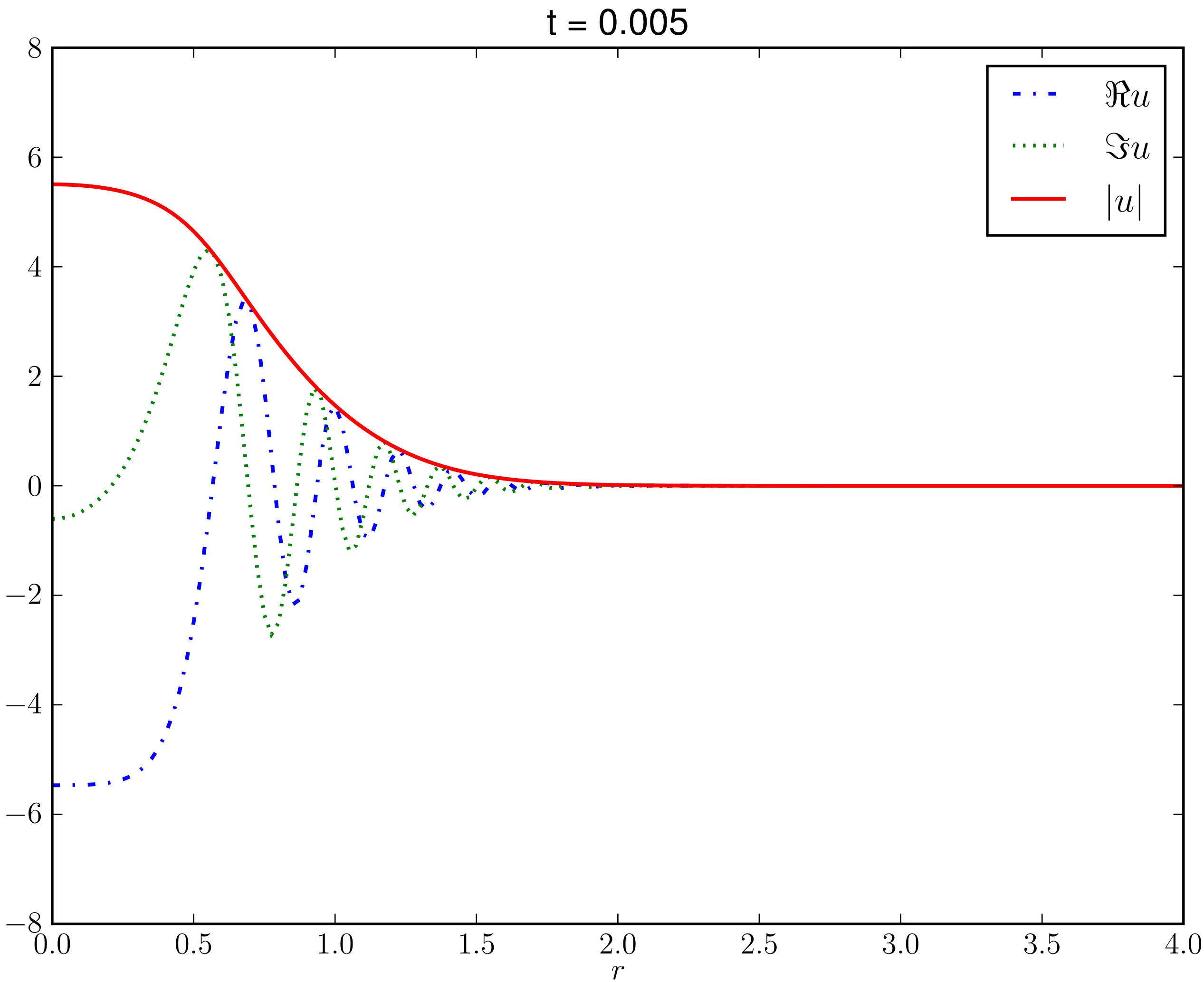

Longer time H^2 norm: Centered Gaussian along nonlinear flow
$\|\Delta u\|_{L^2}$
t
0
200
400
600
800
1000
1200
0.0
0.5
1.0
1.5
2.0
2.5
3.0


Strichartz L^14_x decay asymptotics
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
$\|u\|_{L^{14}}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
t
Simulation
Linear Asymptotics

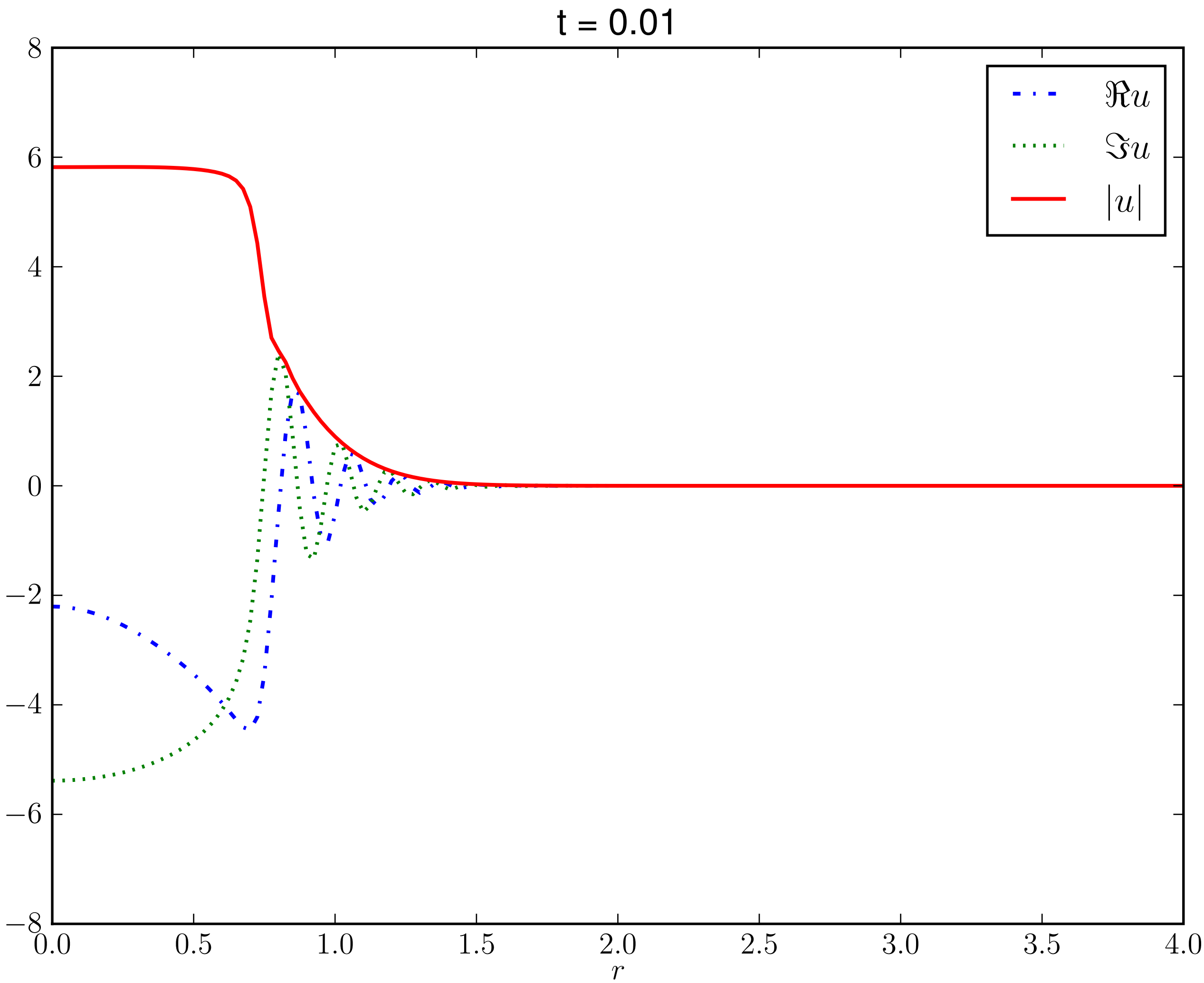

Potential energy norm decay: Centered Gaussian along nonlinear flow
4.5
4.0
3.5
3.0
2.5
2.0
1.5
$\|u\|_{L^6}$
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t



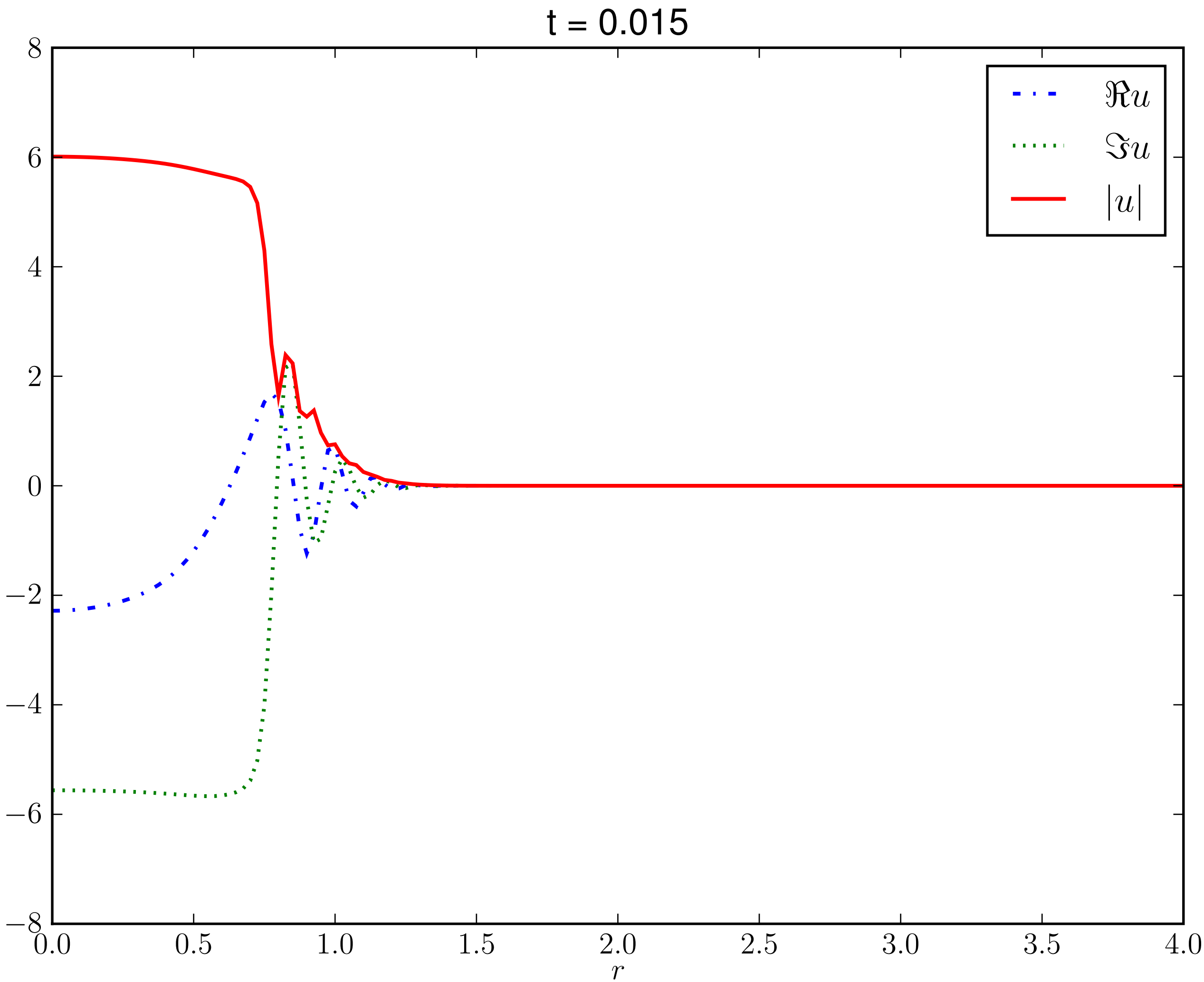
Phased Centered Gaussian Initial Data
Nonlinear Flow
t = 0
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r



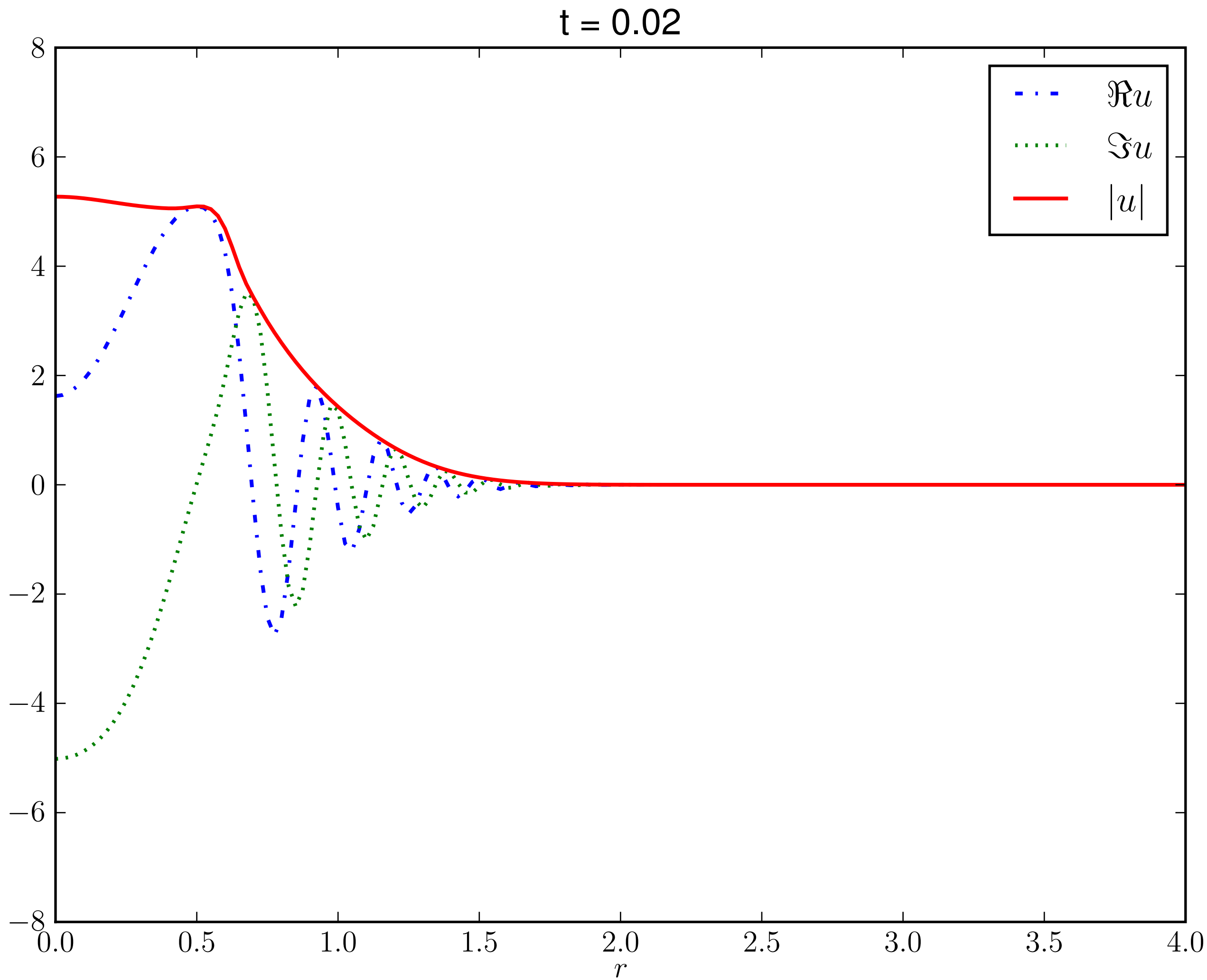
t = 0.005
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



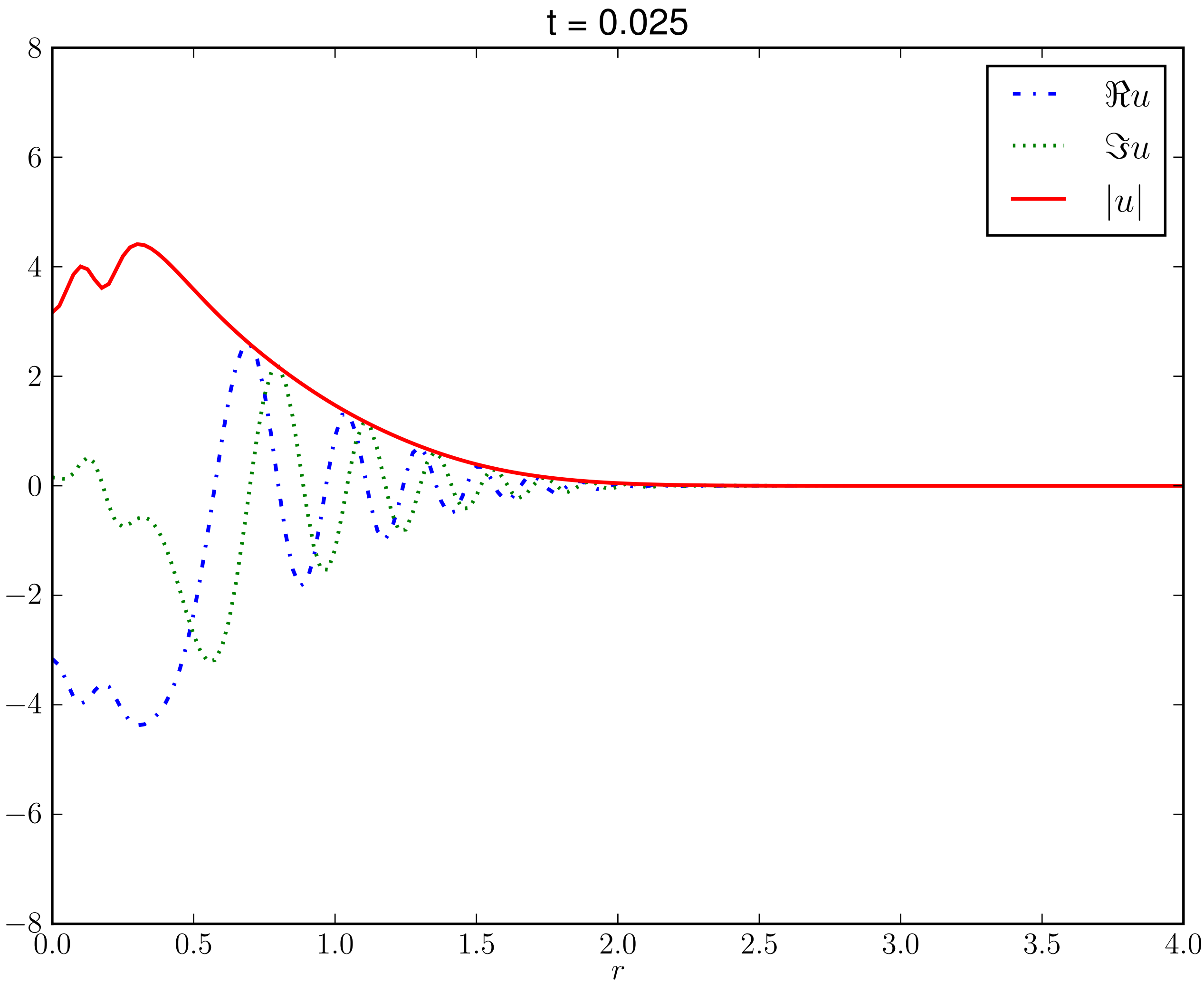
t = 0.01
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



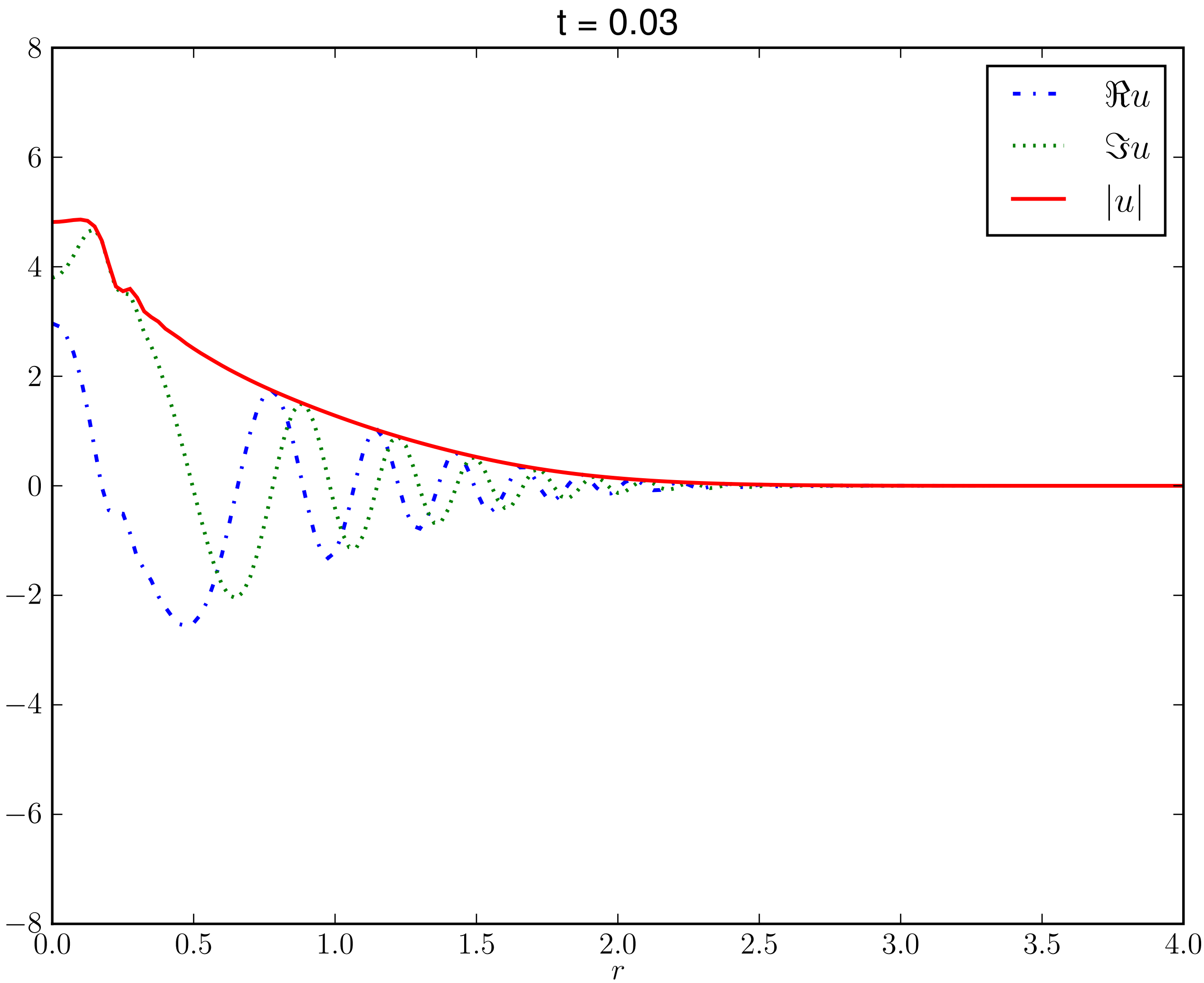
t = 0.015
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



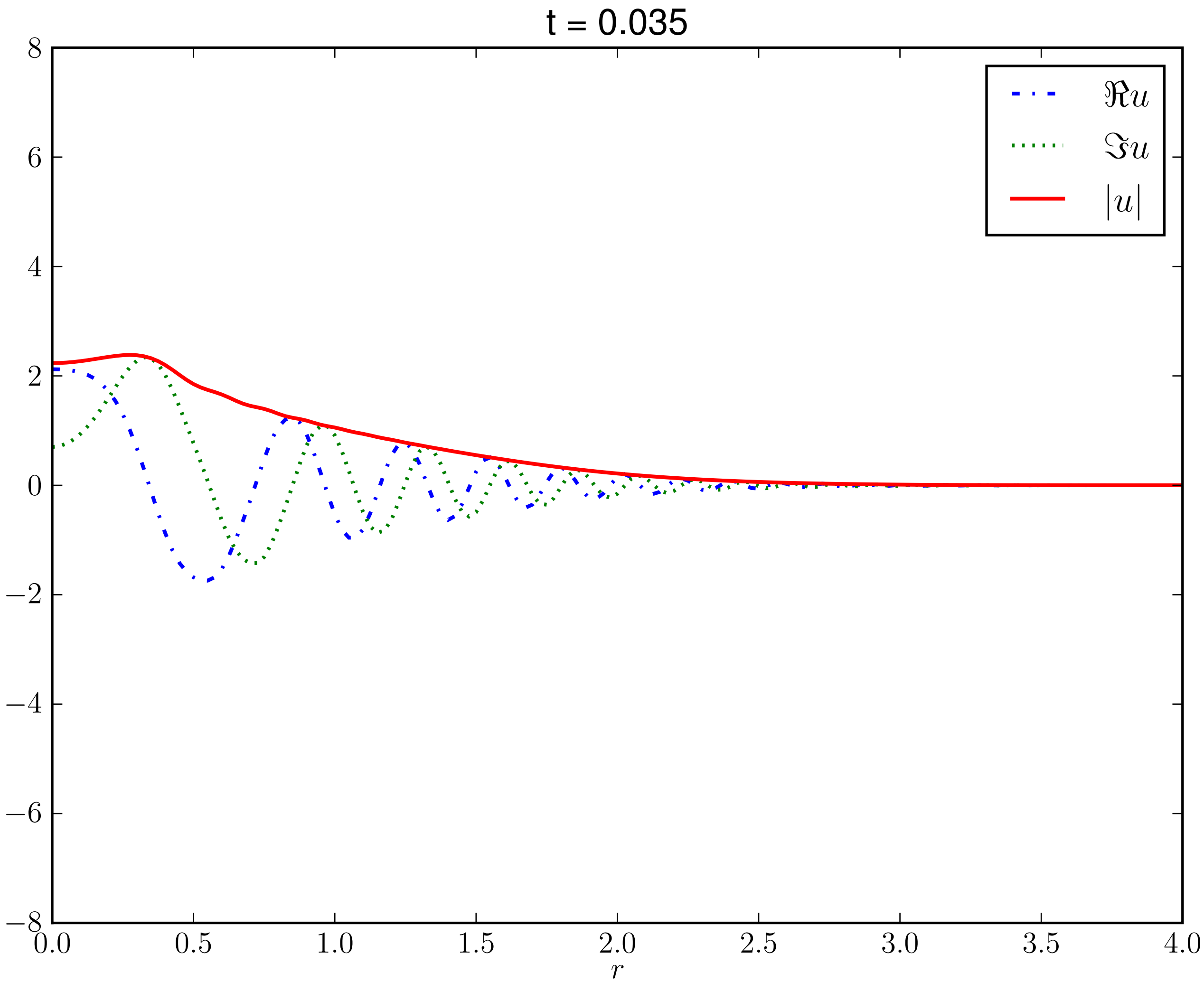
t = 0.02
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

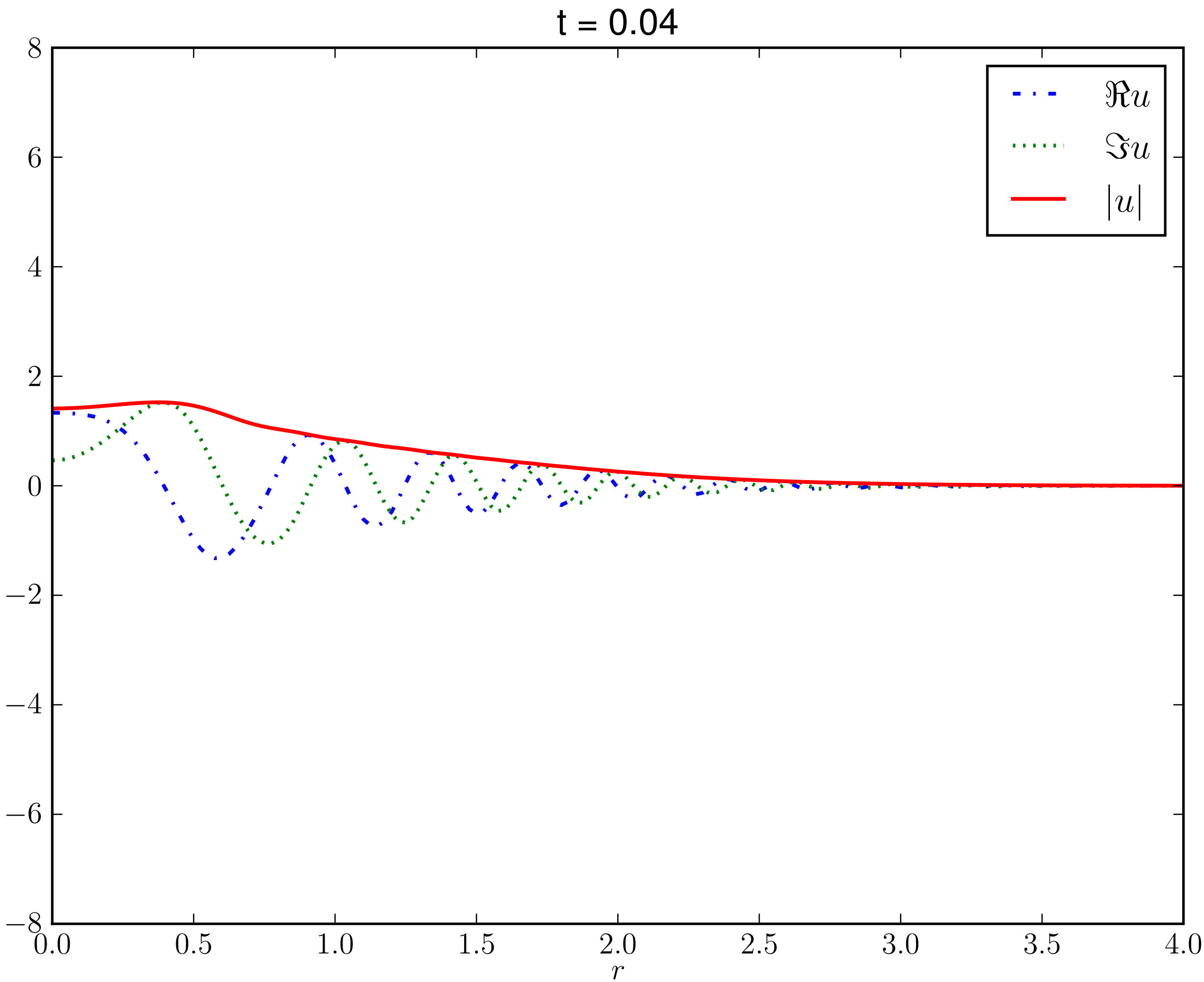


t = 0.025
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

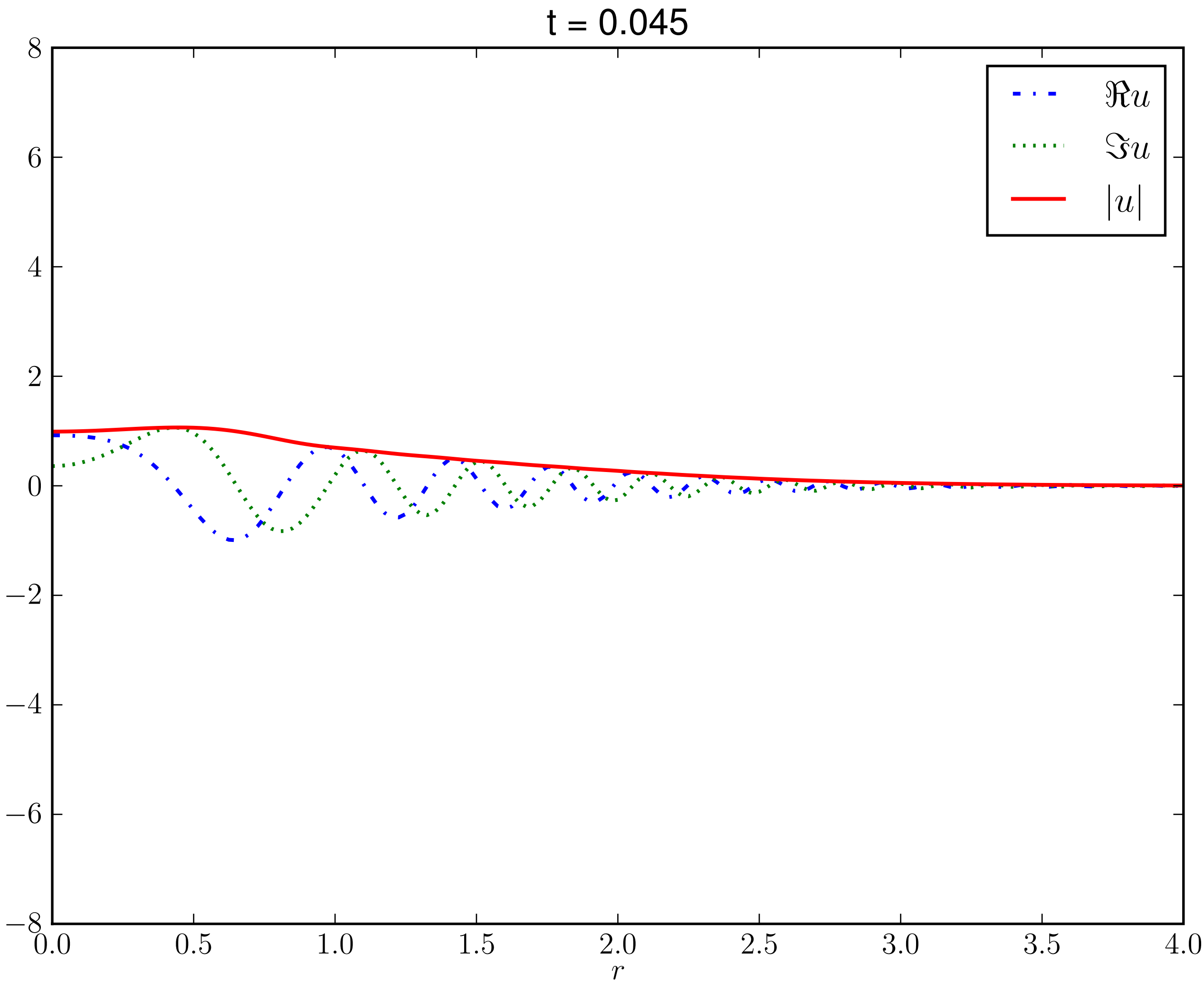


t = 0.03
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

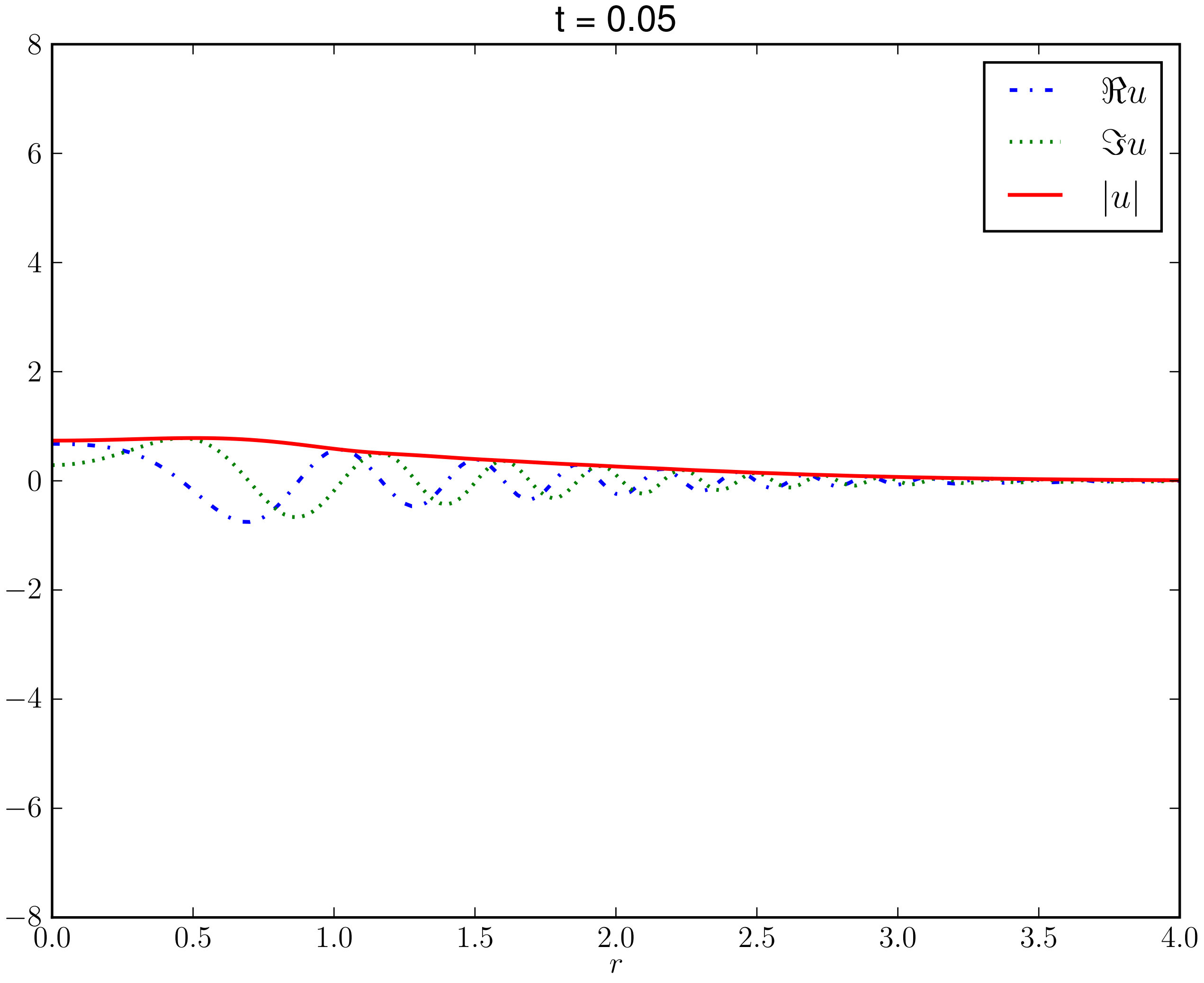

t = 0.035
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

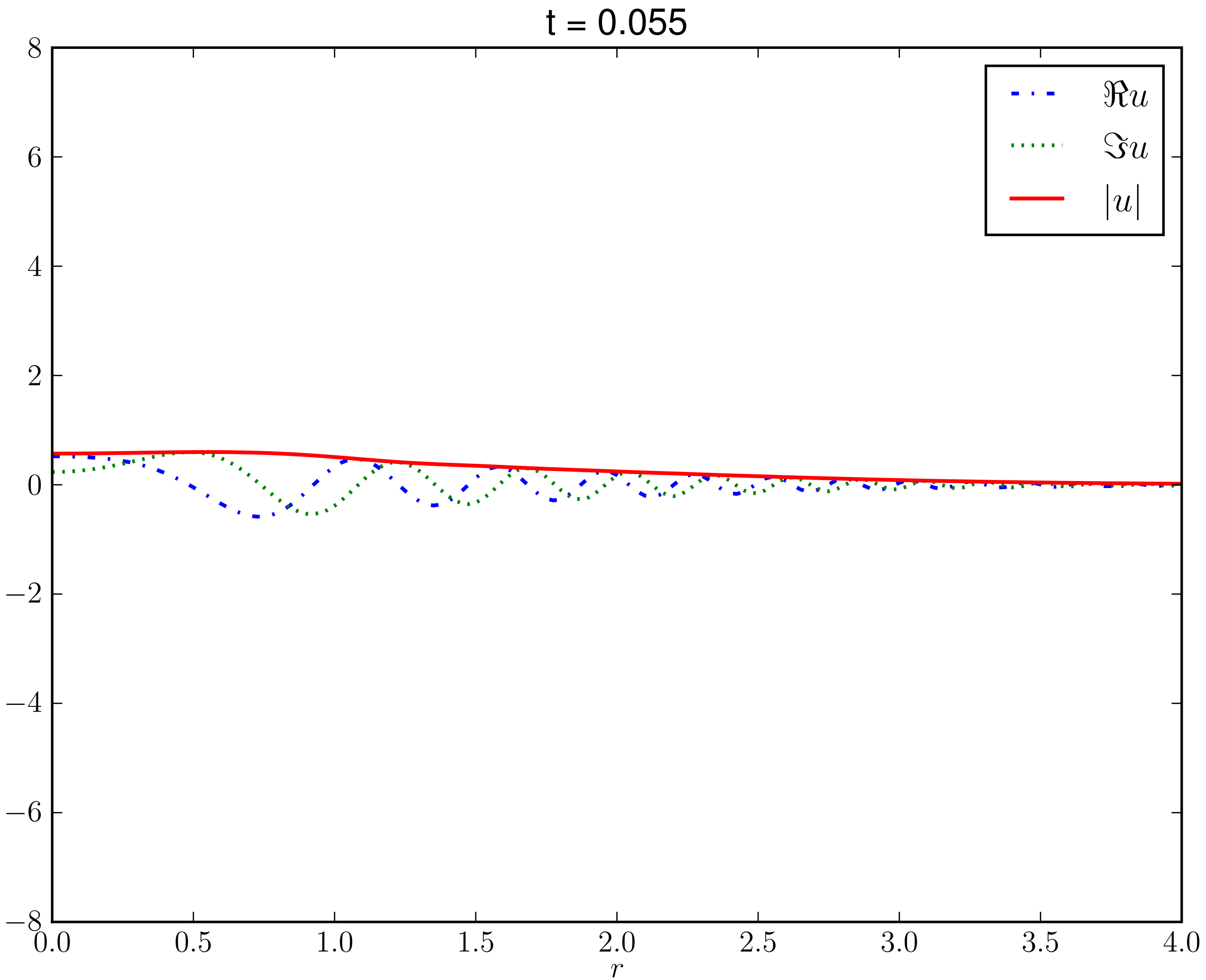


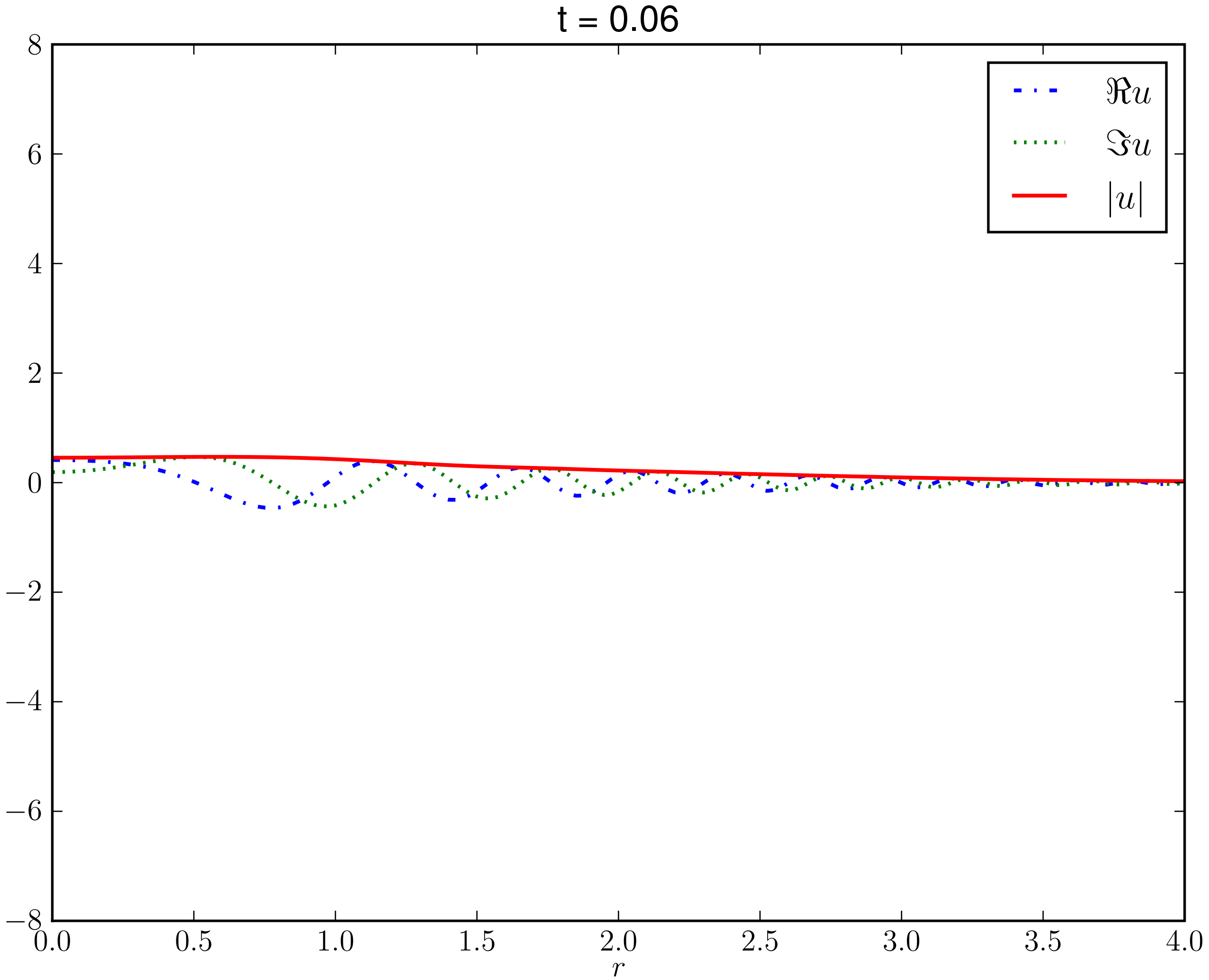
t = 0.04
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

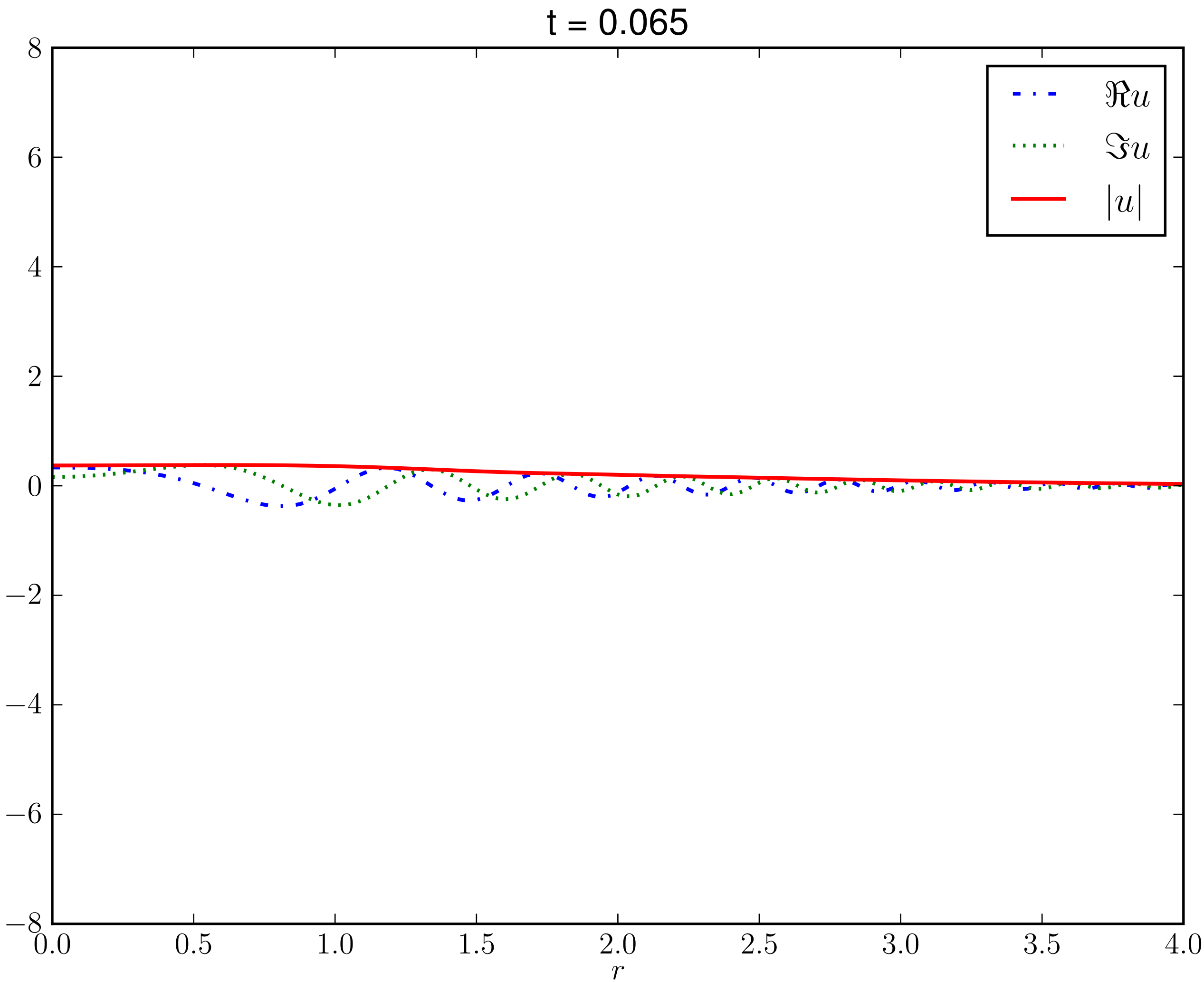


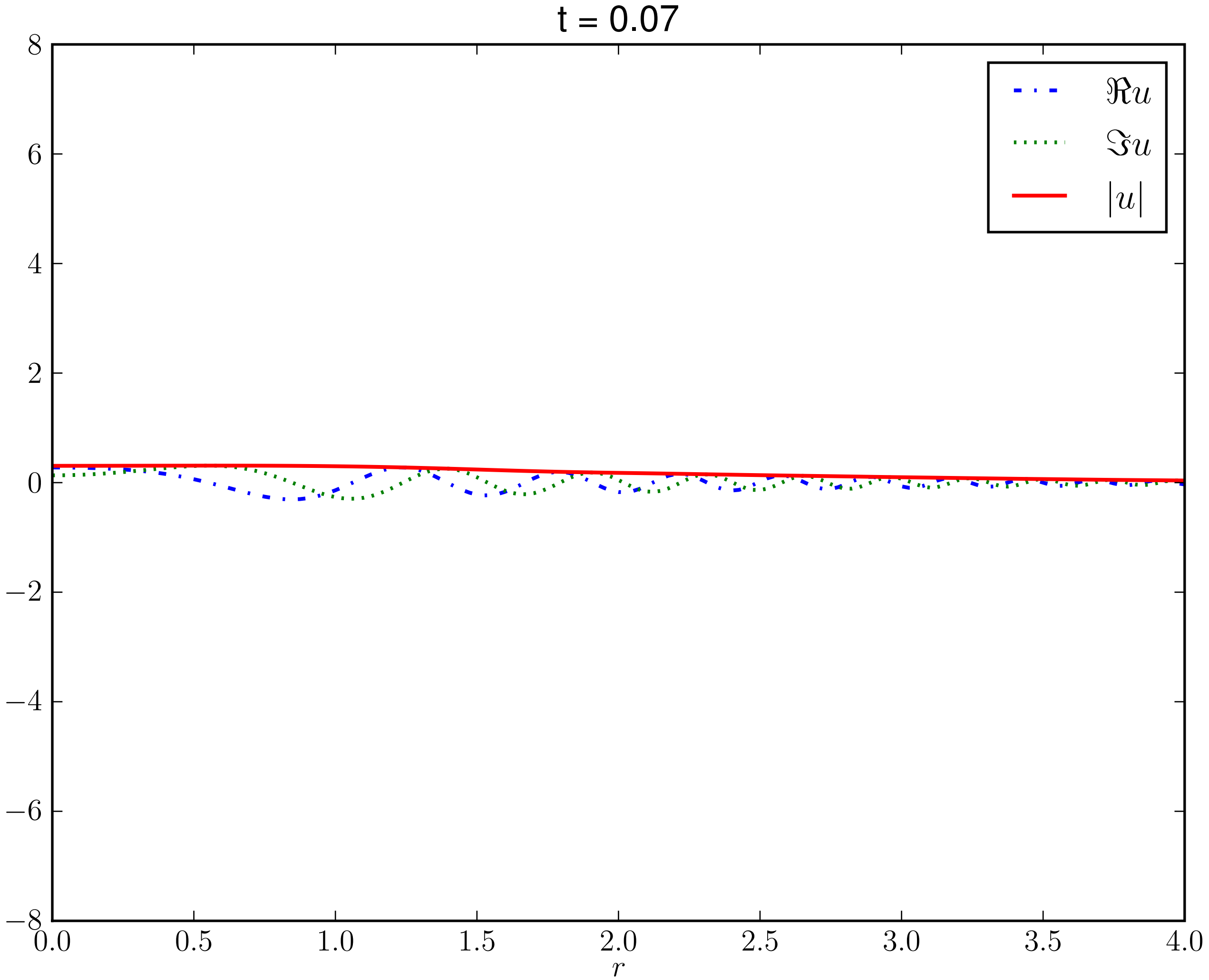
t = 0.045
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

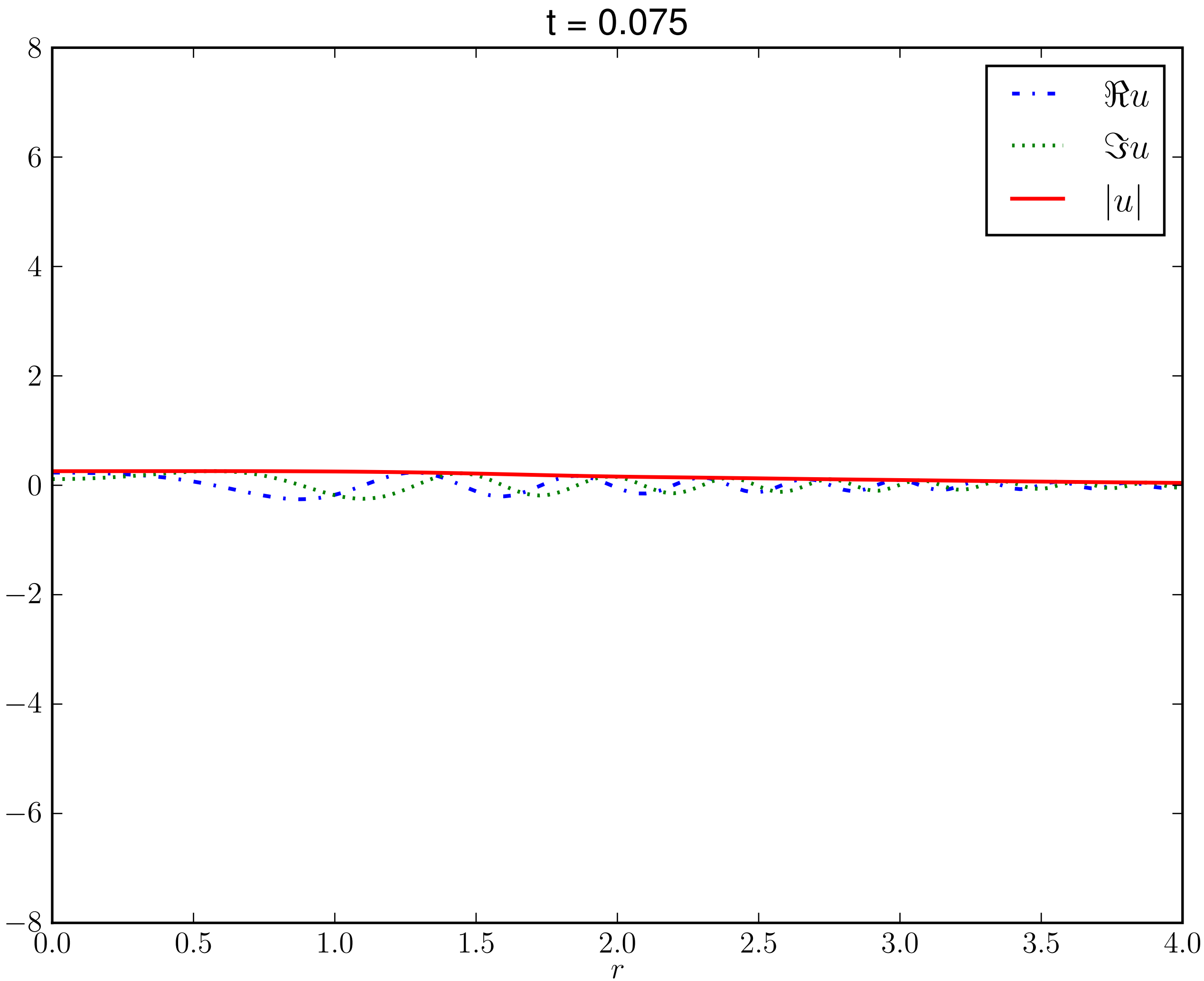

t = 0.05
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r


t = 0.055
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

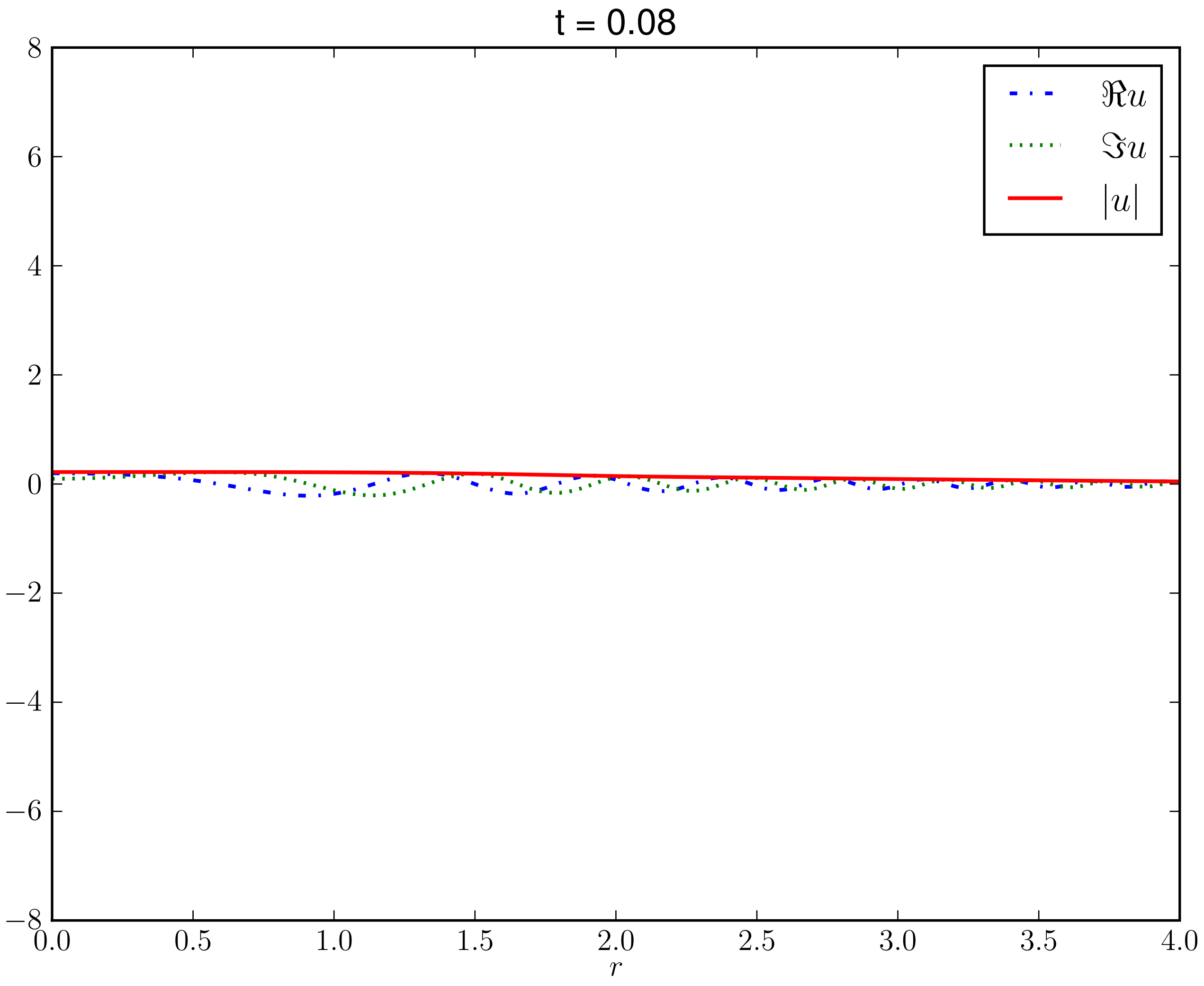

t = 0.06
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.065
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

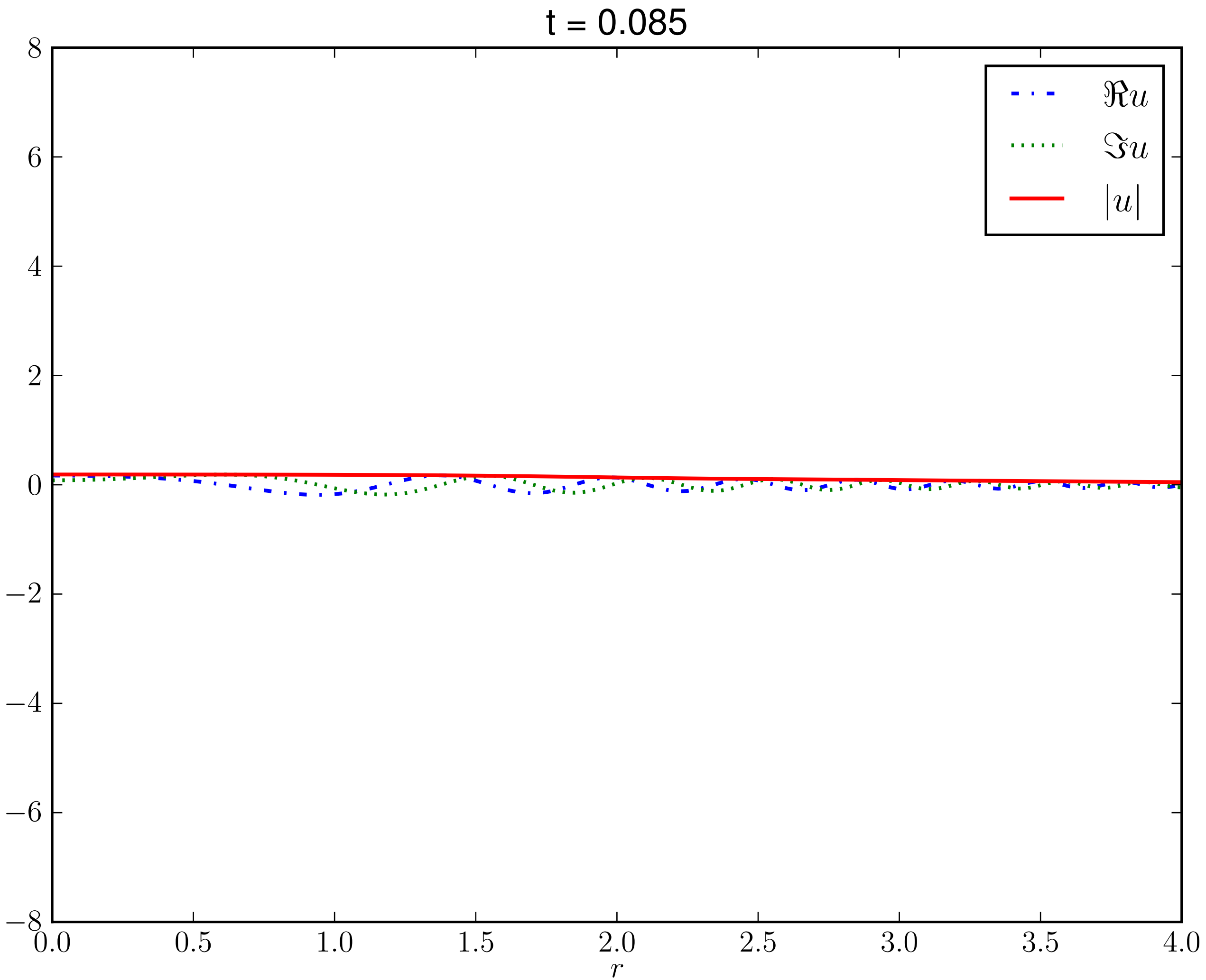

t = 0.07
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



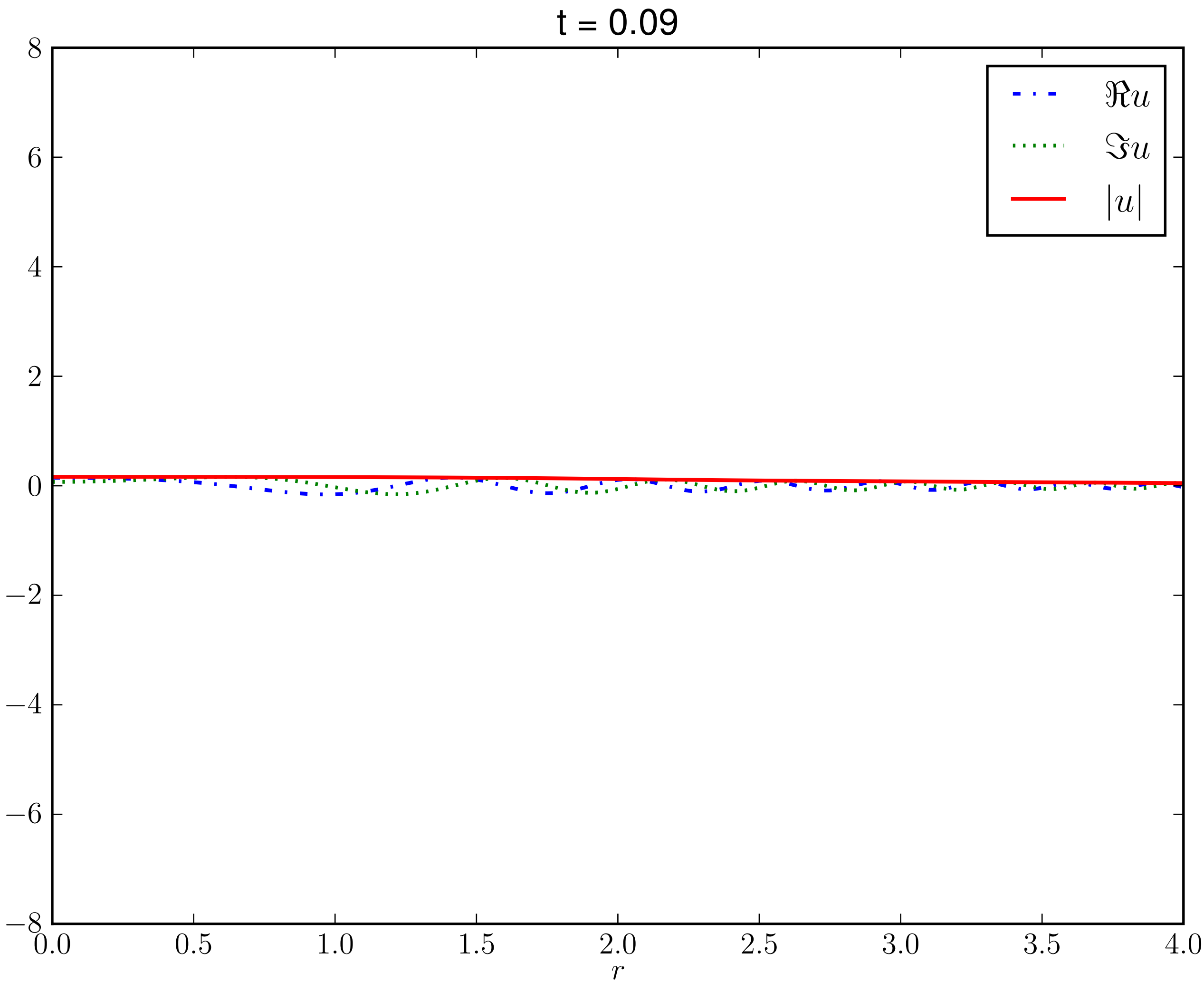
t = 0.075
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

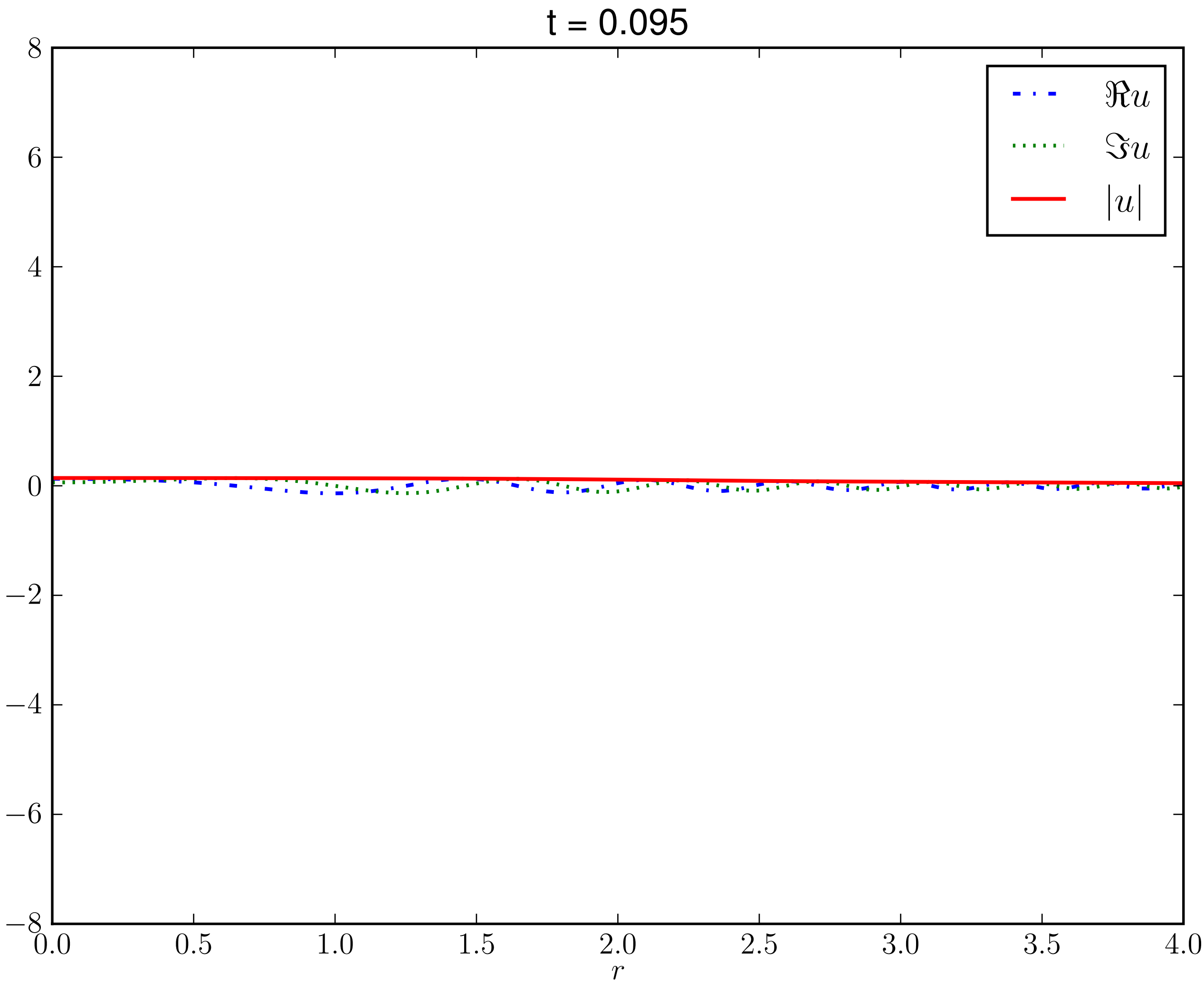


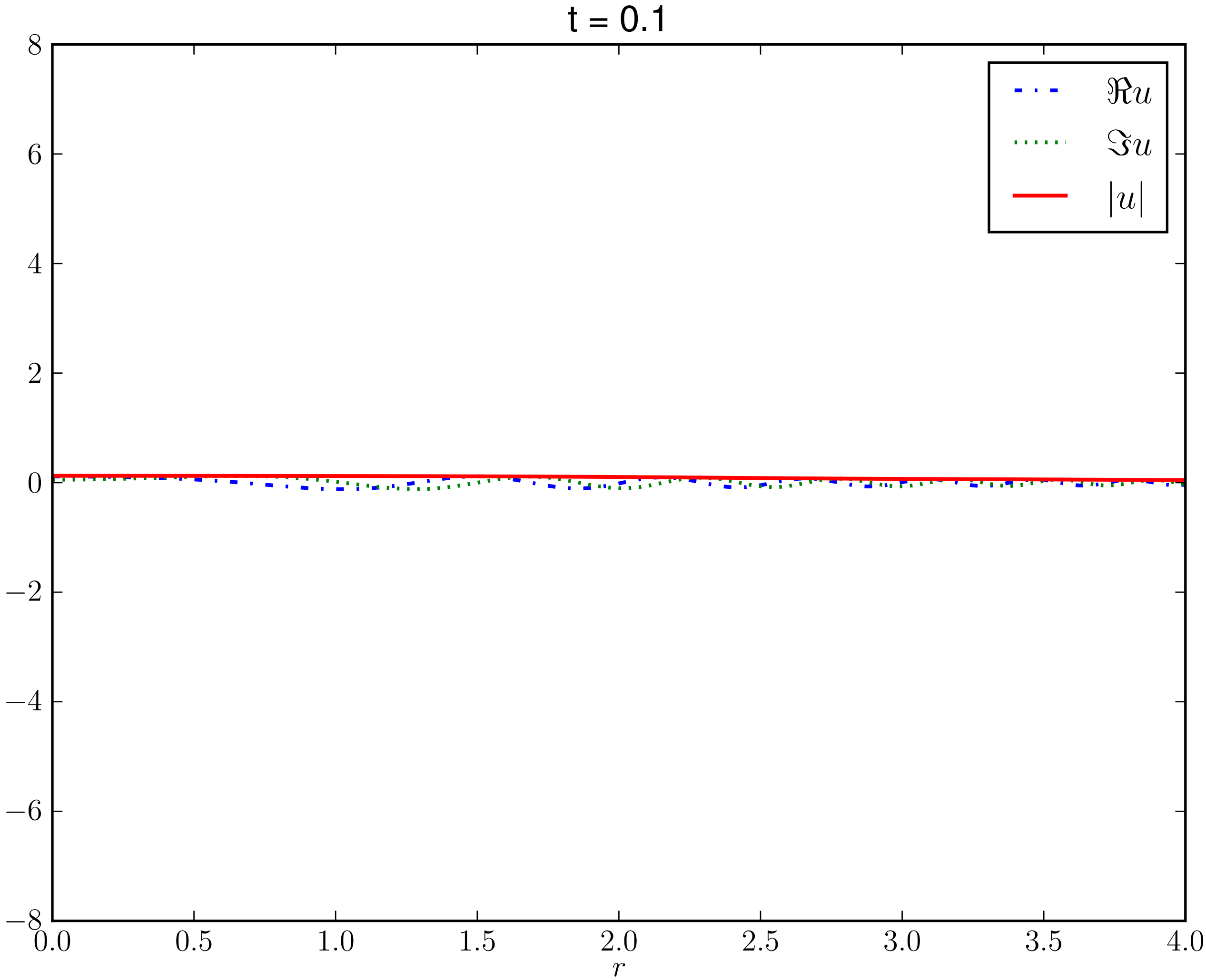
t = 0.08
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

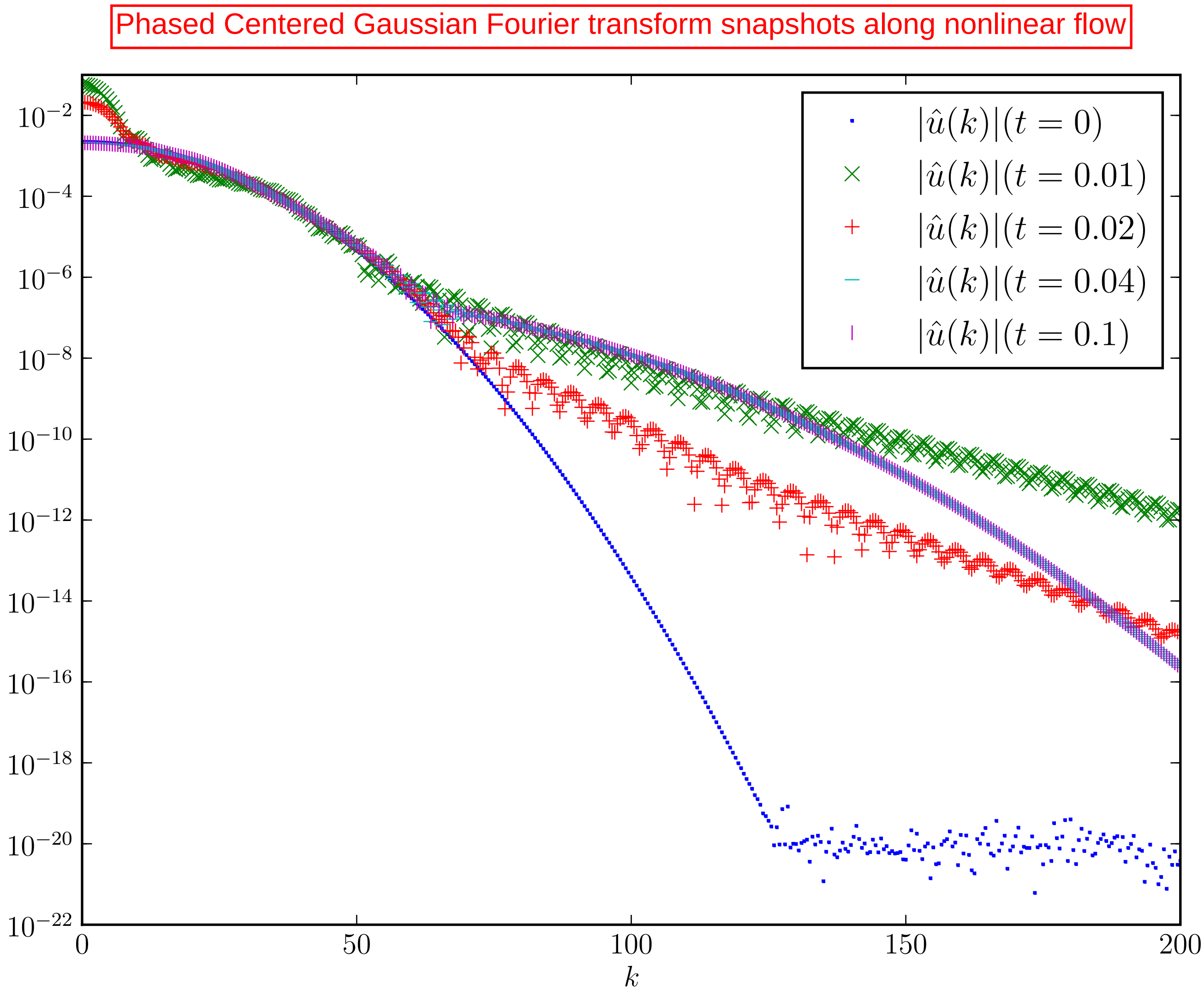


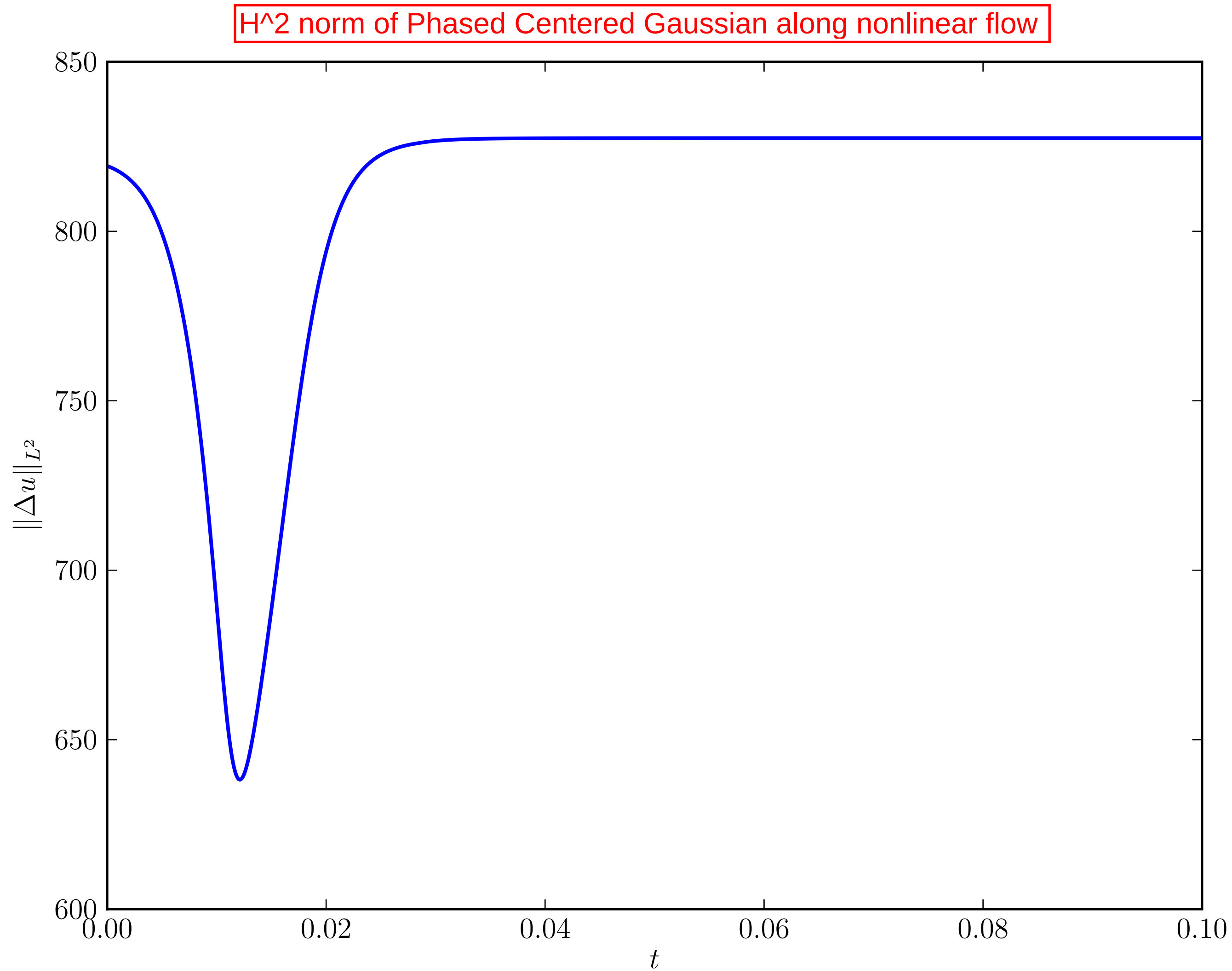
t = 0.085
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

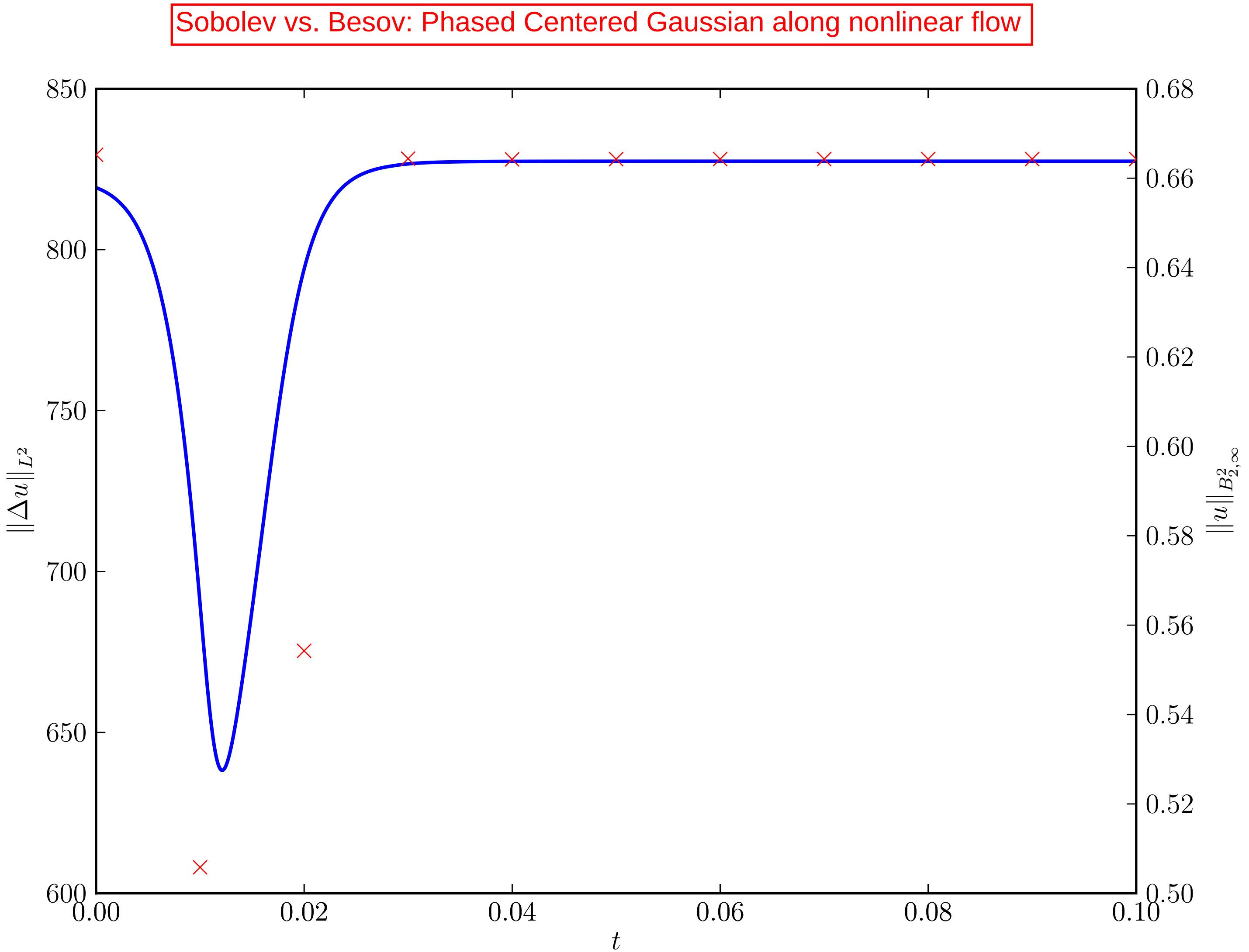

t = 0.09
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.095
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

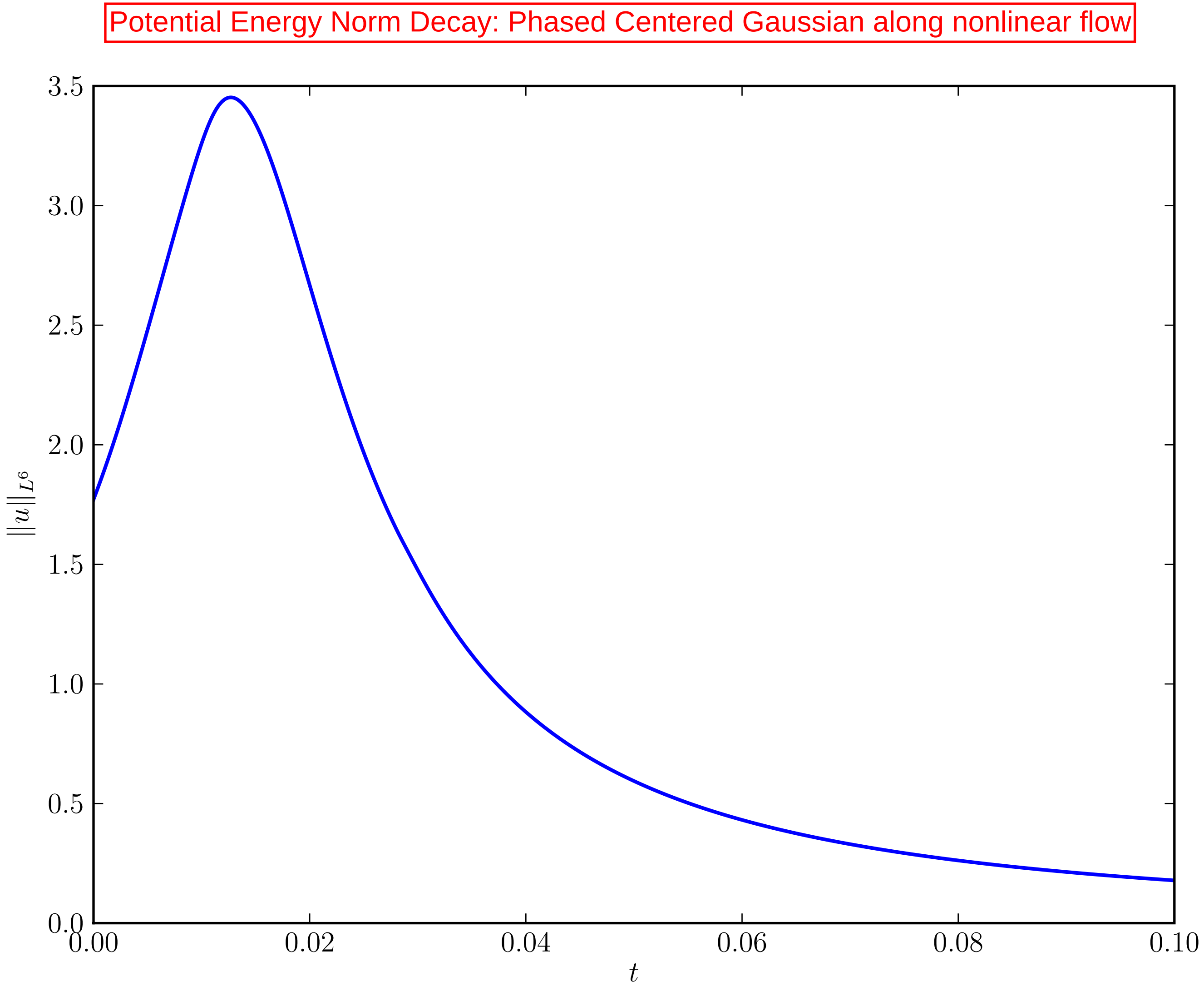

t = 0.1
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

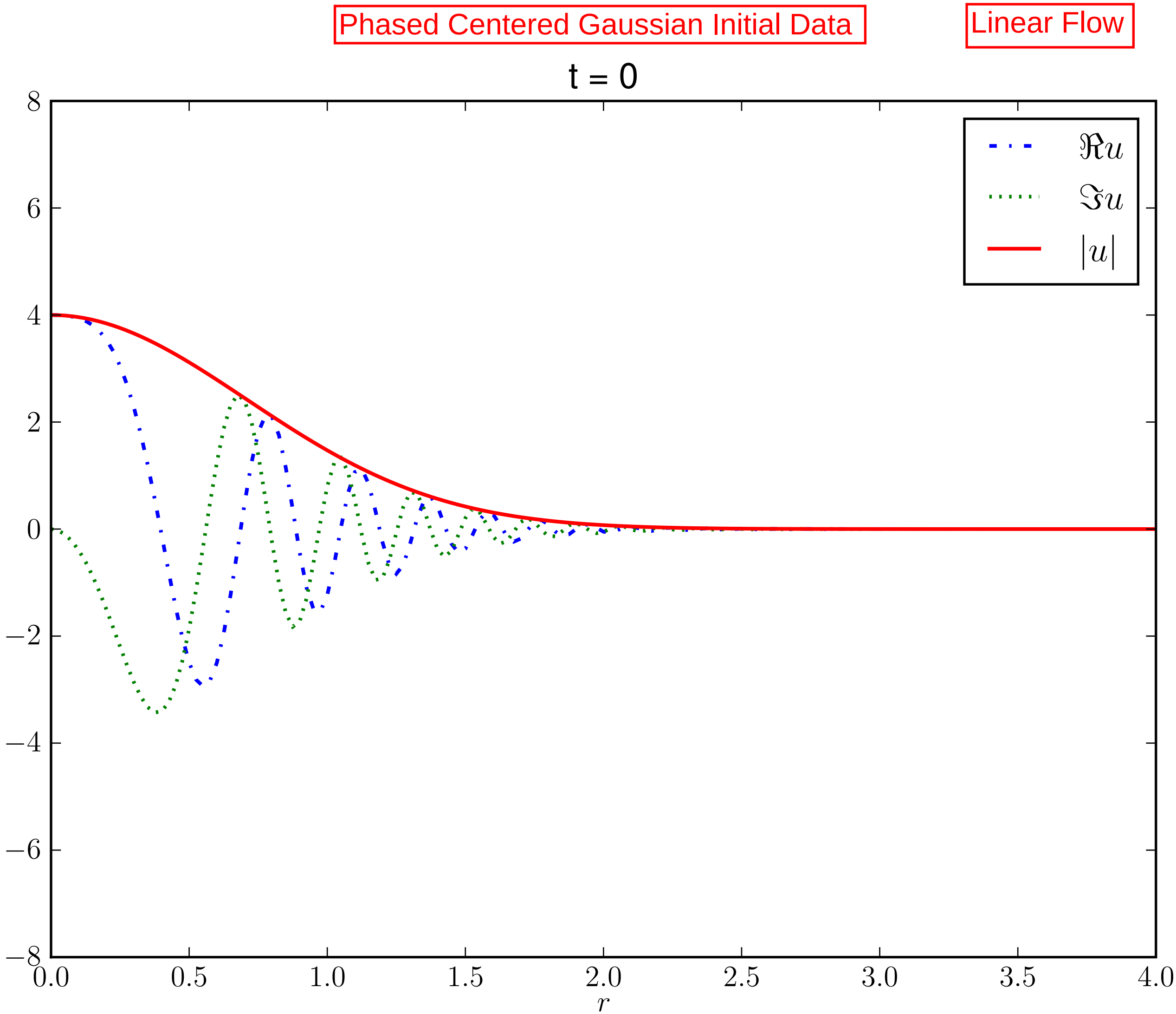

Phased Centered Gaussian Fourier transform snapshots along nonlinear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.01)$
$|\hat{u}(k)|(t=0.02)$
$|\hat{u}(k)|(t=0.04)$
$|\hat{u}(k)|(t=0.1)$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k


H^2 norm of Phased Centered Gaussian along nonlinear flow
850
800
750
700
650
600
$\|\Delta u\|_{L^2}$
0.00
0.02
0.04
0.06
0.08
0.10
t

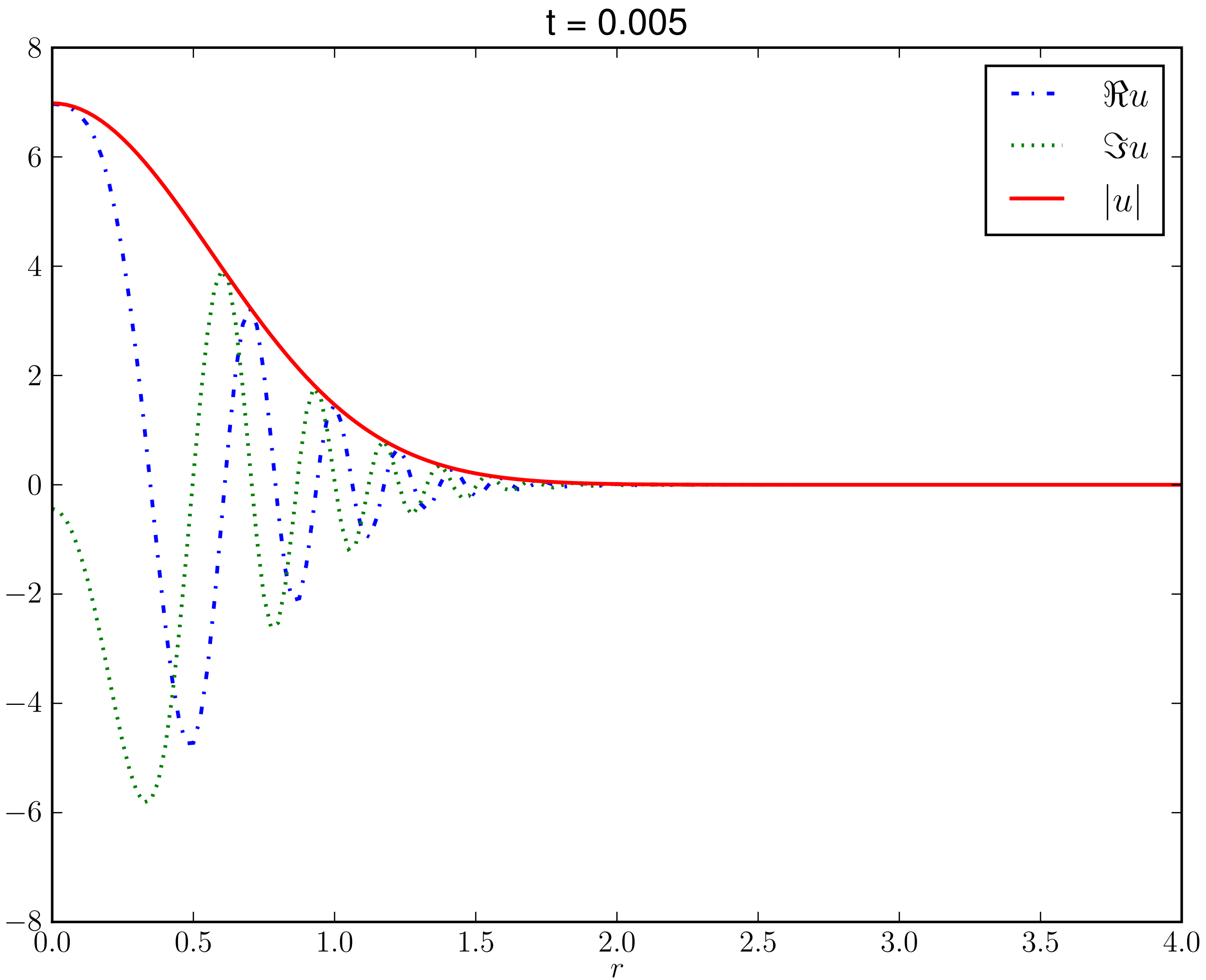


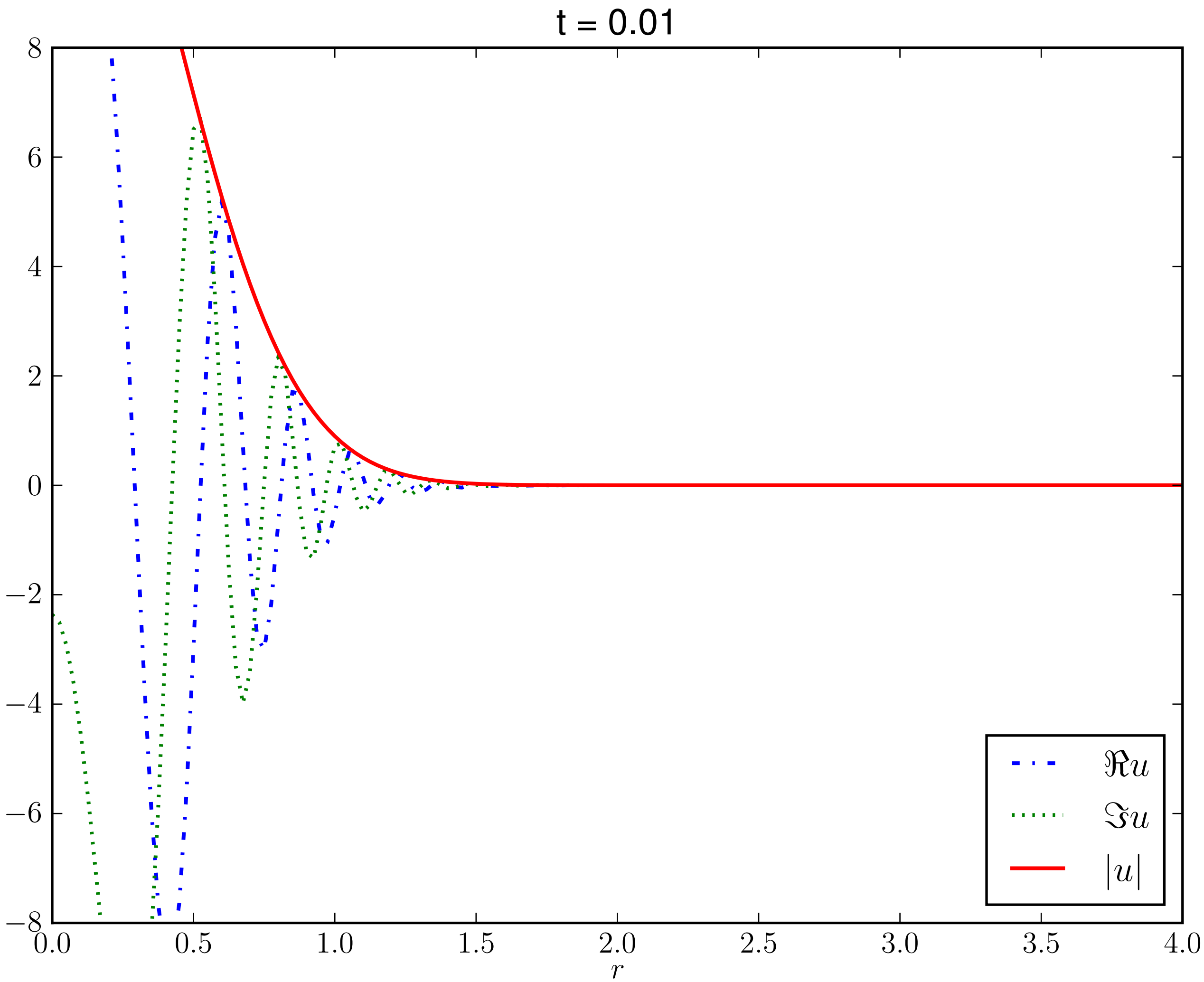
Sobolev vs. Besov: Phased Centered Gaussian along nonlinear flow
850
800
750
700
650
600
0.00
0.02
0.04
0.06
0.08
0.10
t
$\|\Delta u\|_{L^2}$
0.68
0.66
0.64
0.62
0.60
0.58
0.56
0.54
0.52
0.50
$\|u\|_{B^2_{2,\infty}}$

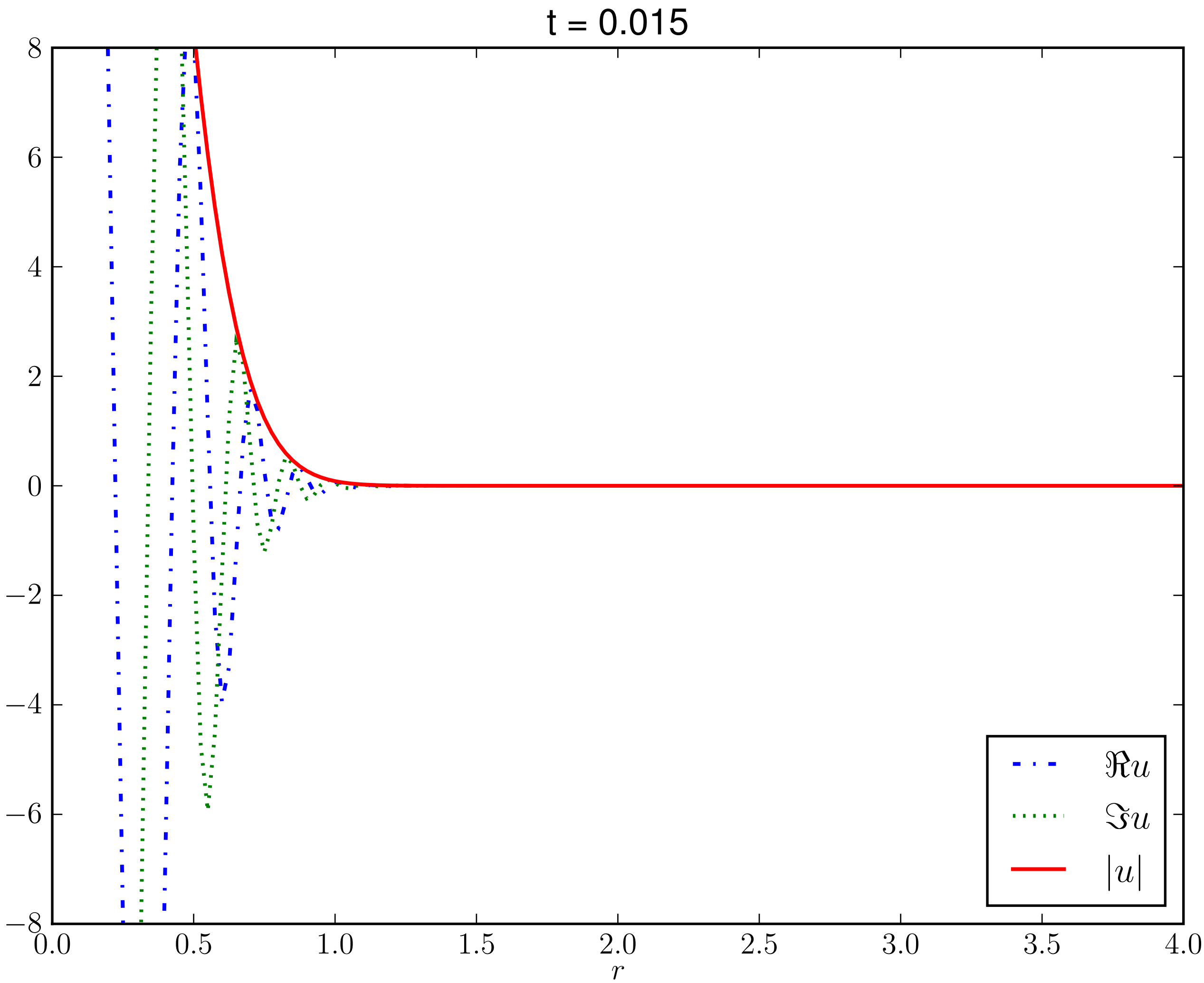

Potential Energy Norm Decay: Phased Centered Gaussian along nonlinear flow
$\|u\|_{L^6}$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.02
0.04
0.06
0.08
0.10
t



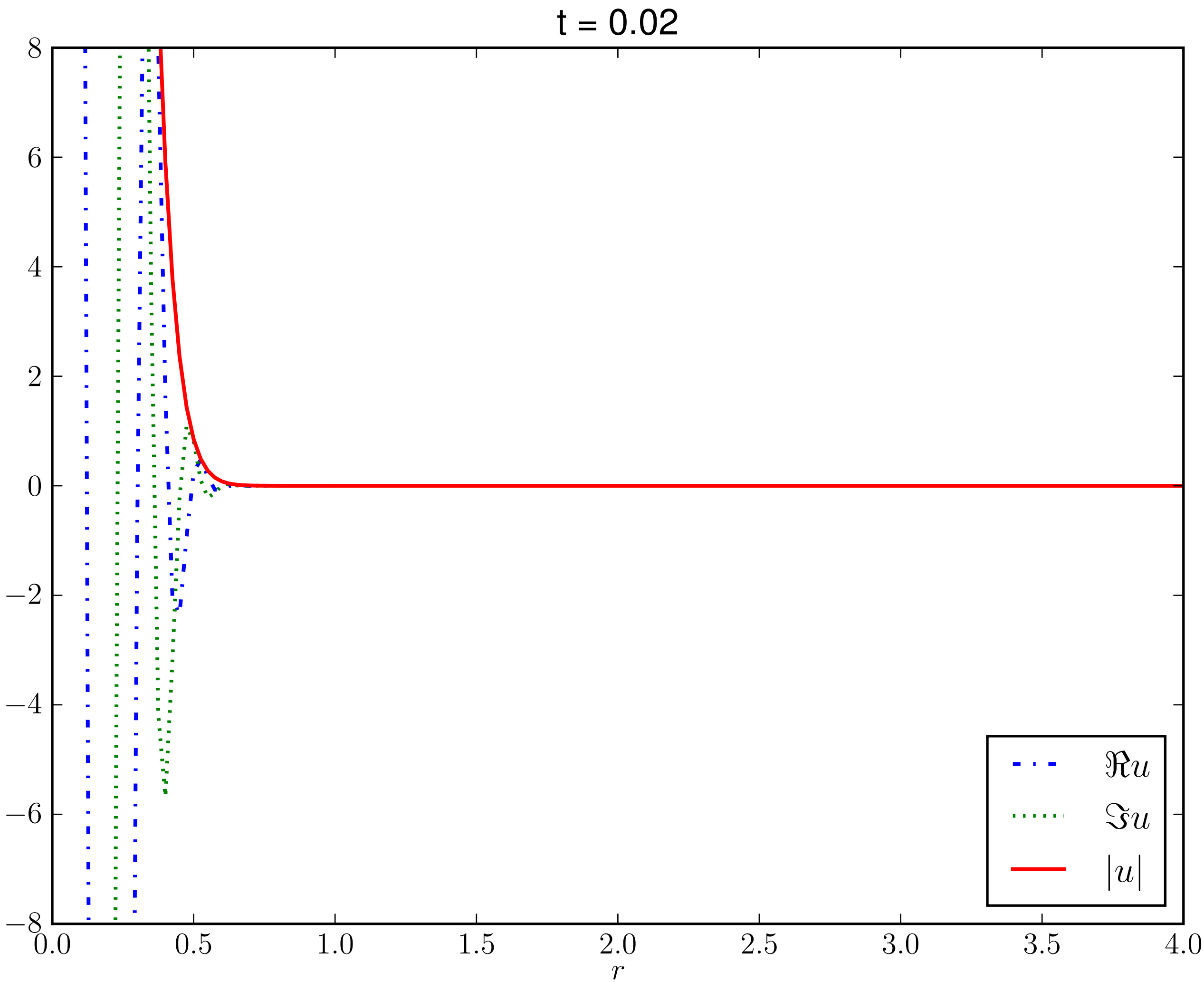
Phased Centered Gaussian Initial Data
Linear Flow
t = 0
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r


t = 0.005
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

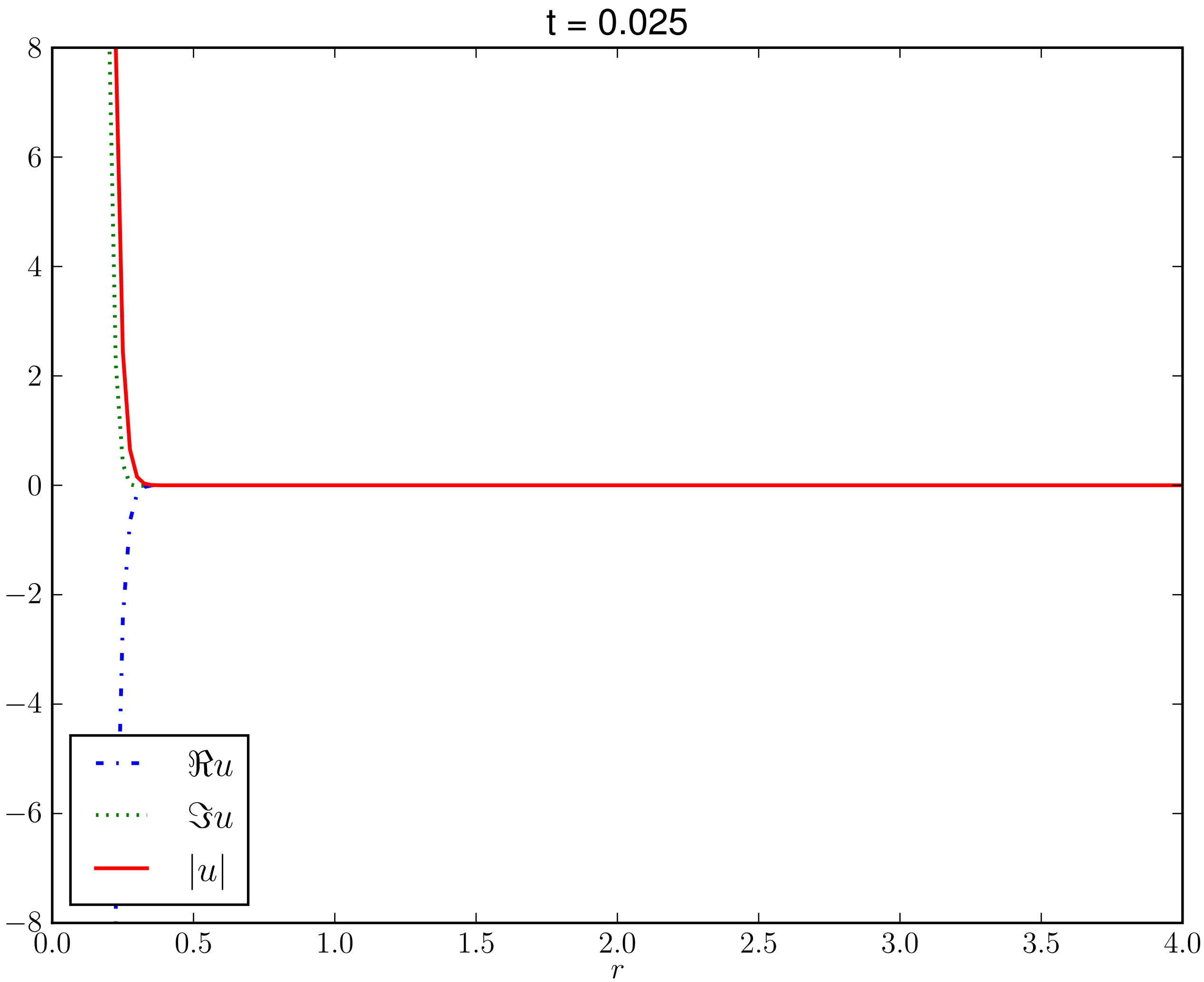

t = 0.01
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



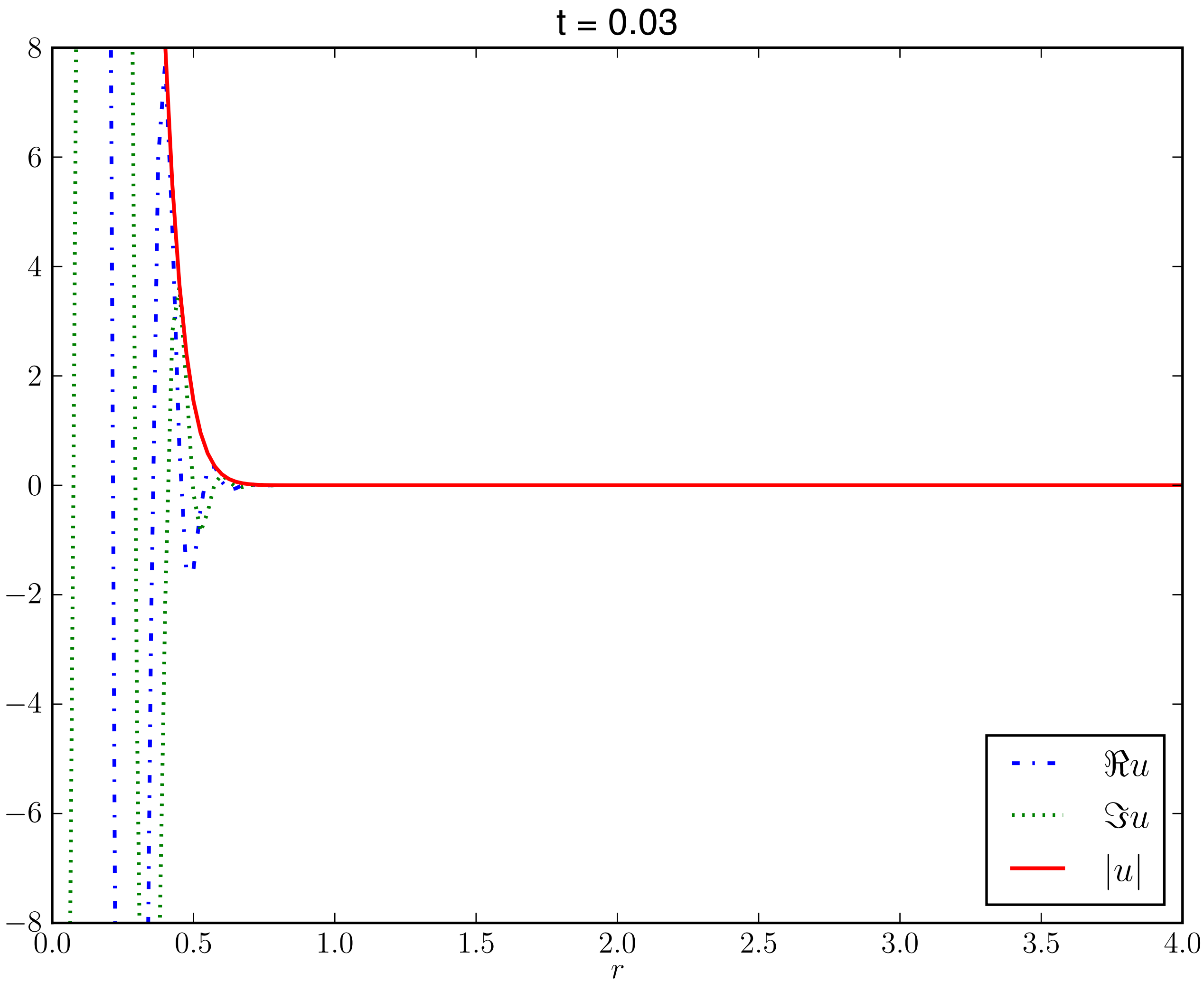
t = 0.015
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

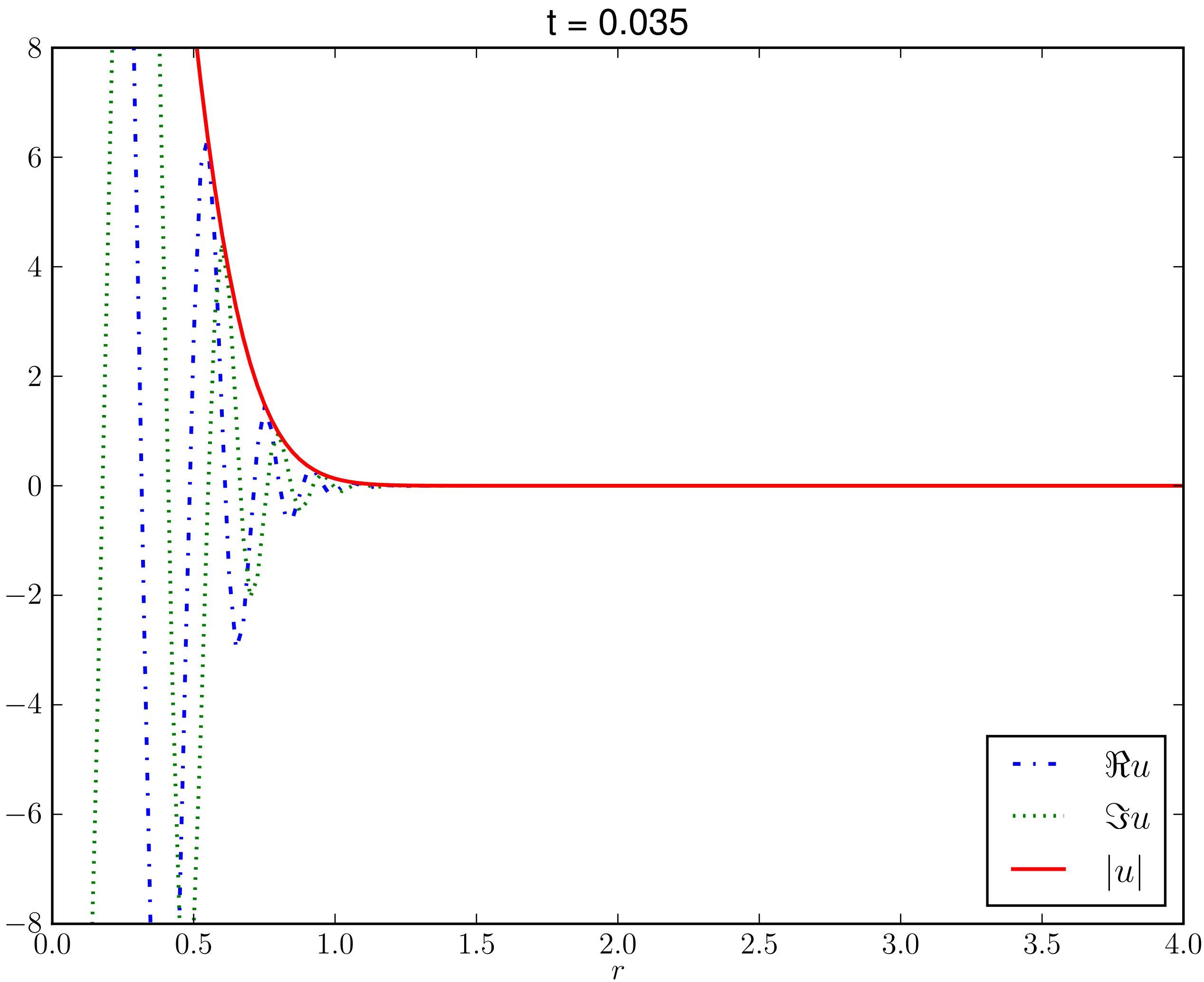


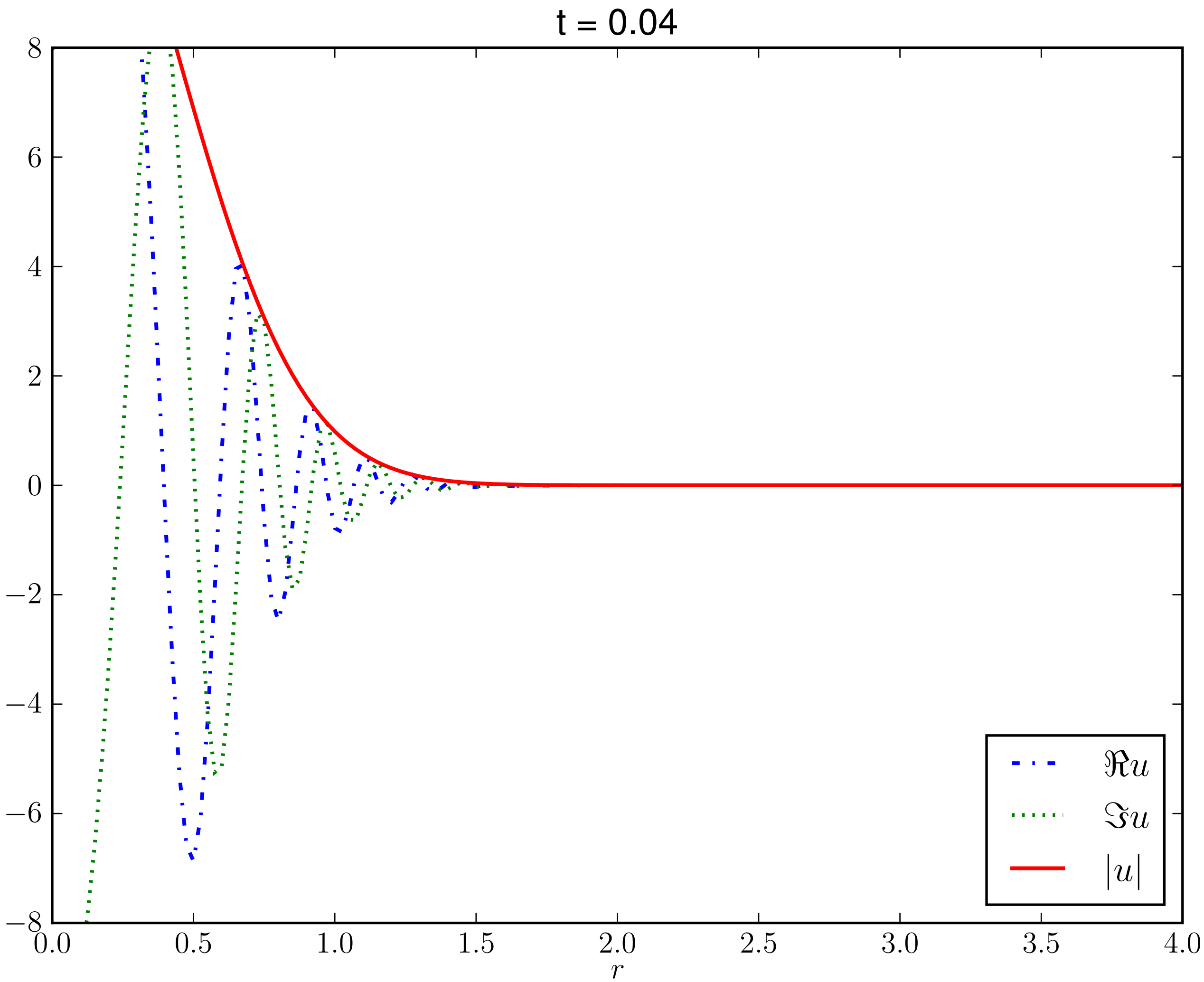
t = 0.02
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



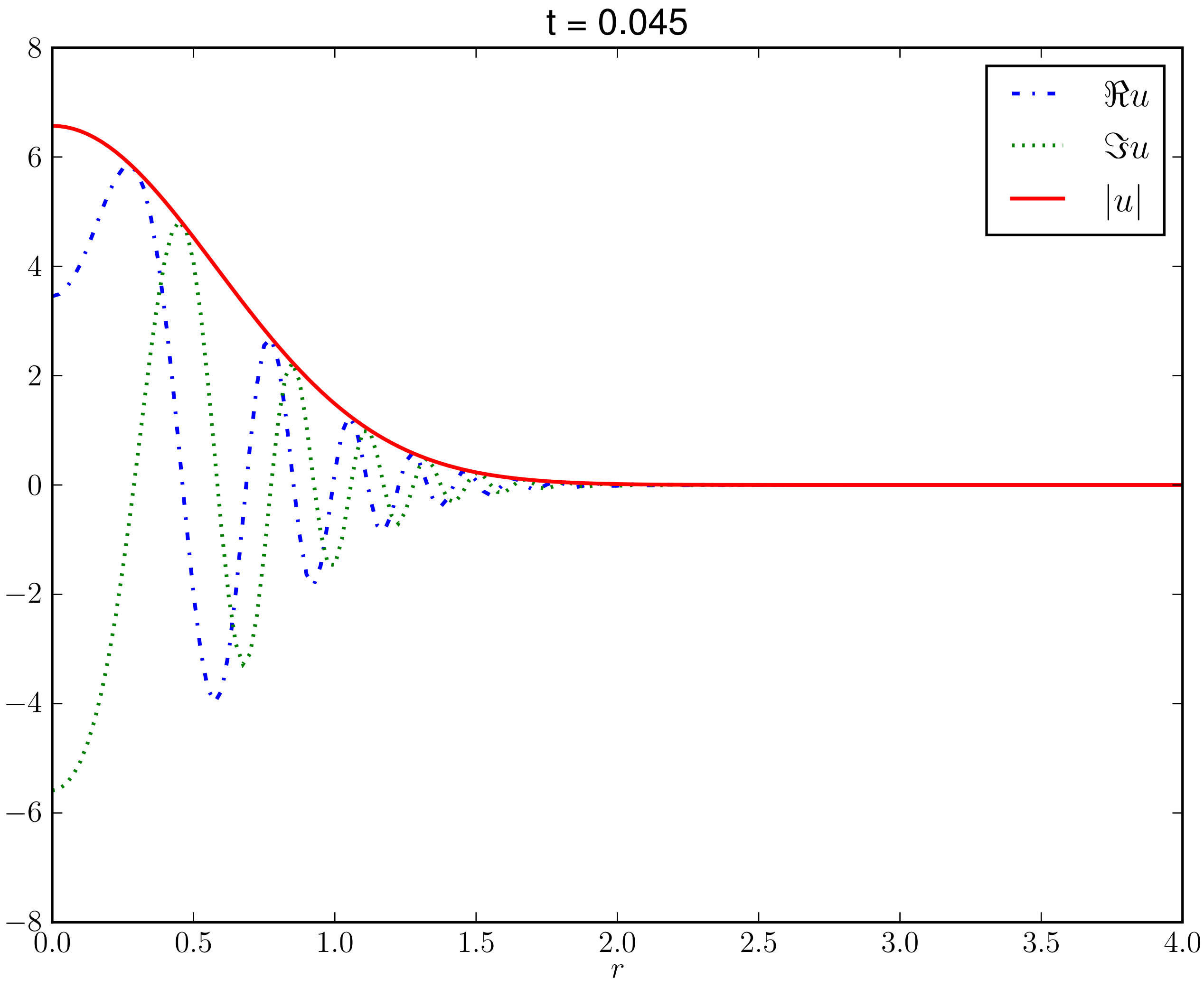
t = 0.025
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

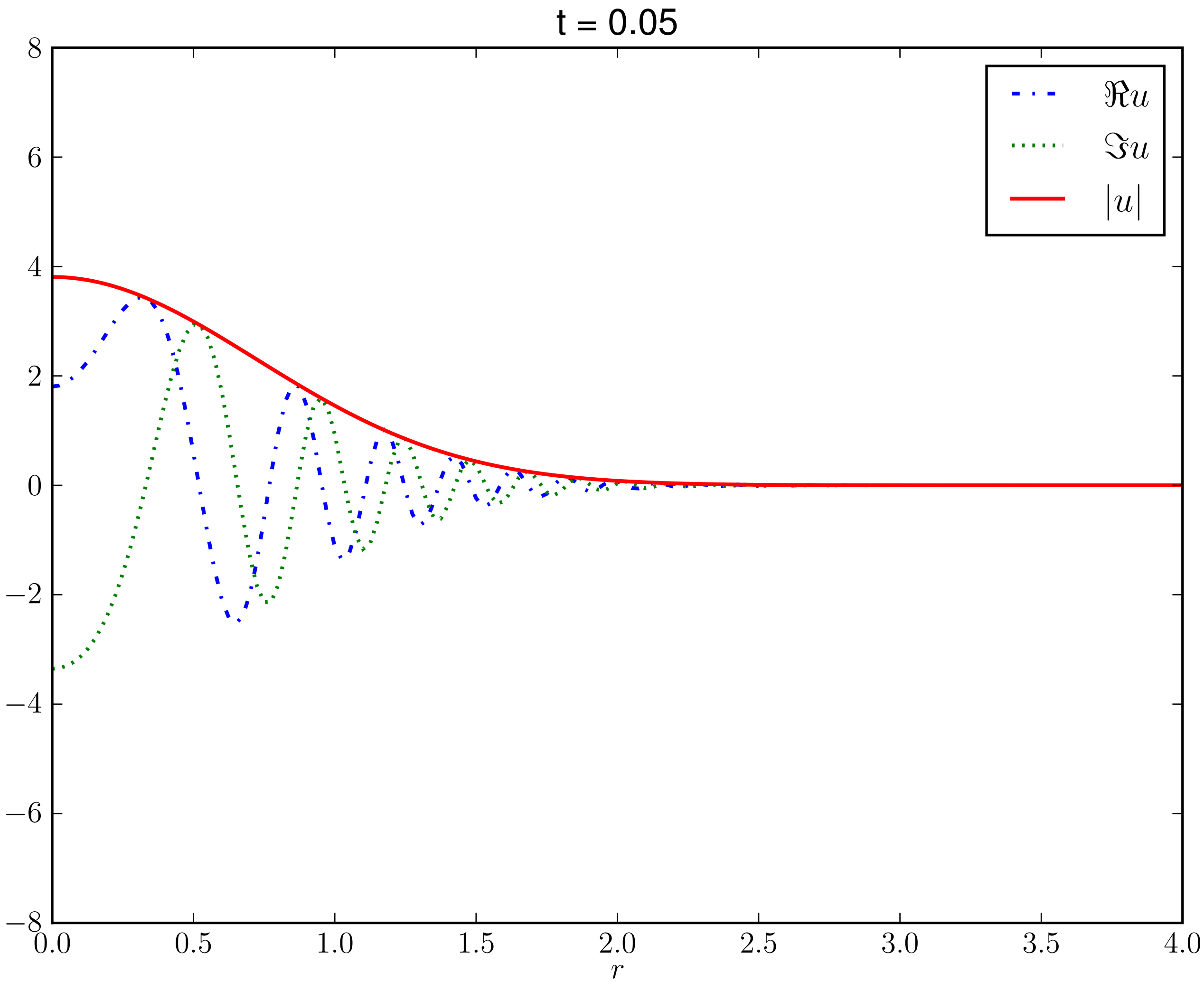

t = 0.03
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

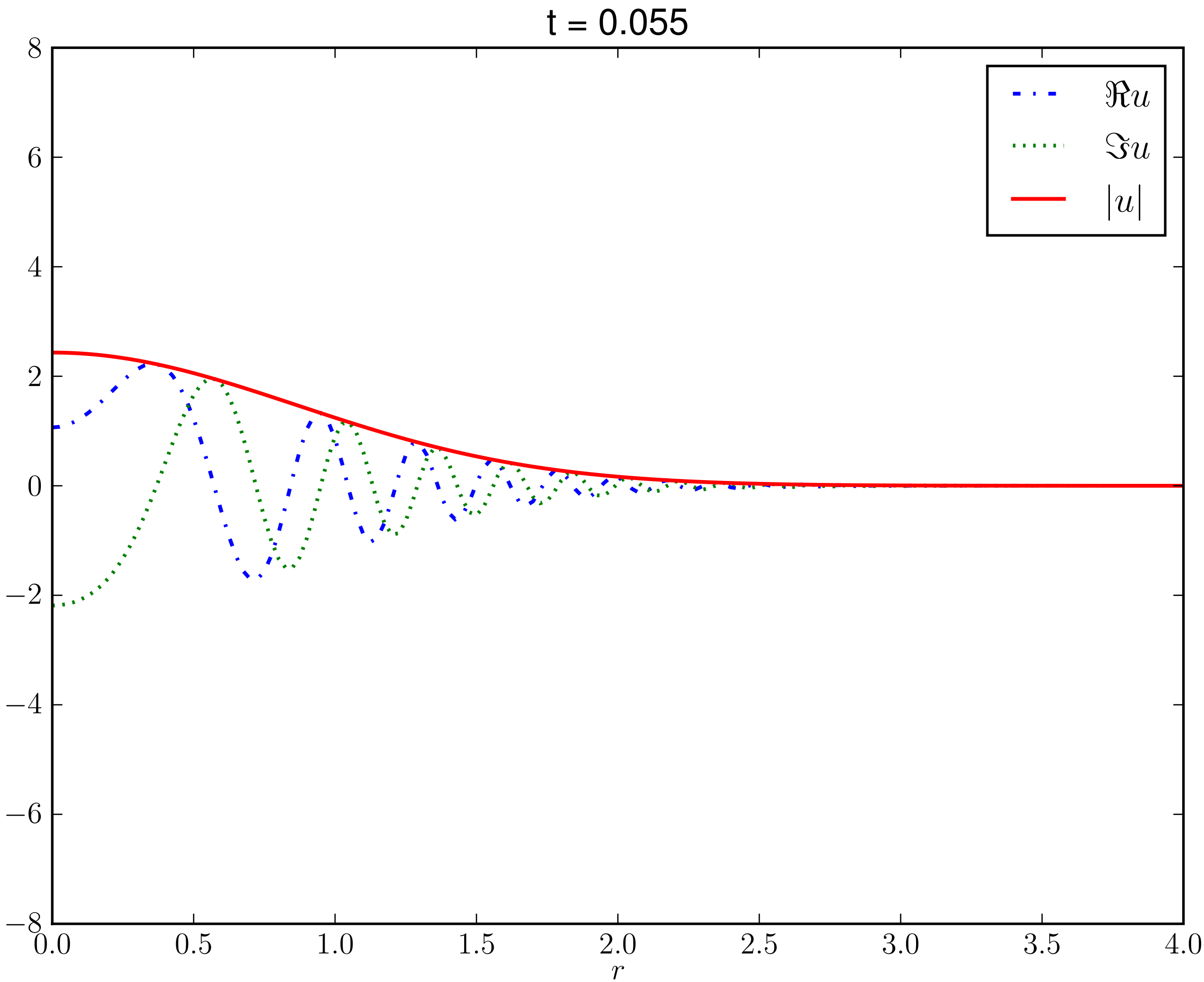

t = 0.035
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

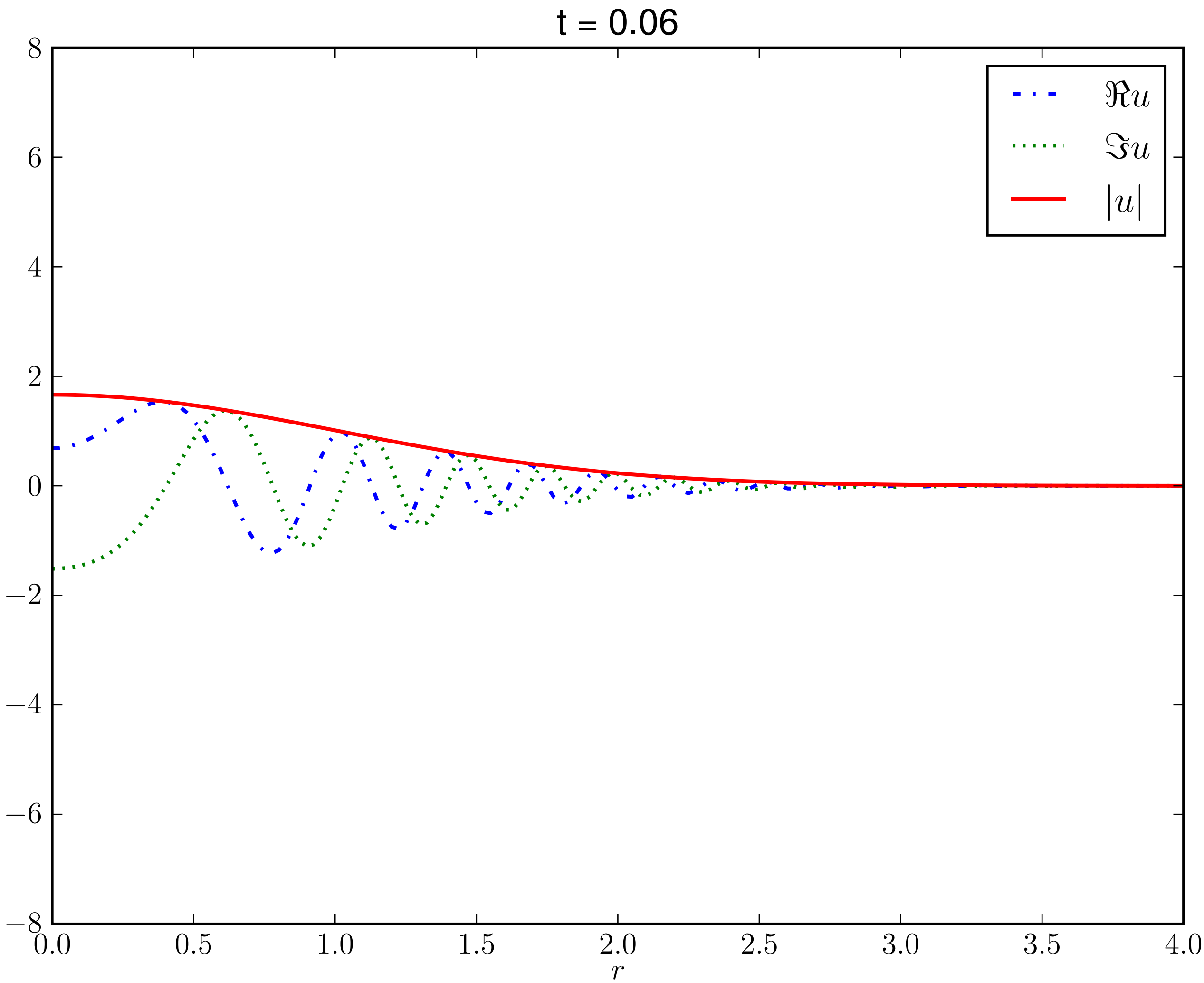


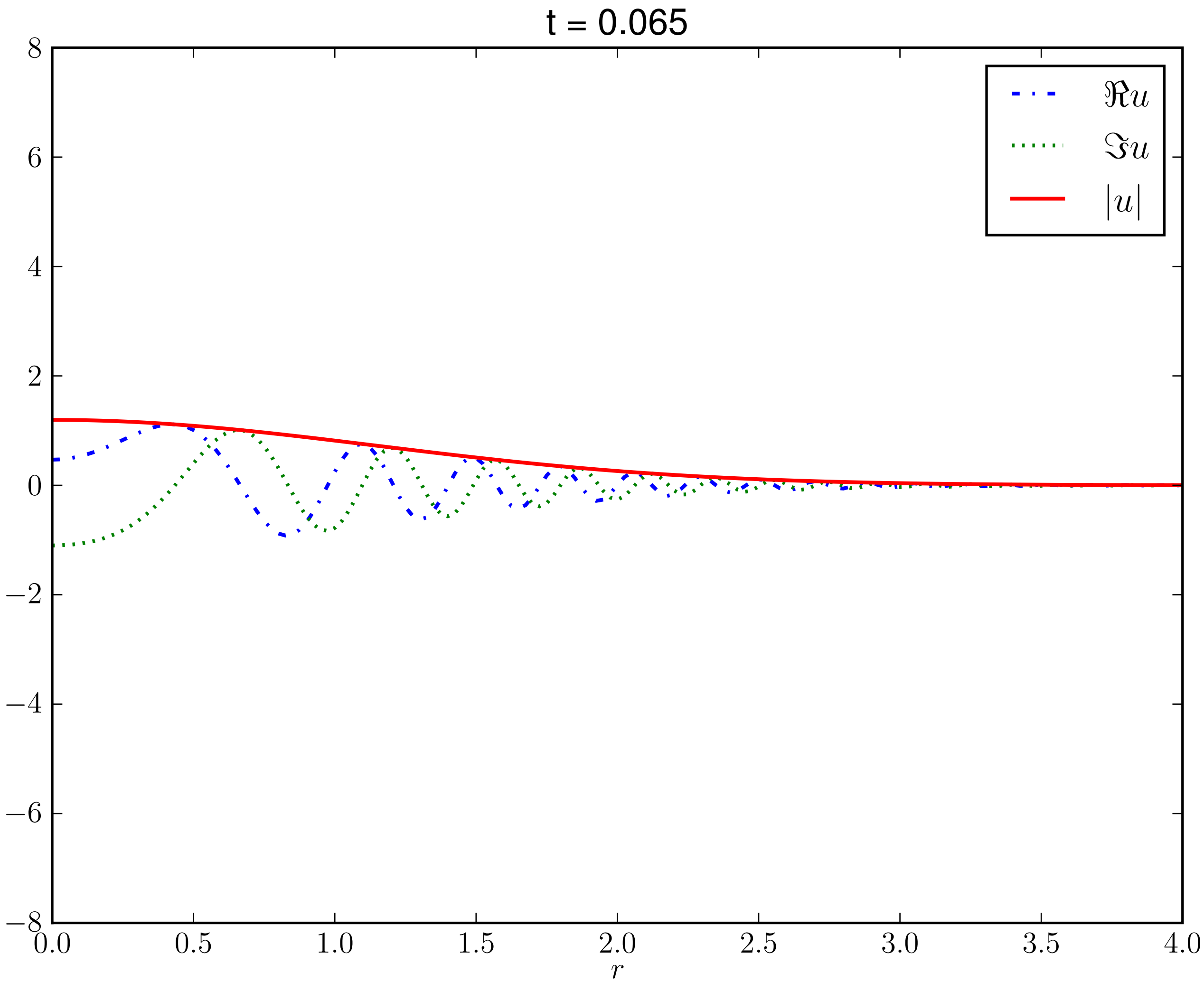
t = 0.04
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

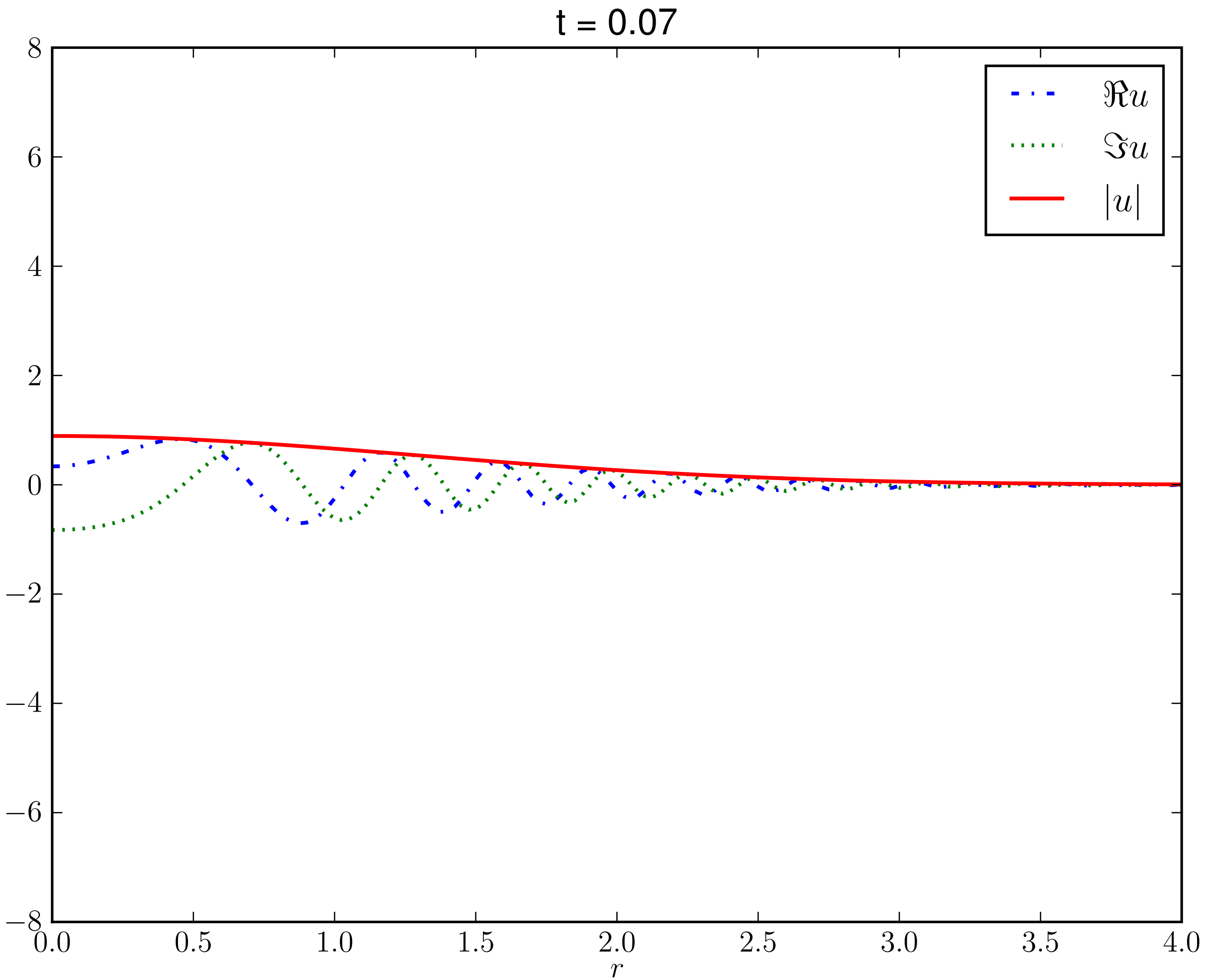

t = 0.045
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.05
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

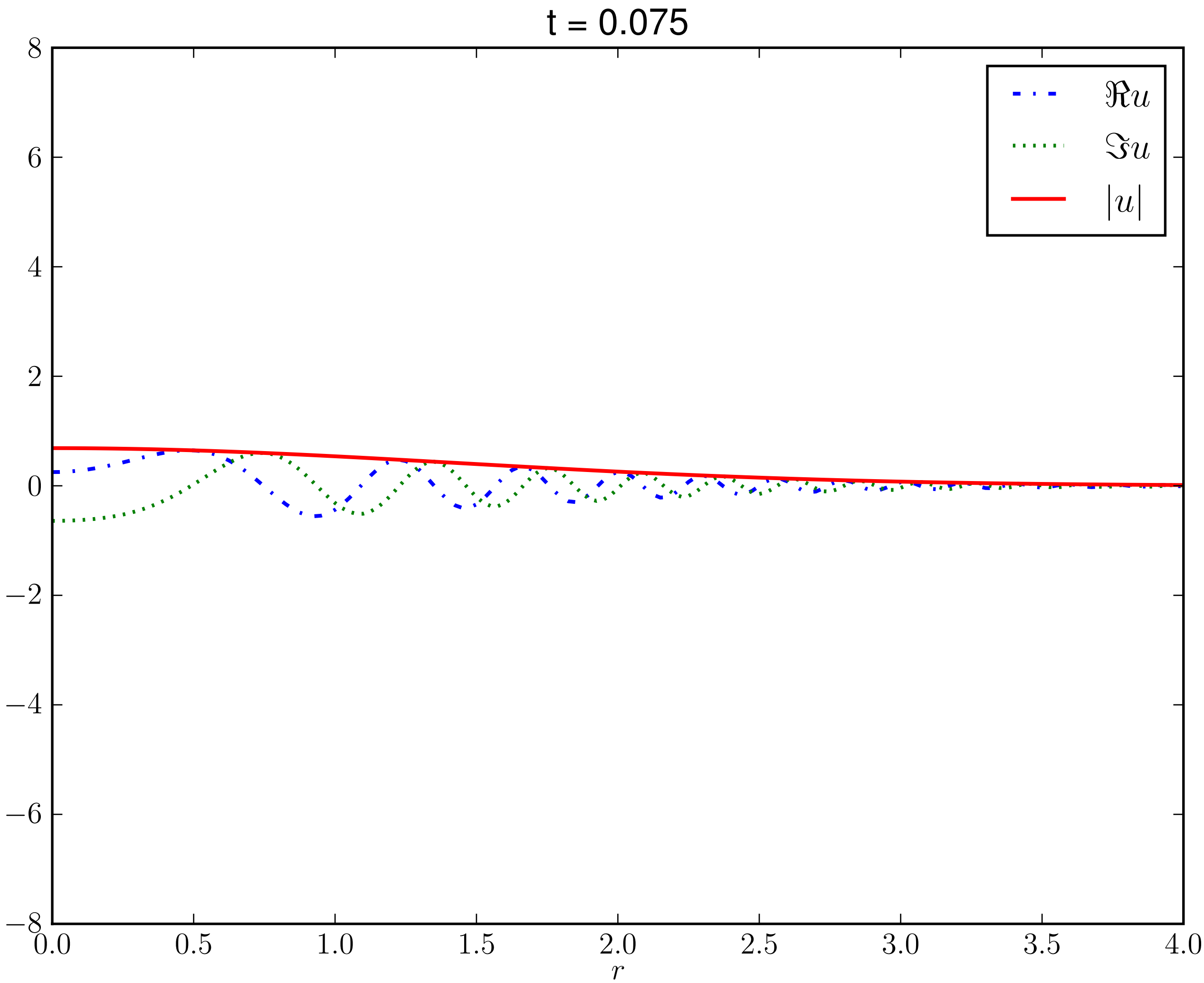

t = 0.055
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.06
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

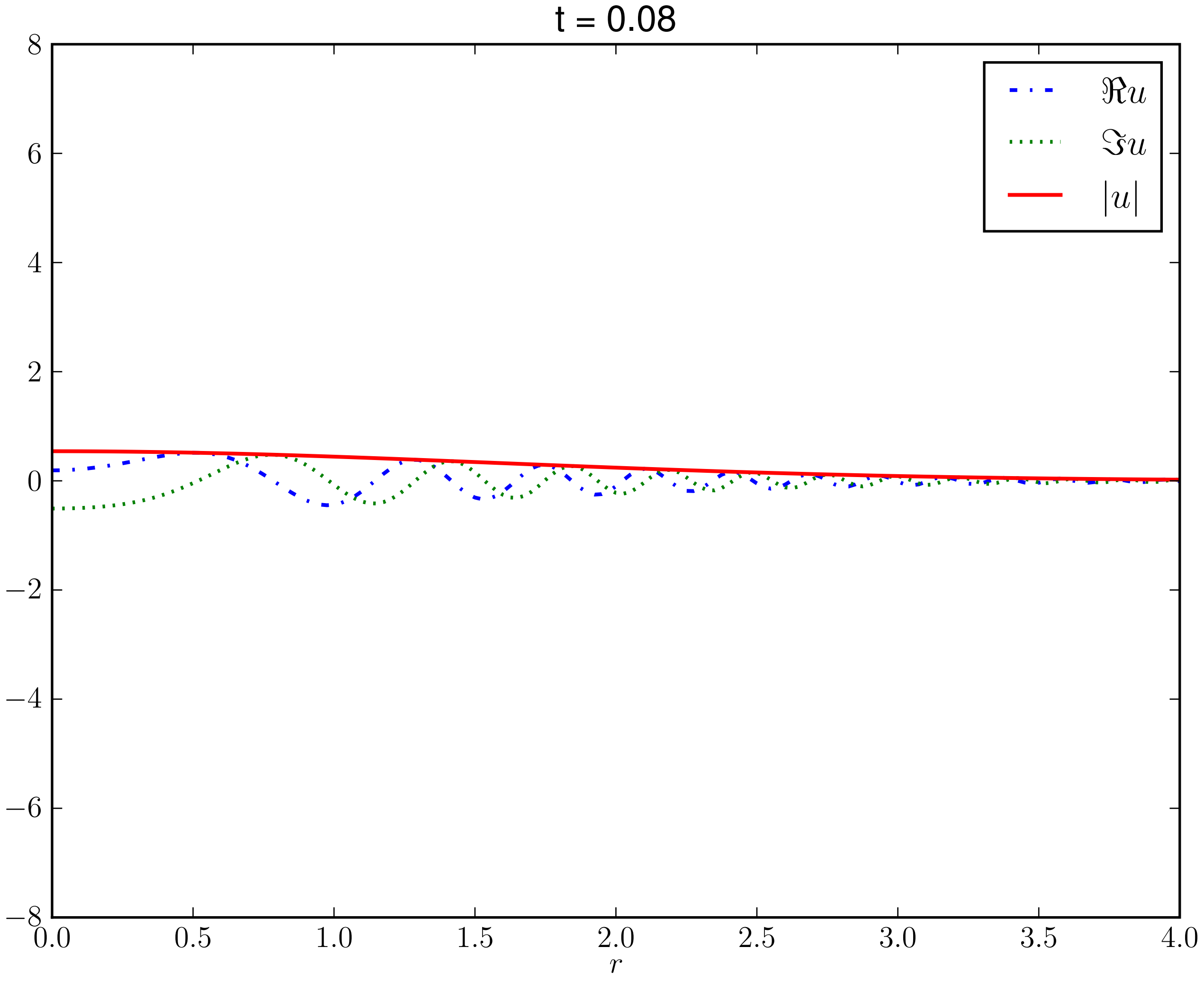

t = 0.065
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

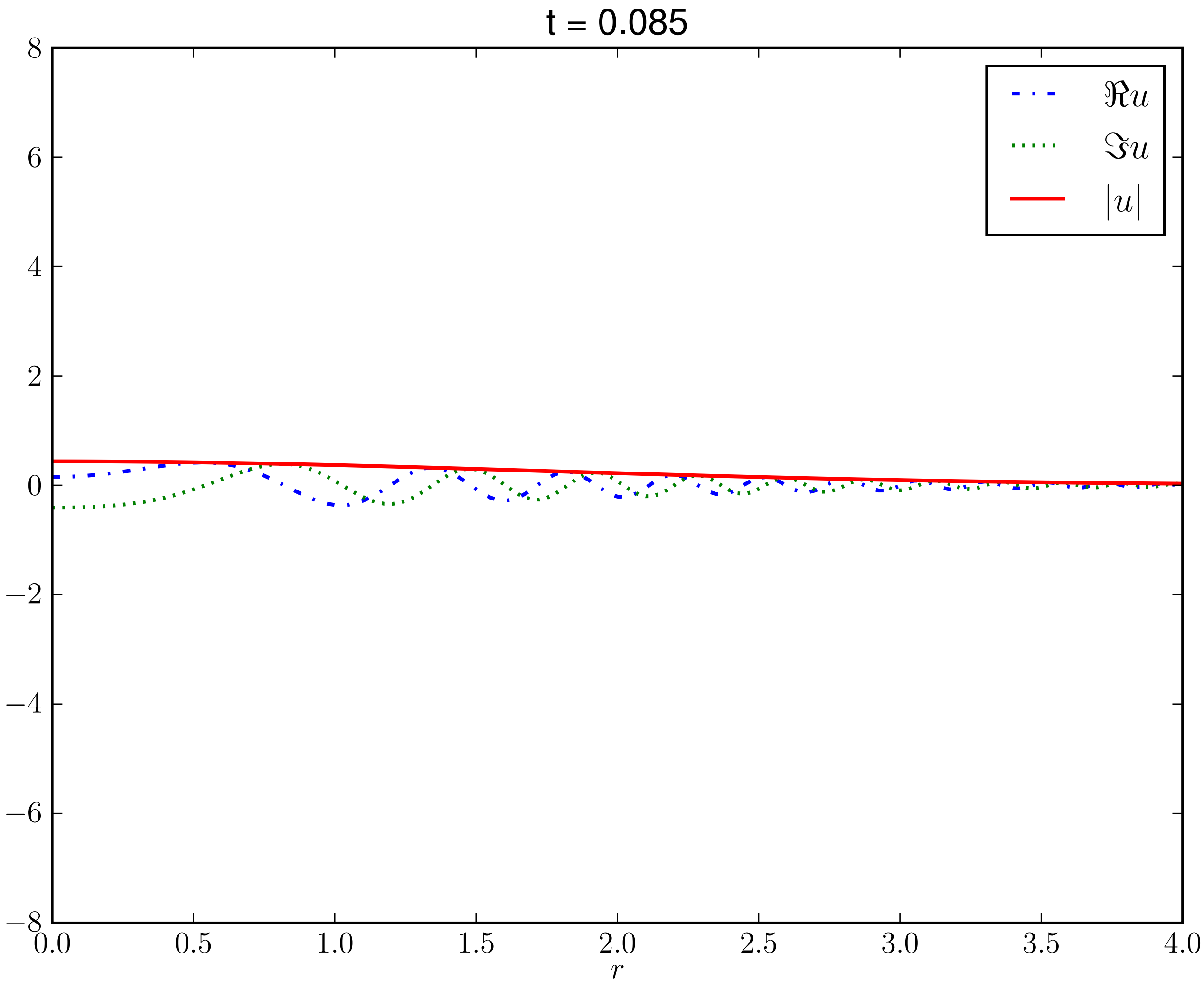


t = 0.07
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

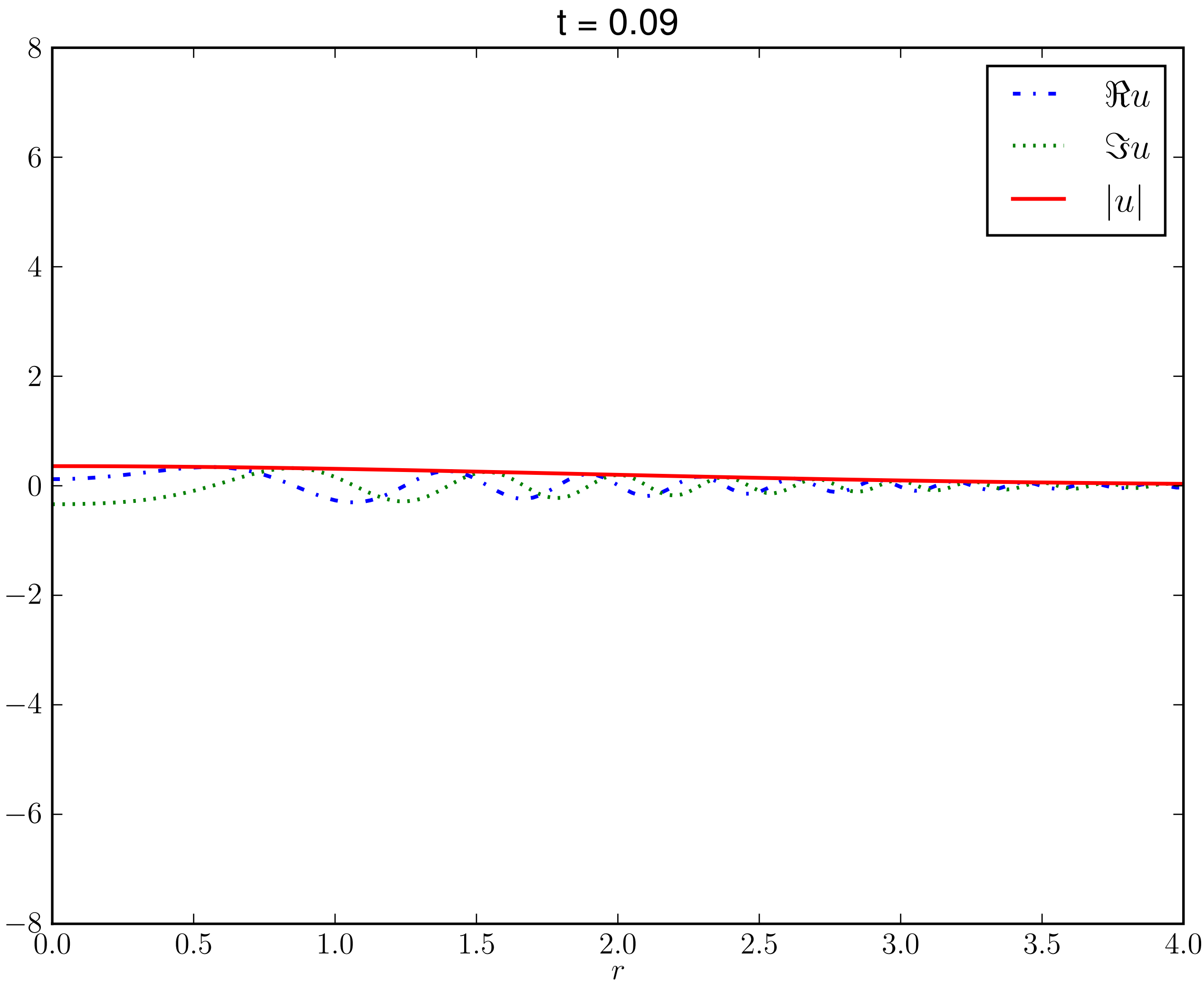


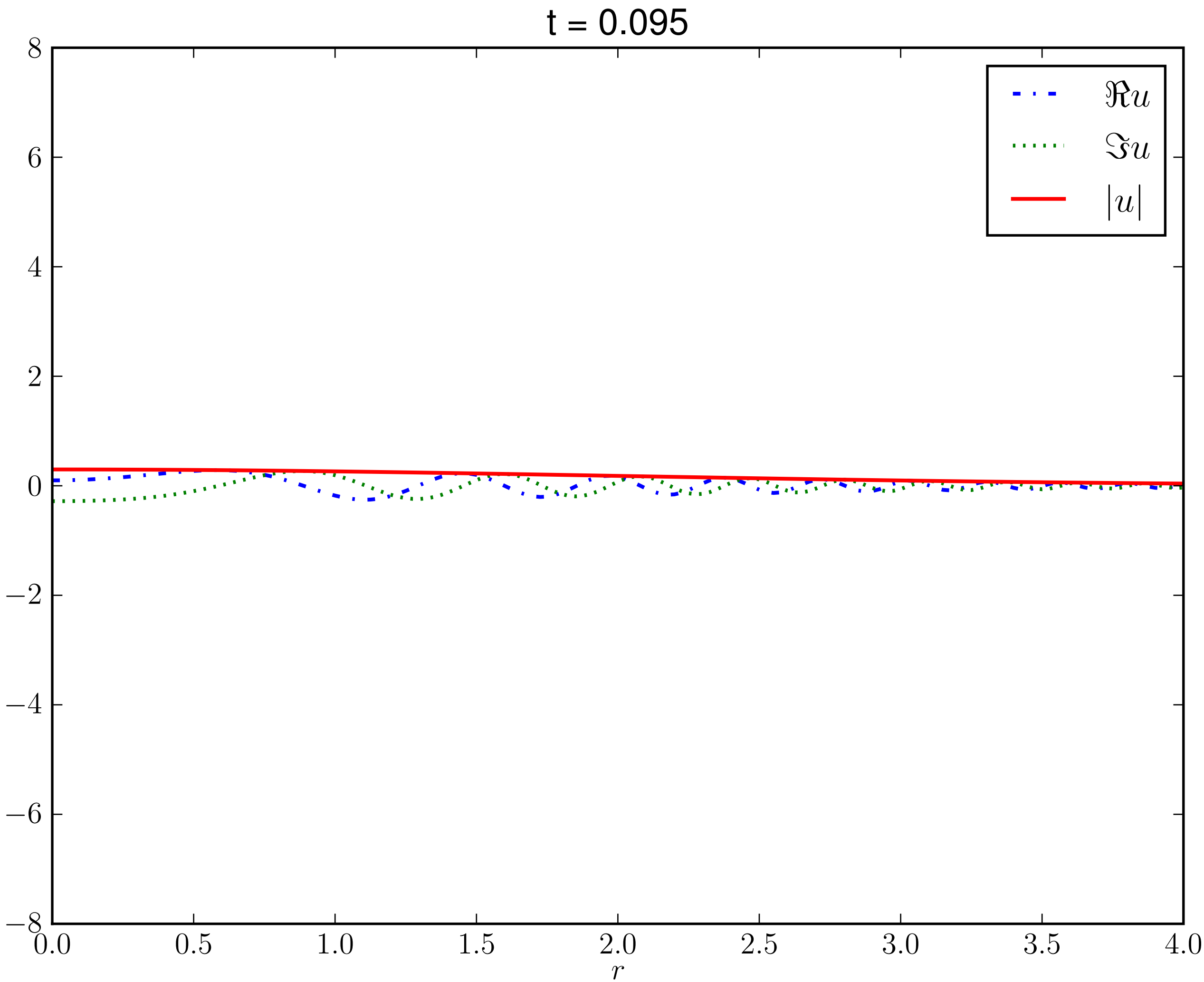
t = 0.075
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

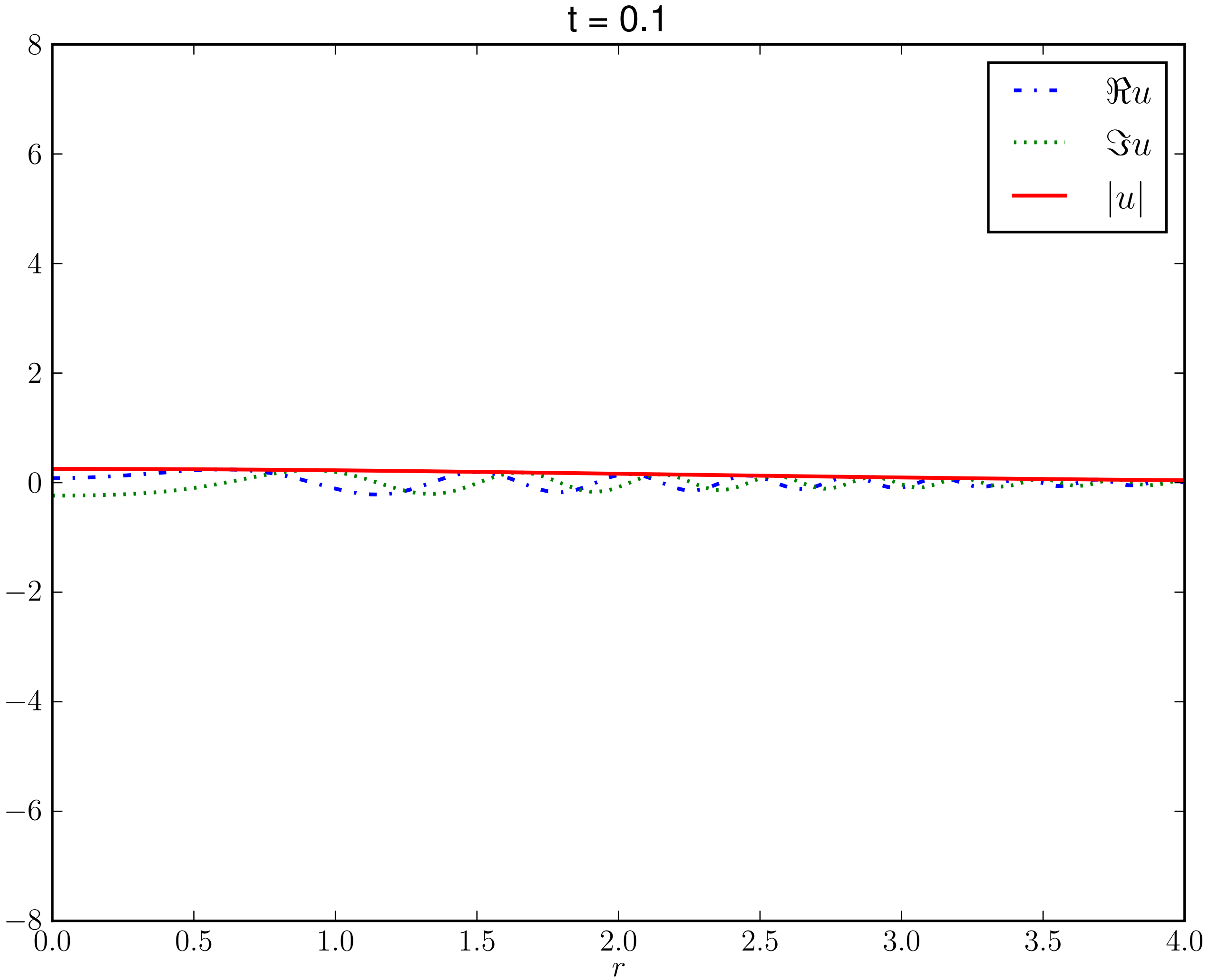

t = 0.08
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r



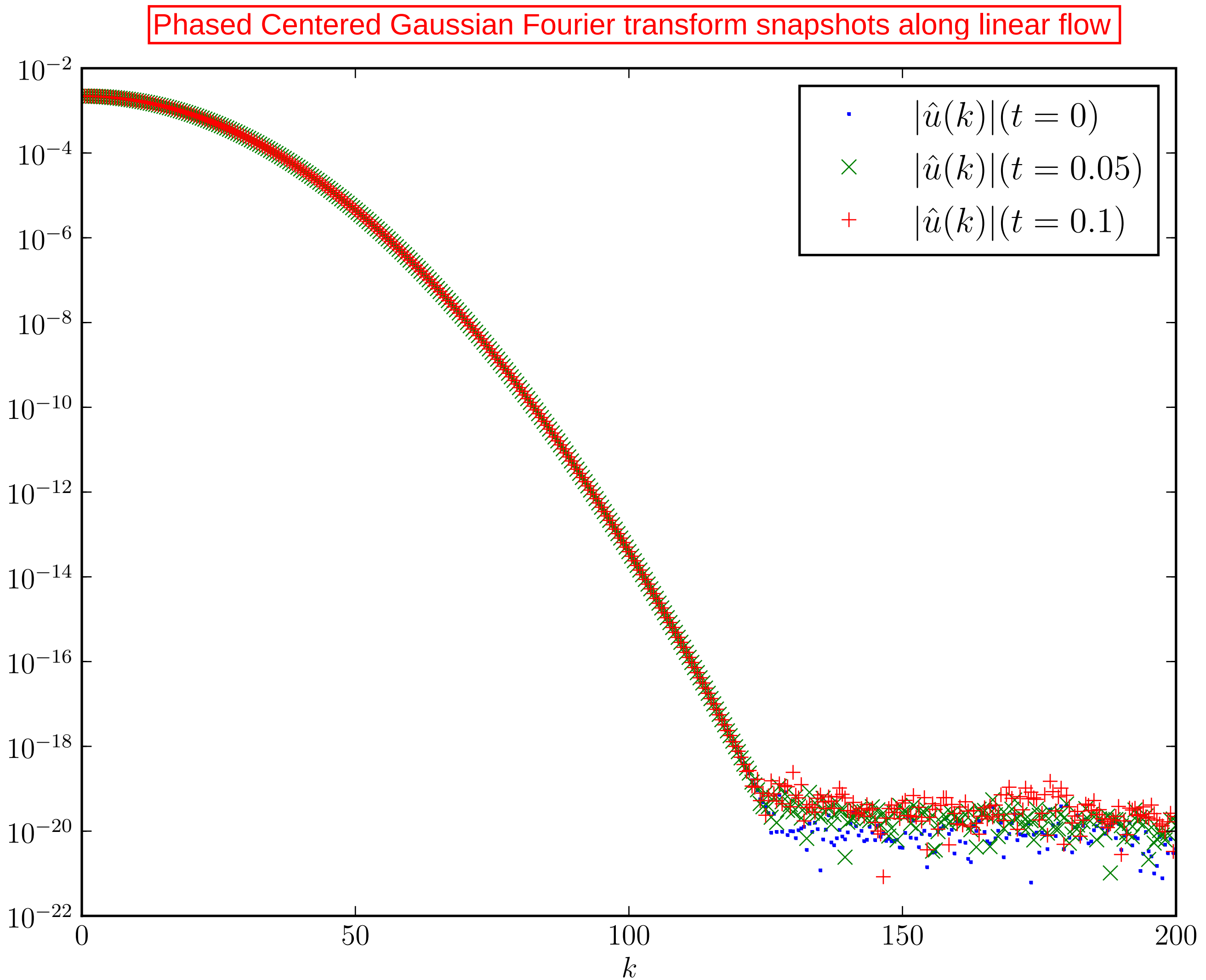
t = 0.085
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

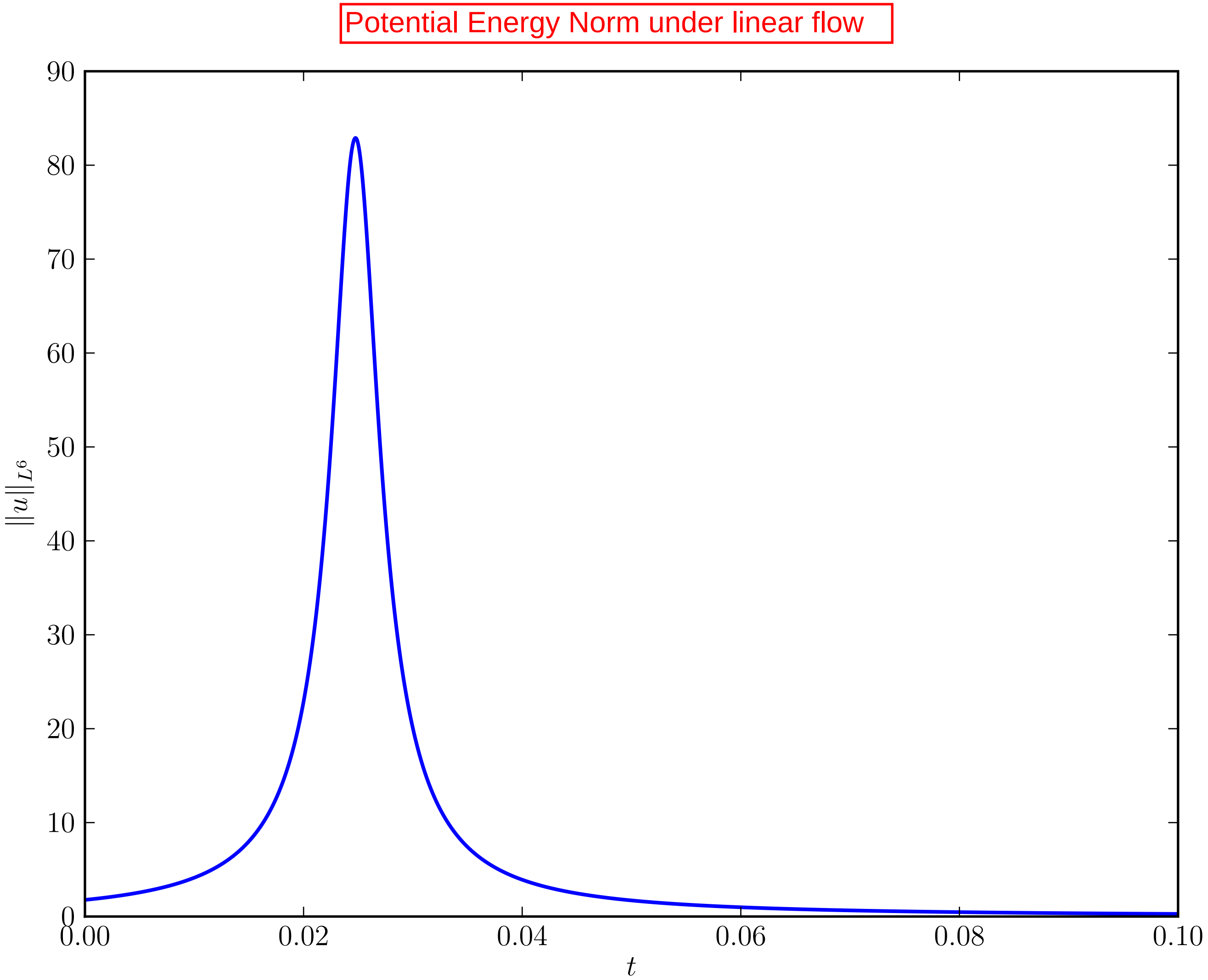

t = 0.09
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.095
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

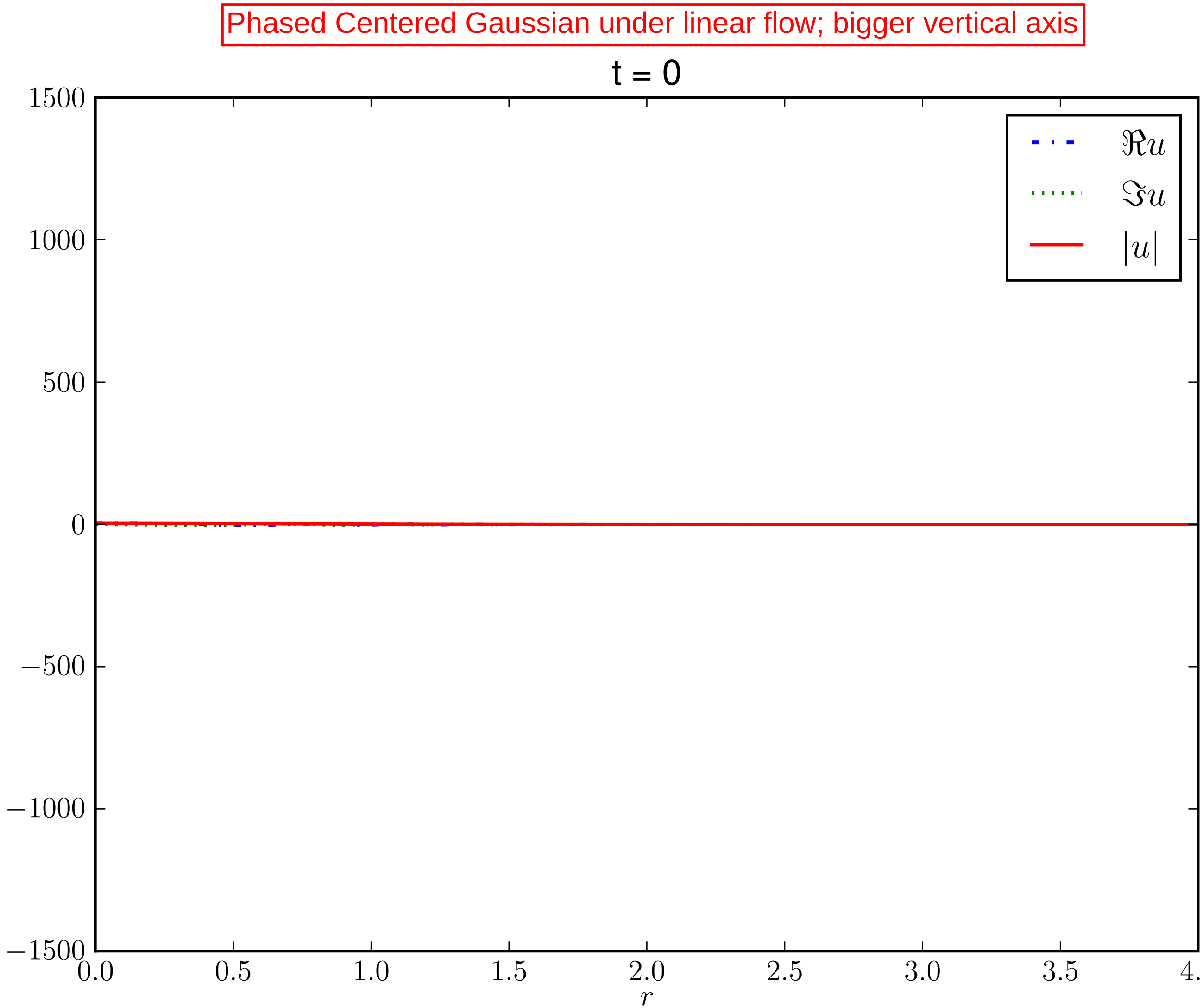


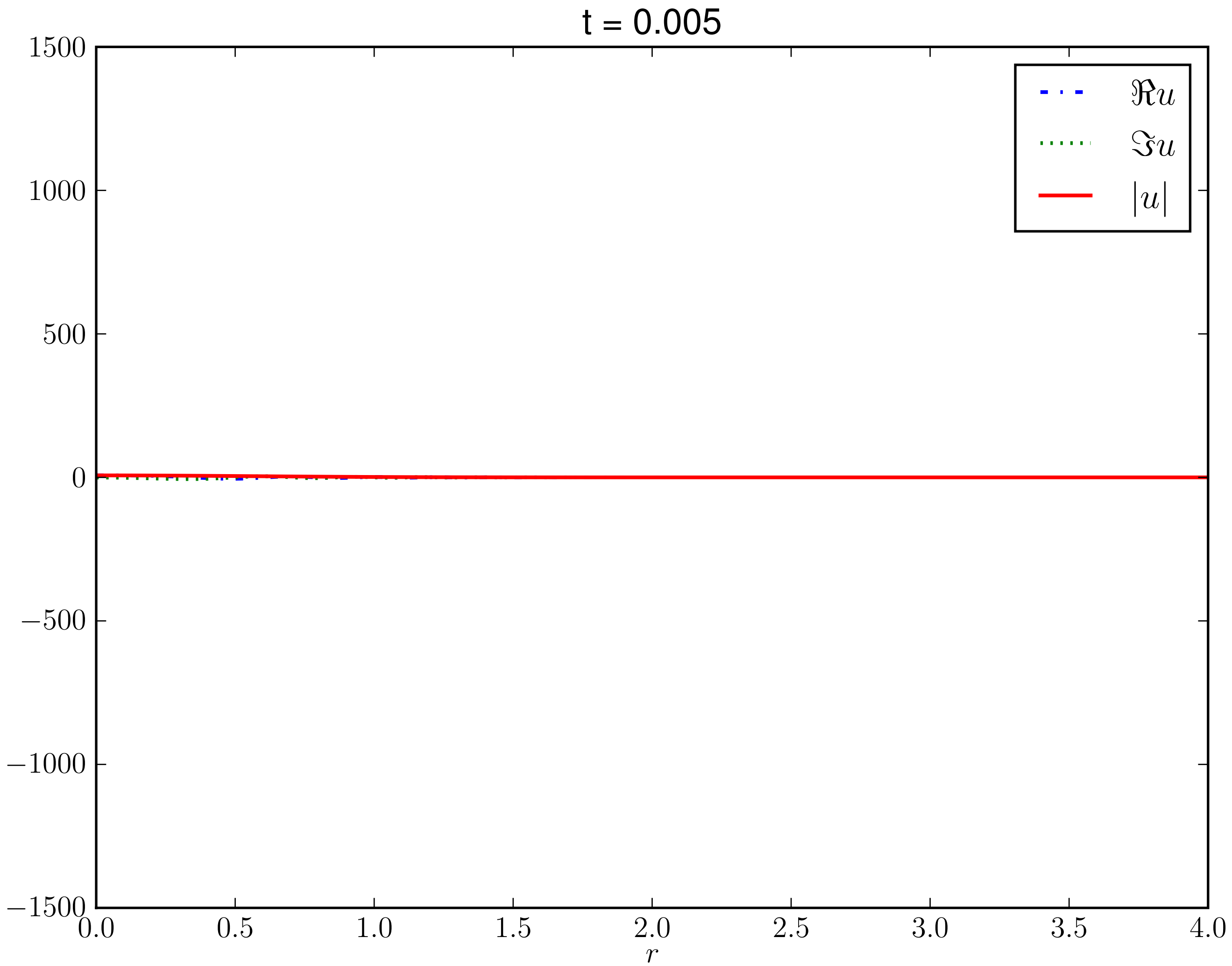
t = 0.1
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


Phased Centered Gaussian Fourier transform snapshots along linear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.05)$
$|\hat{u}(k)|(t=0.1)$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k

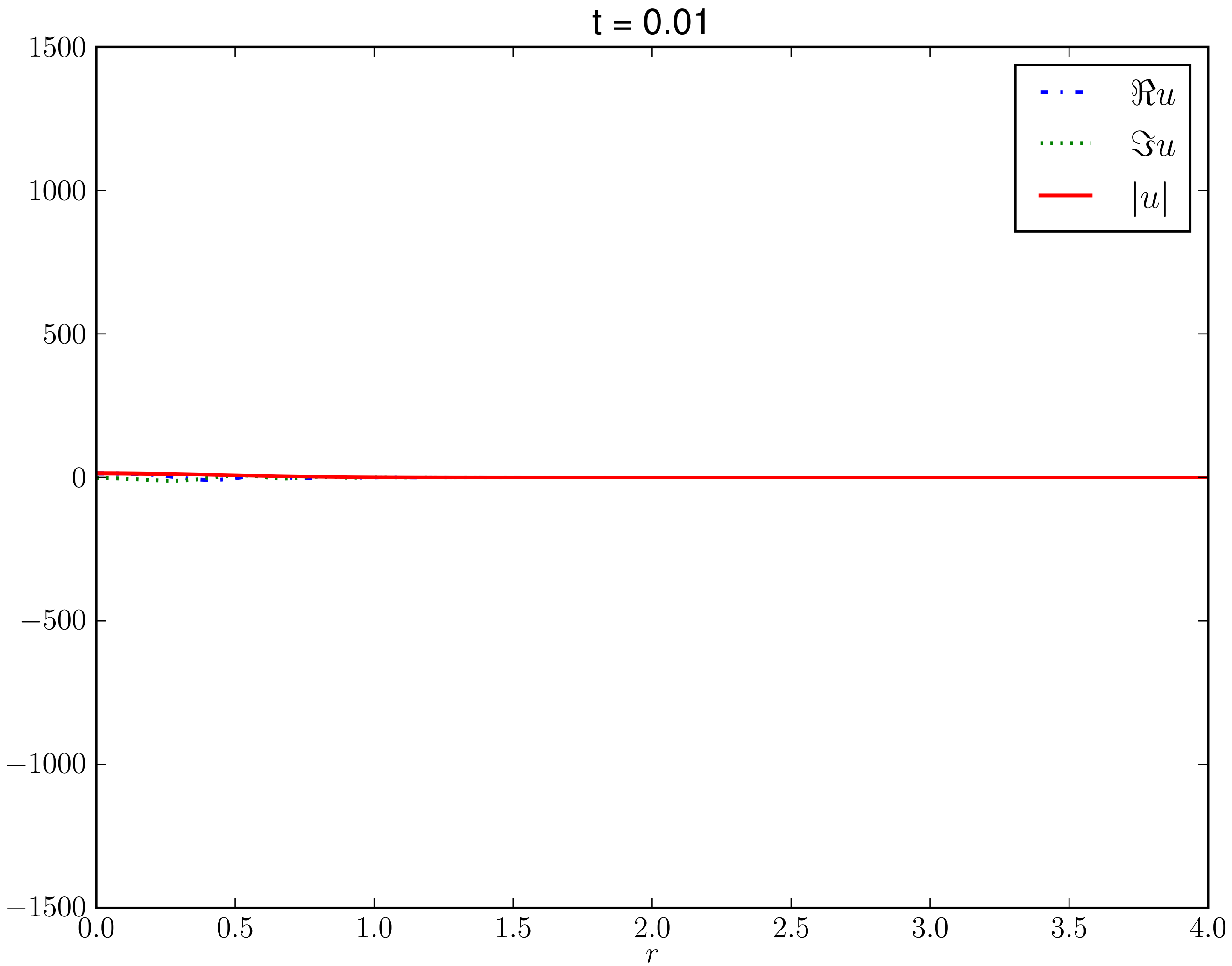


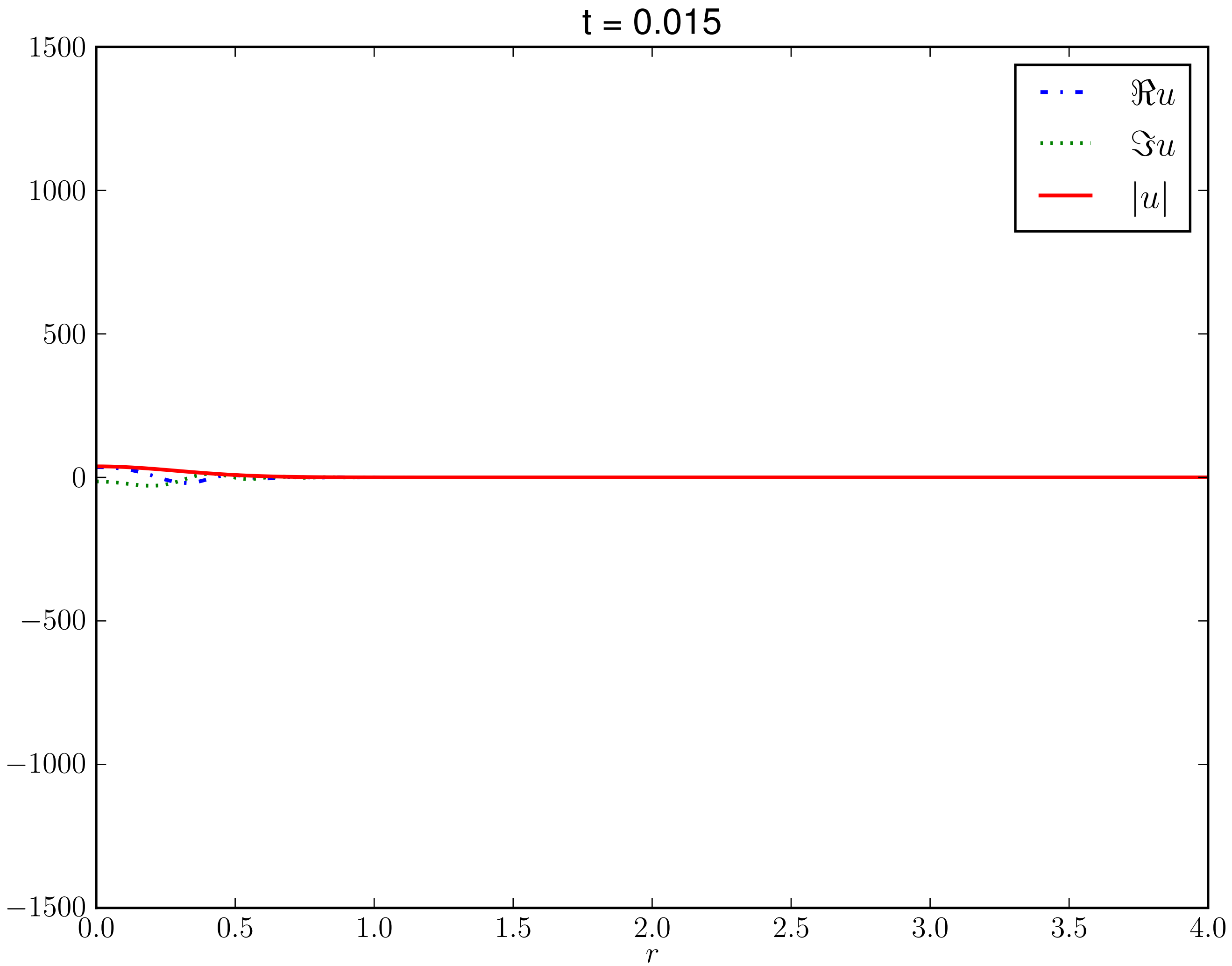
Potential Energy Norm under linear flow
90
80
70
60
50
40
30
20
10
0
$\|u\|_{L^6}$
0.00
0.02
0.04
0.06
0.08
0.10
t

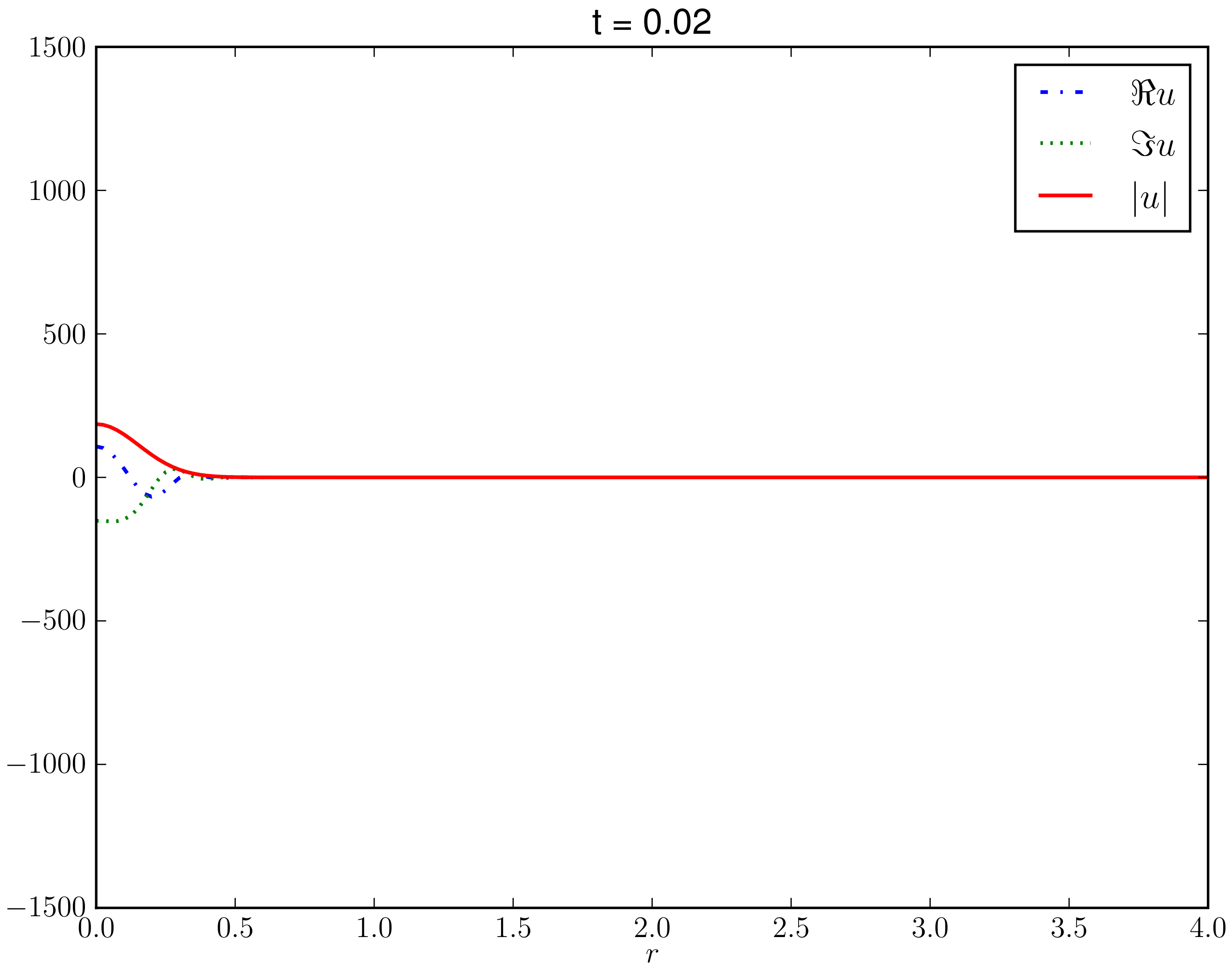

Phased Centered Gaussian under linear flow; bigger vertical axis
t = 0
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



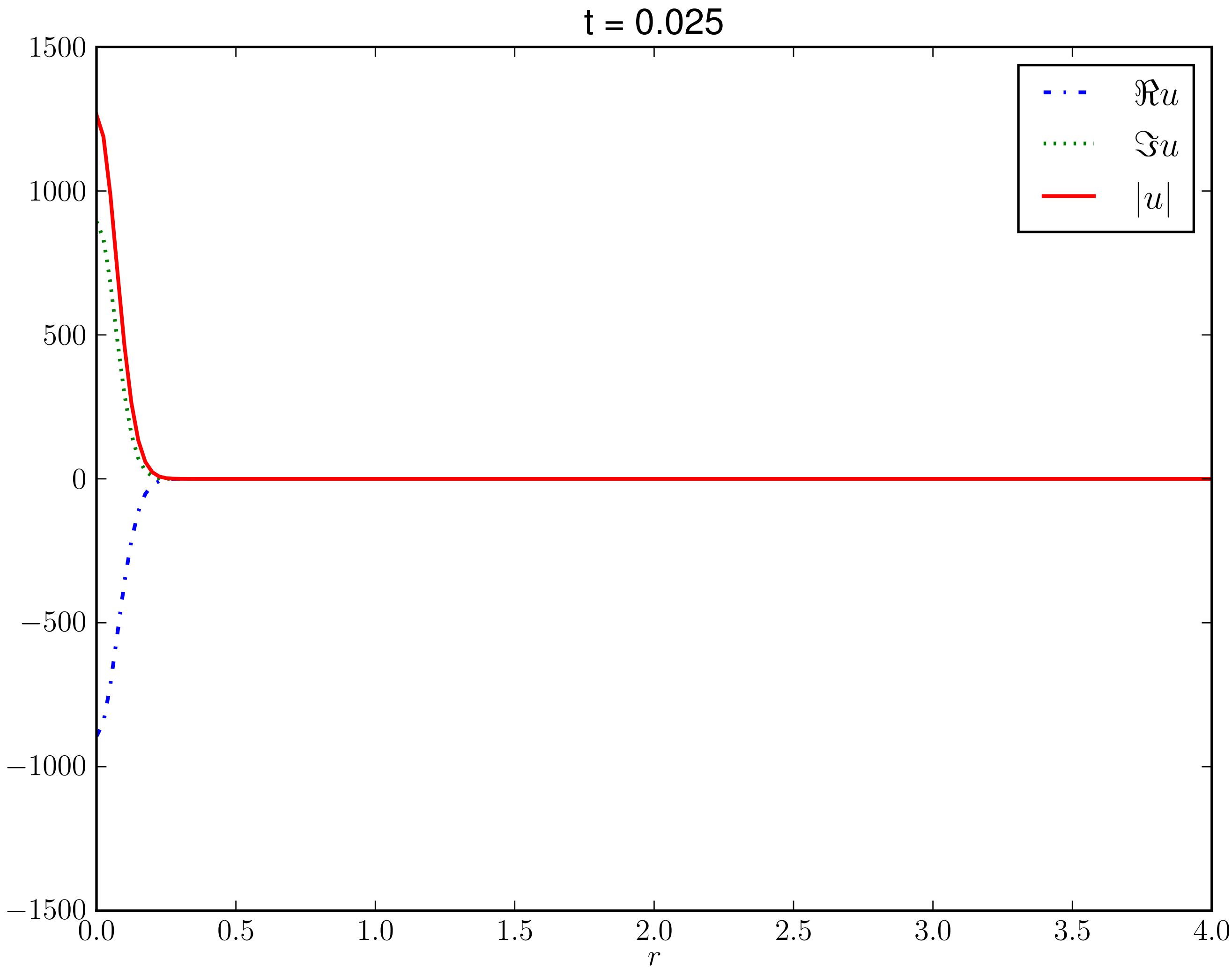
t = 0.005
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

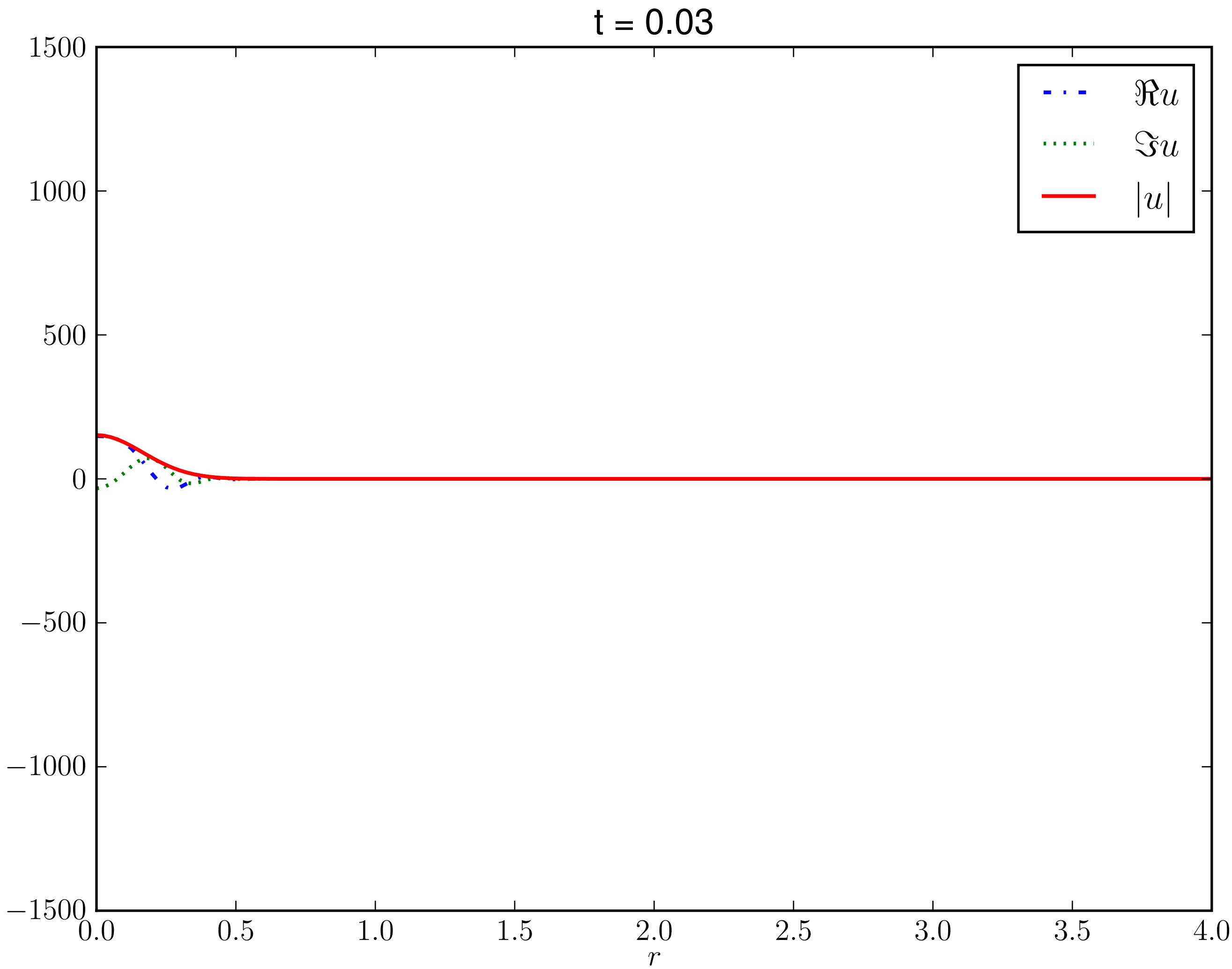

t = 0.01
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.015
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

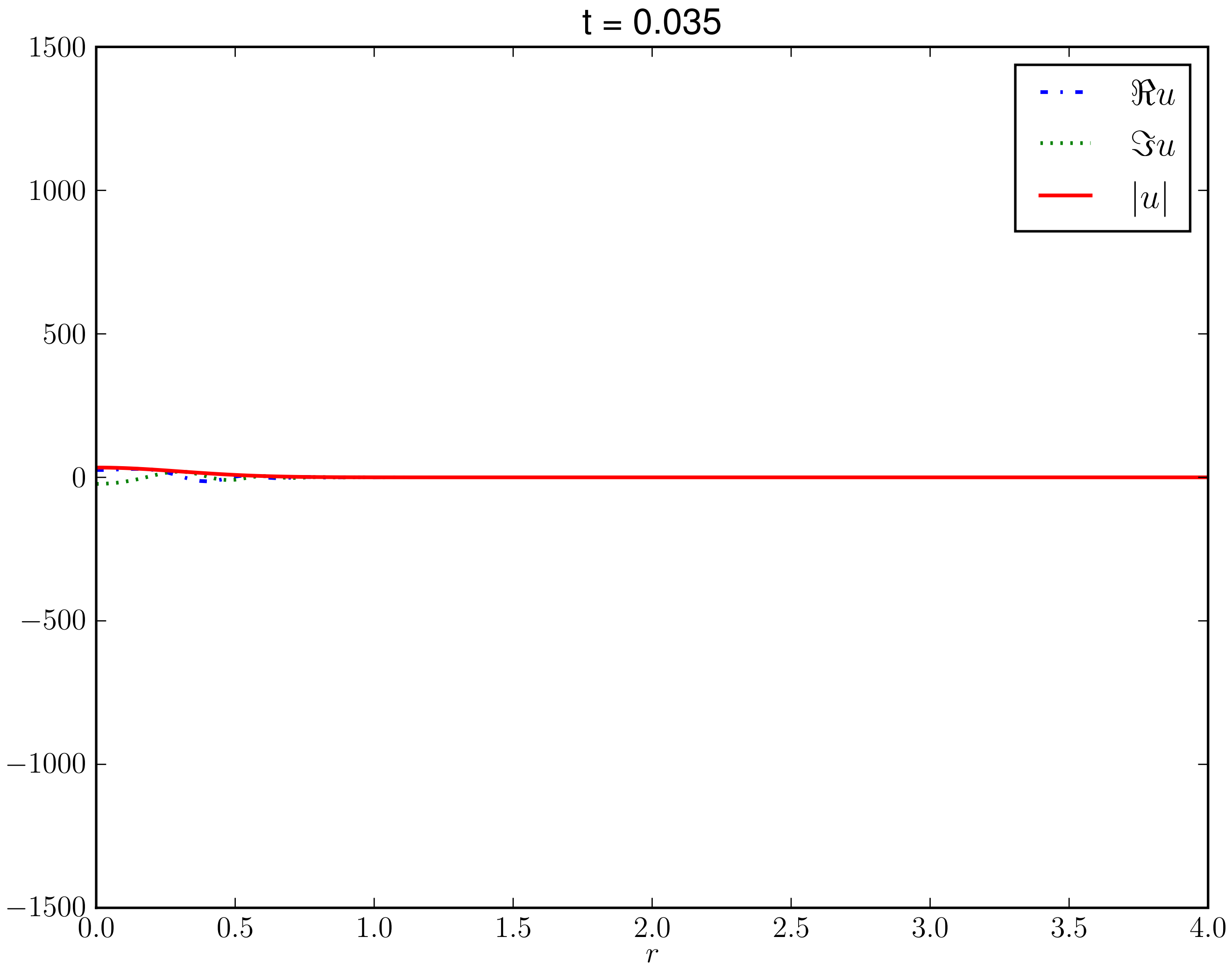


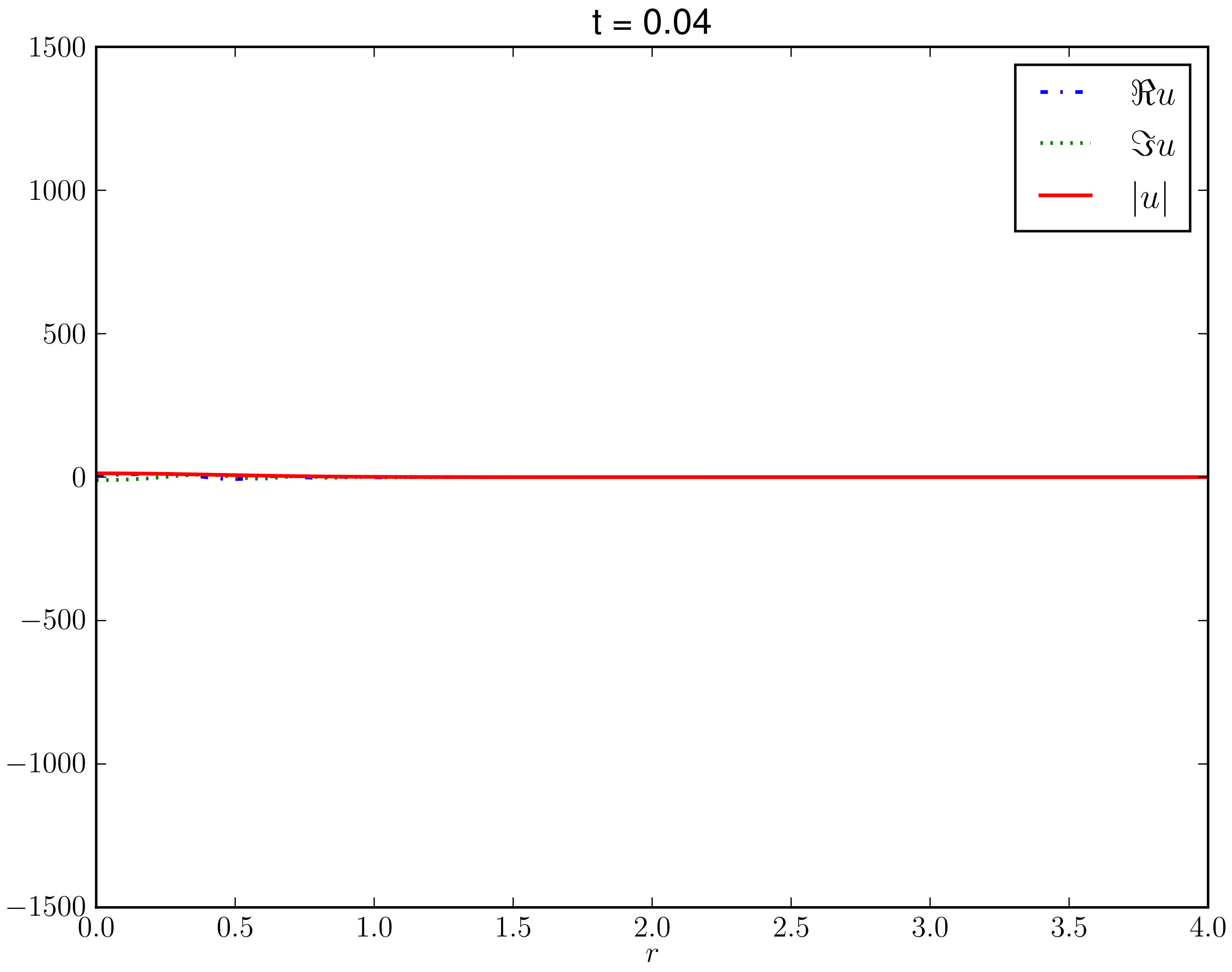
t = 0.02
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.025
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

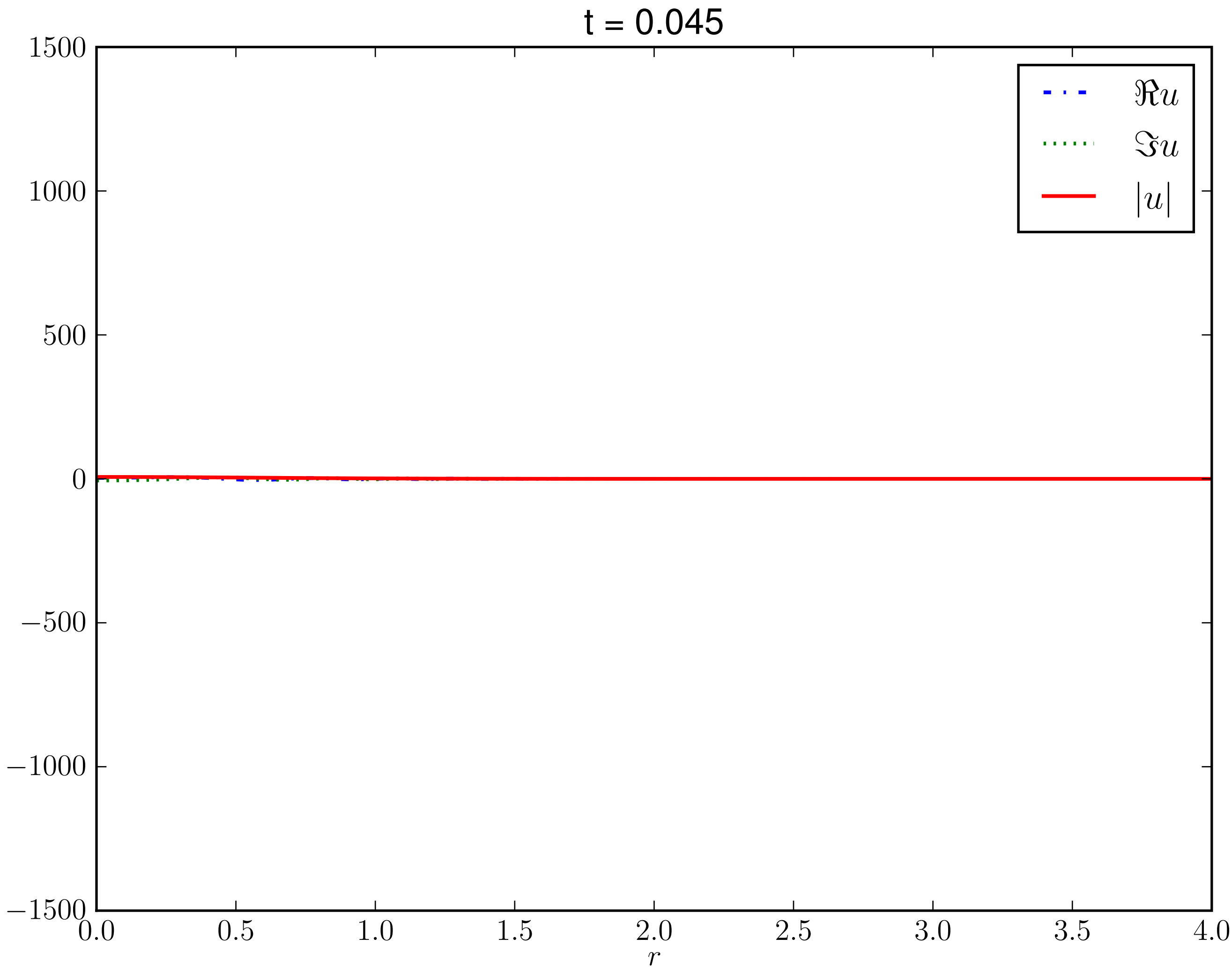


t = 0.03
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

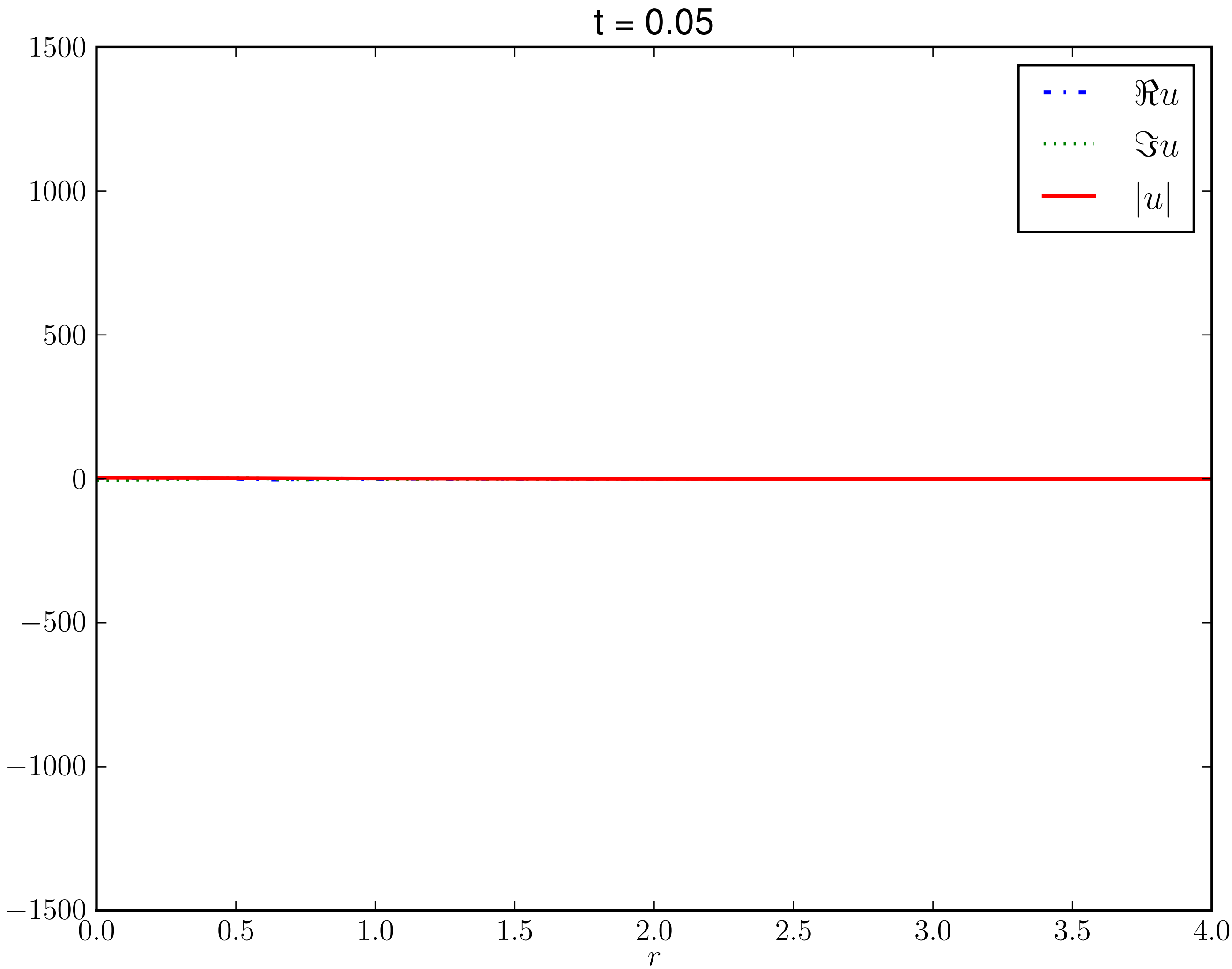

t = 0.035
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

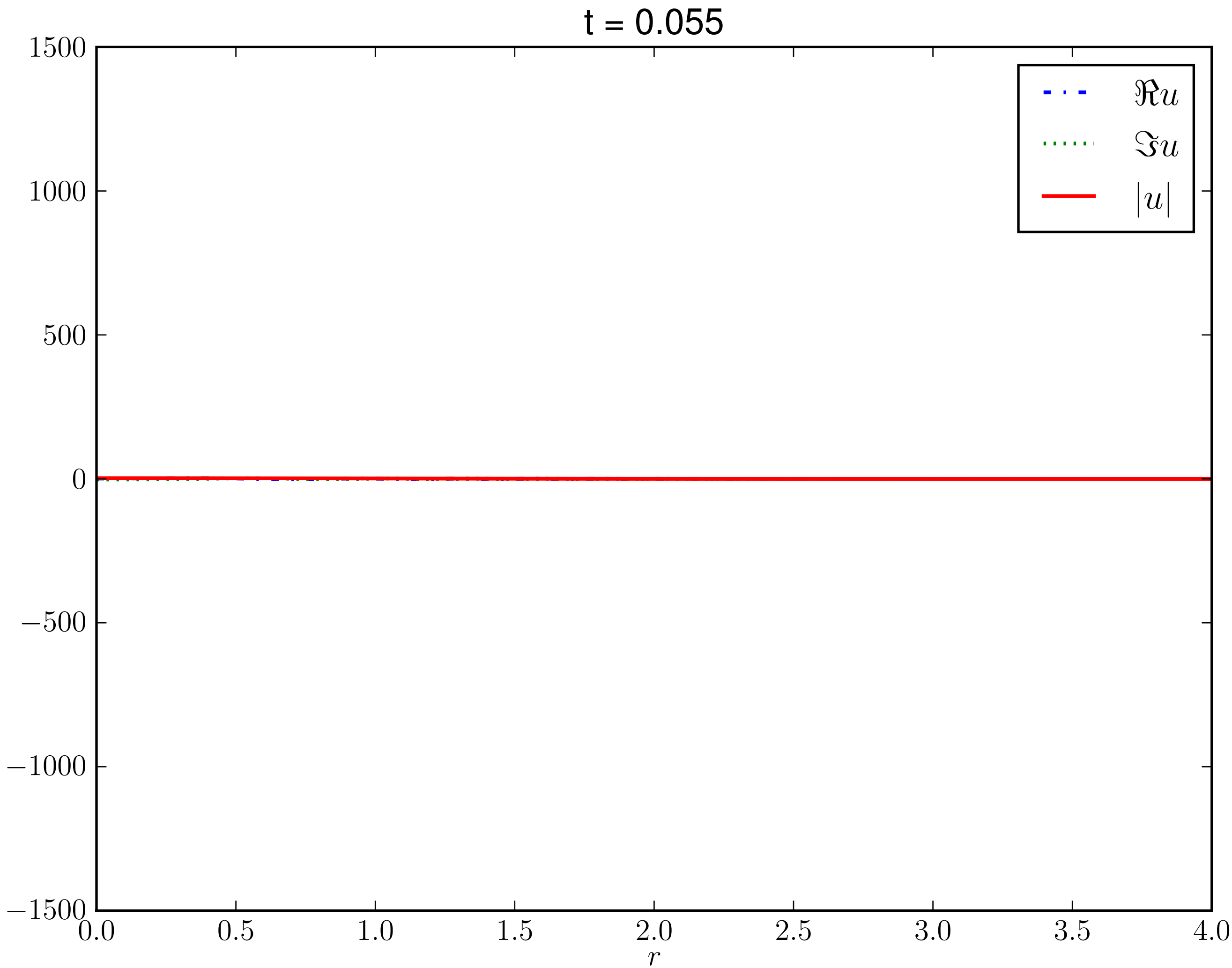


t = 0.04
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

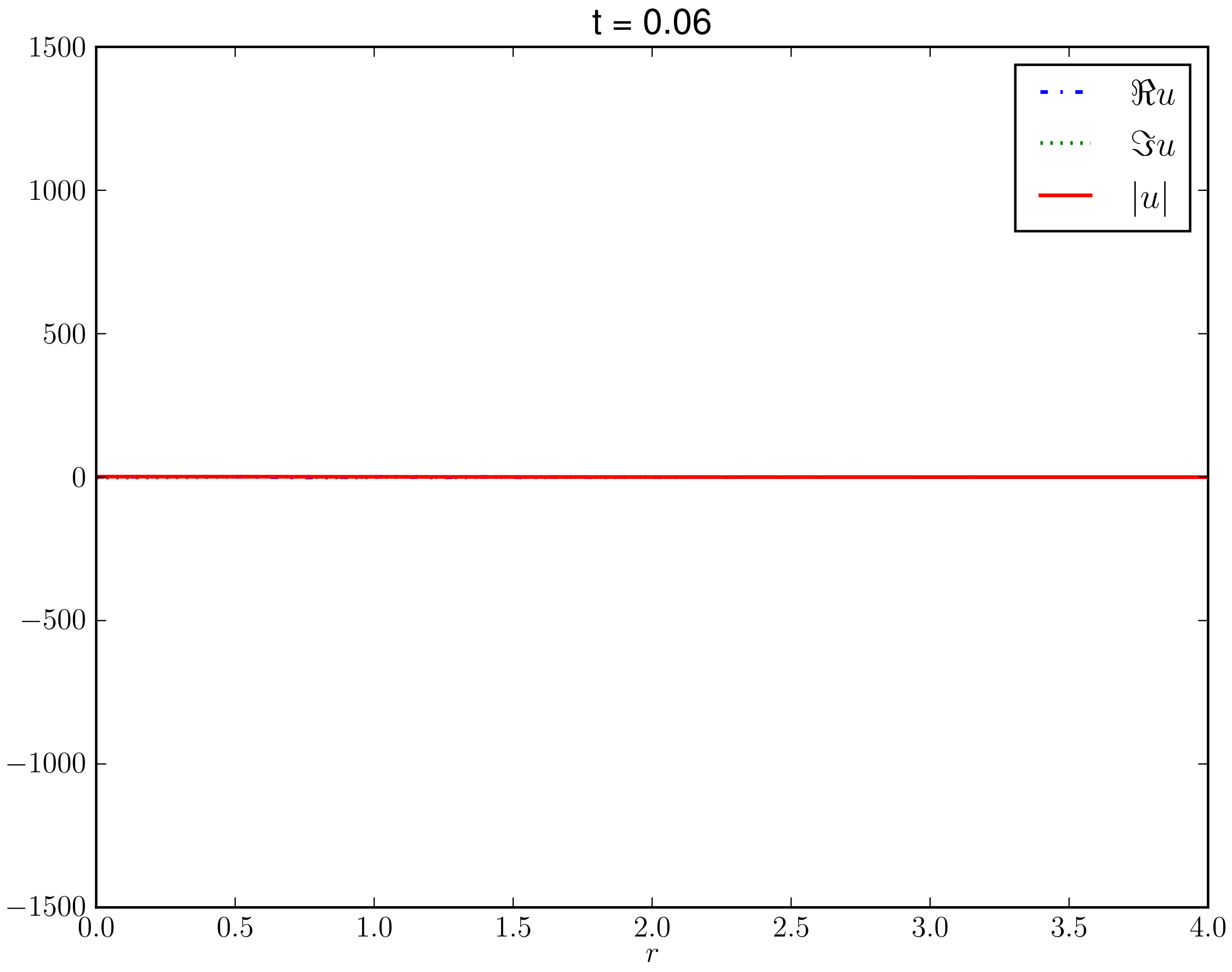


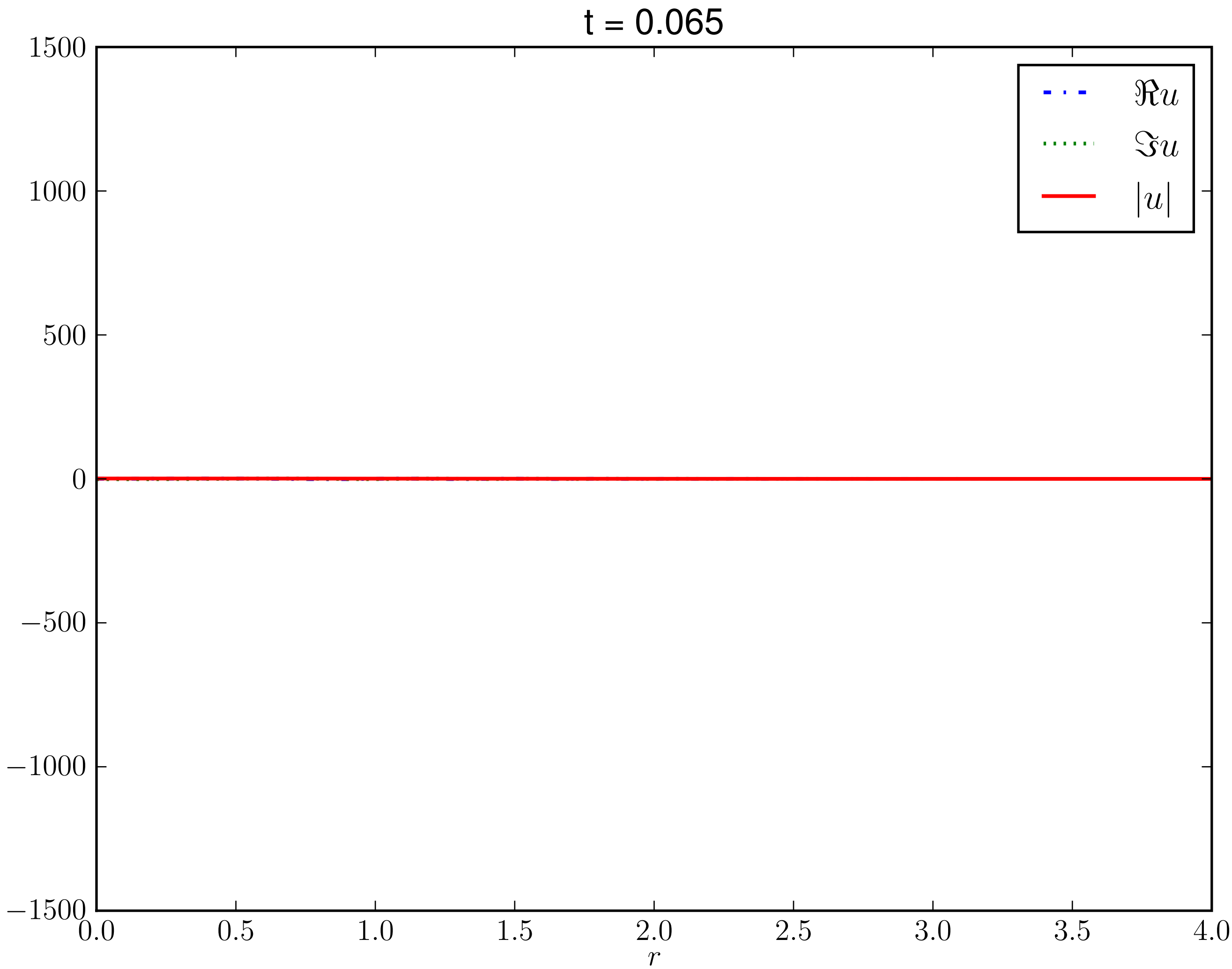
t = 0.045
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

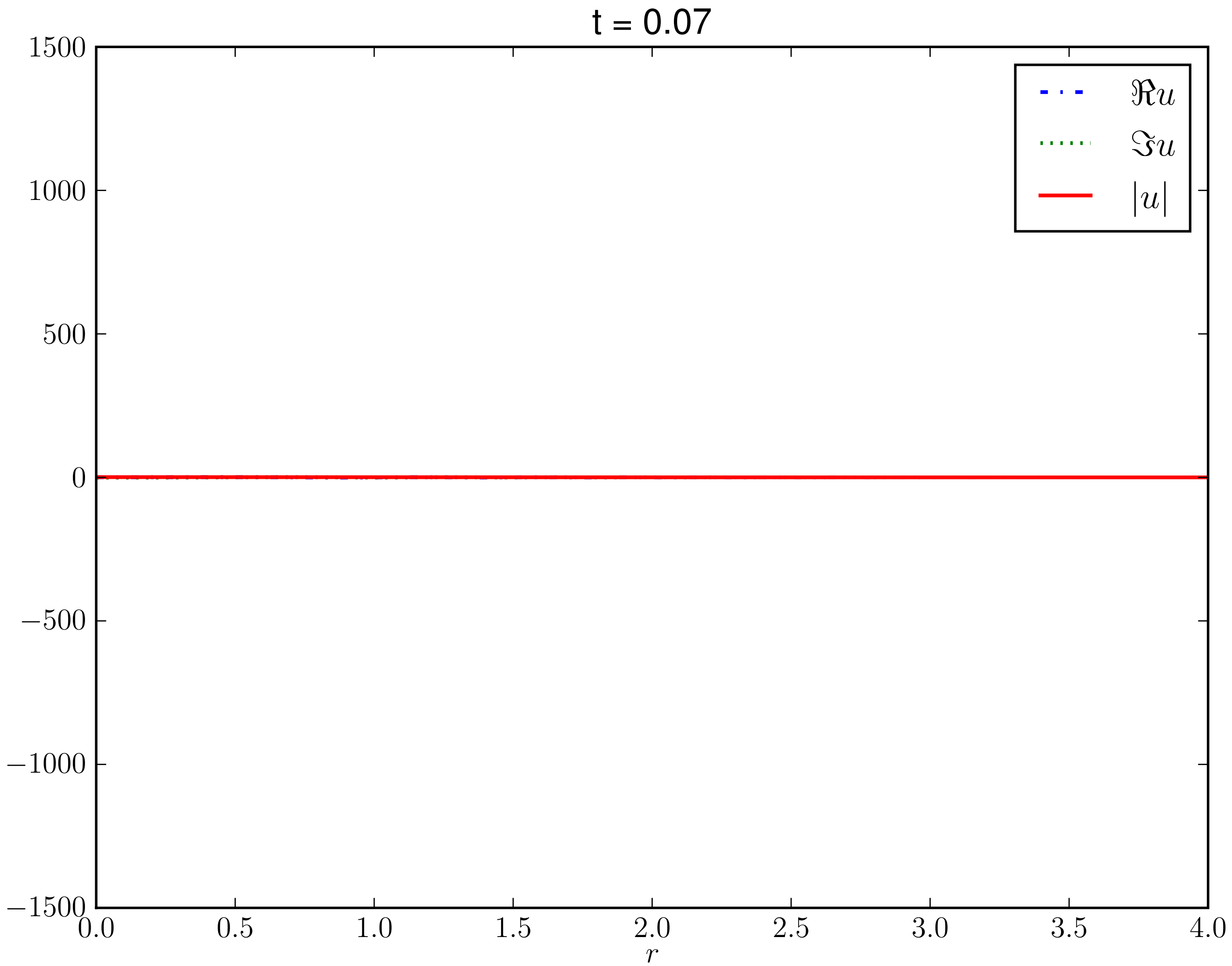

t = 0.05
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

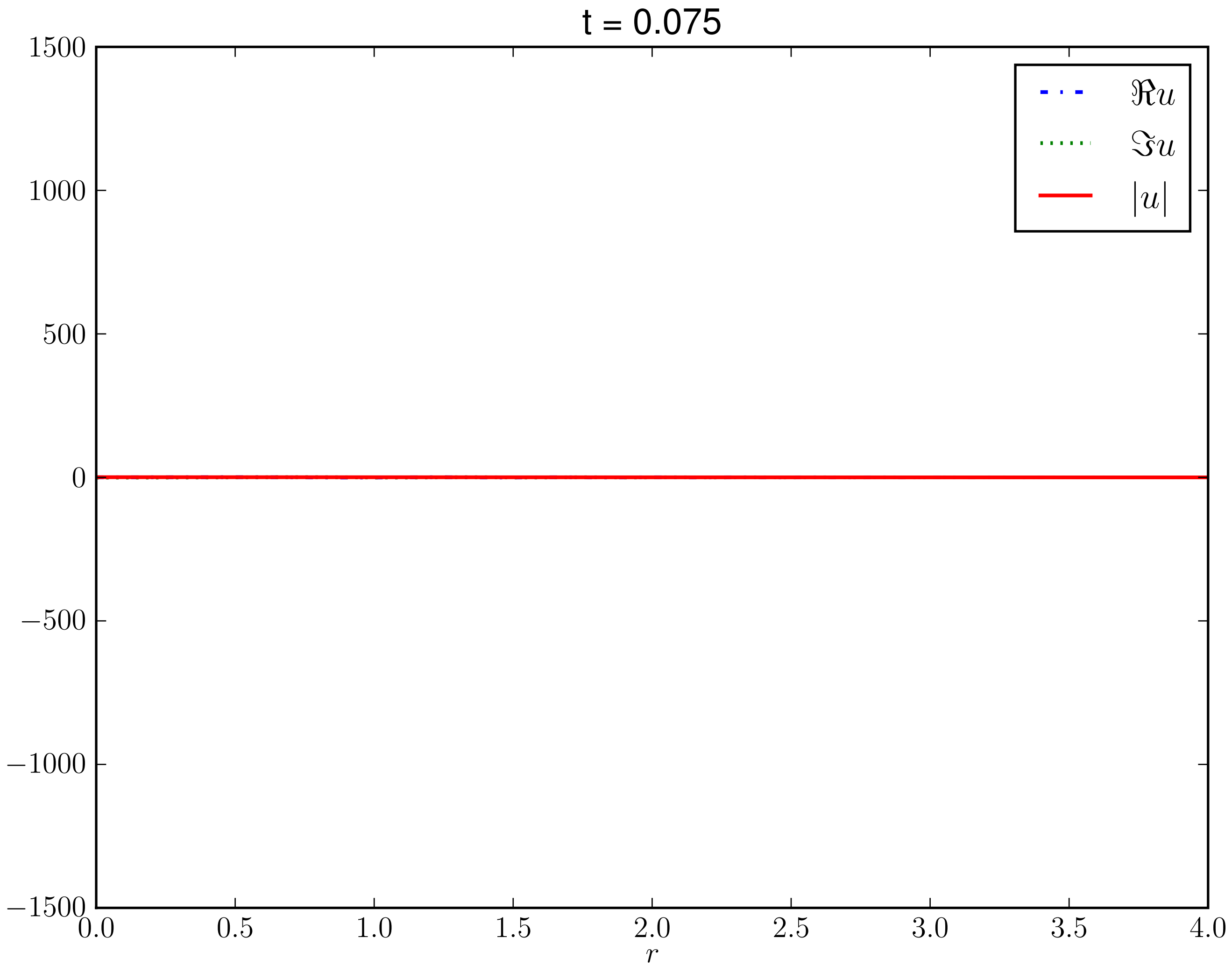


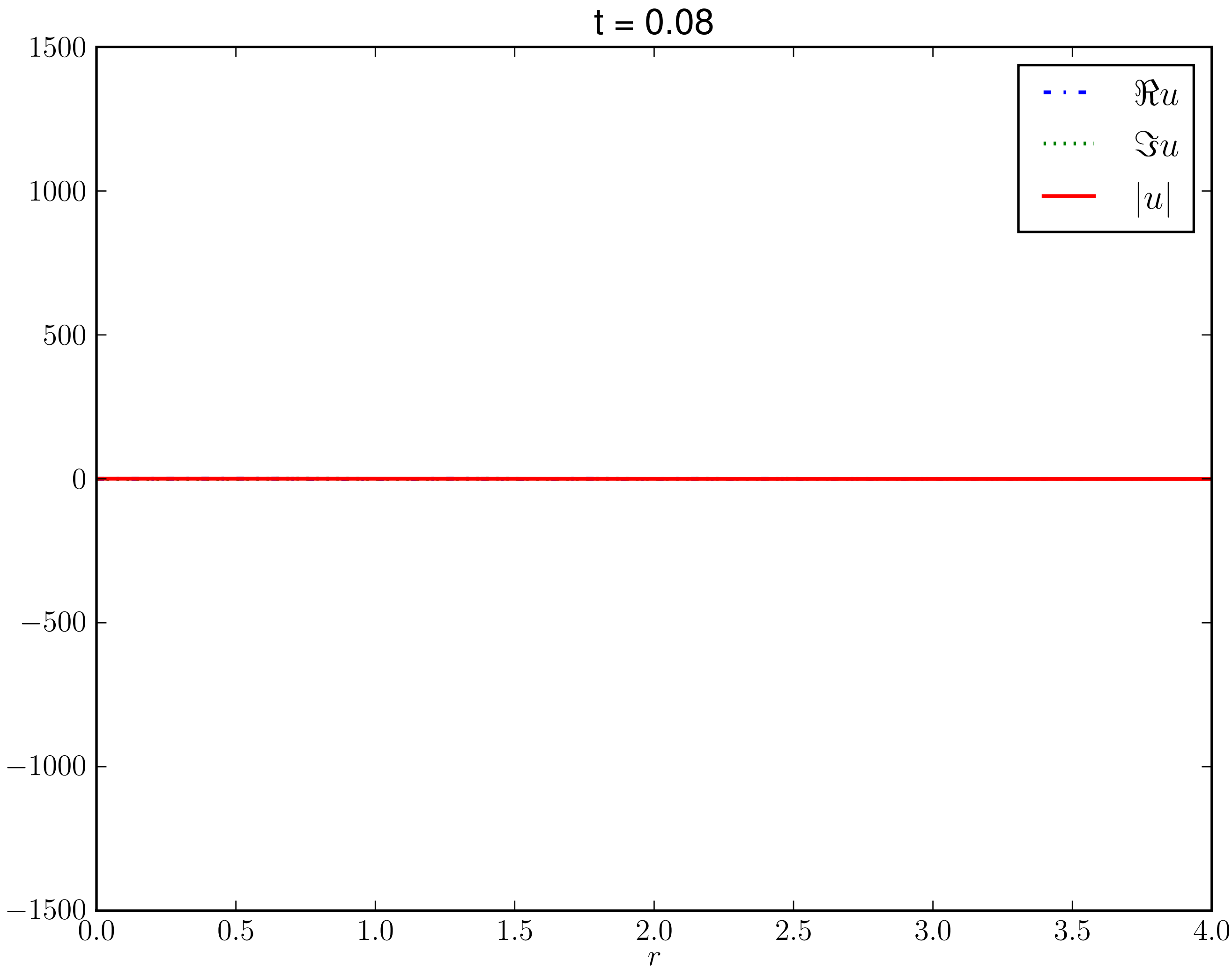
t = 0.055
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$


t = 0.06
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

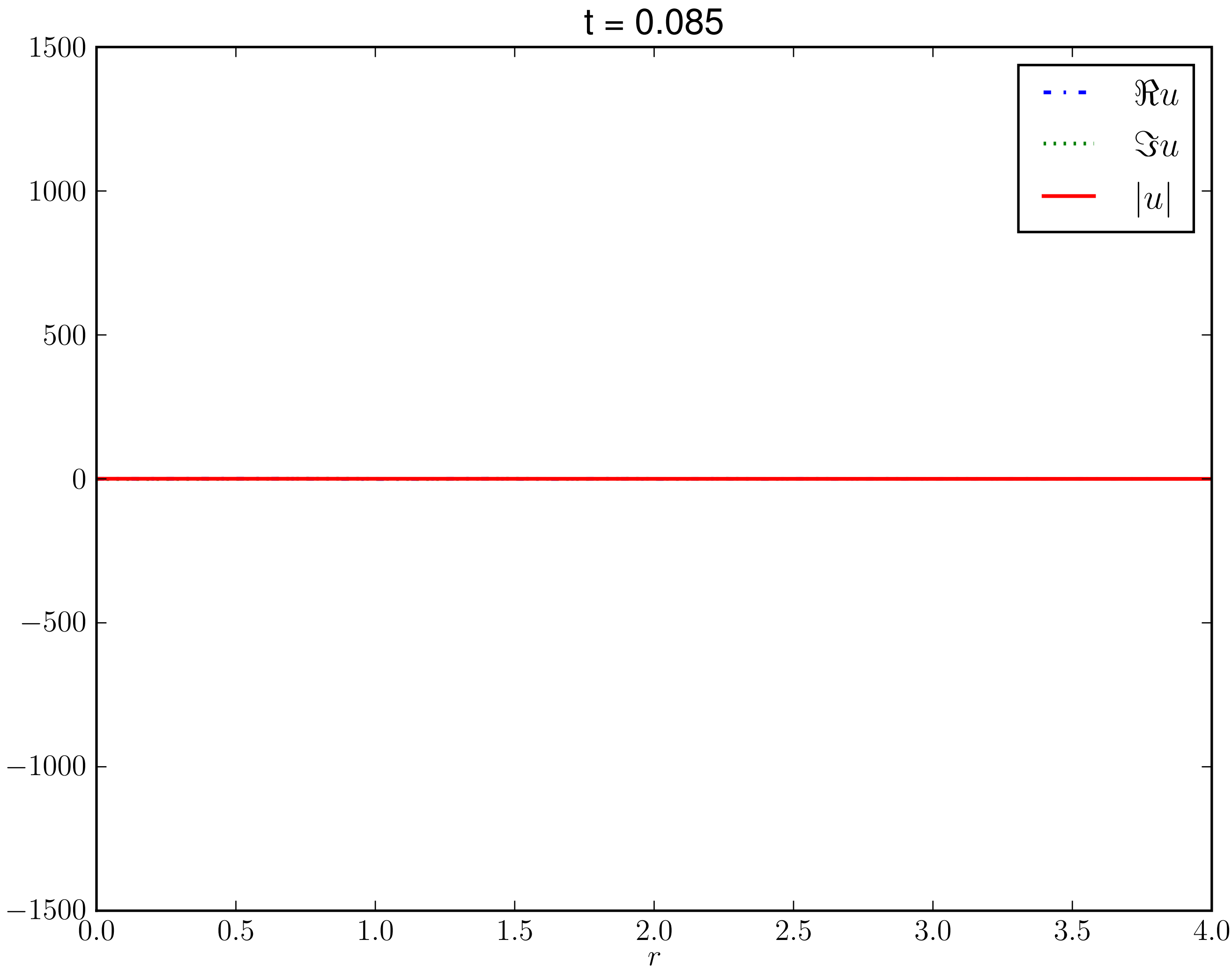

t = 0.065
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

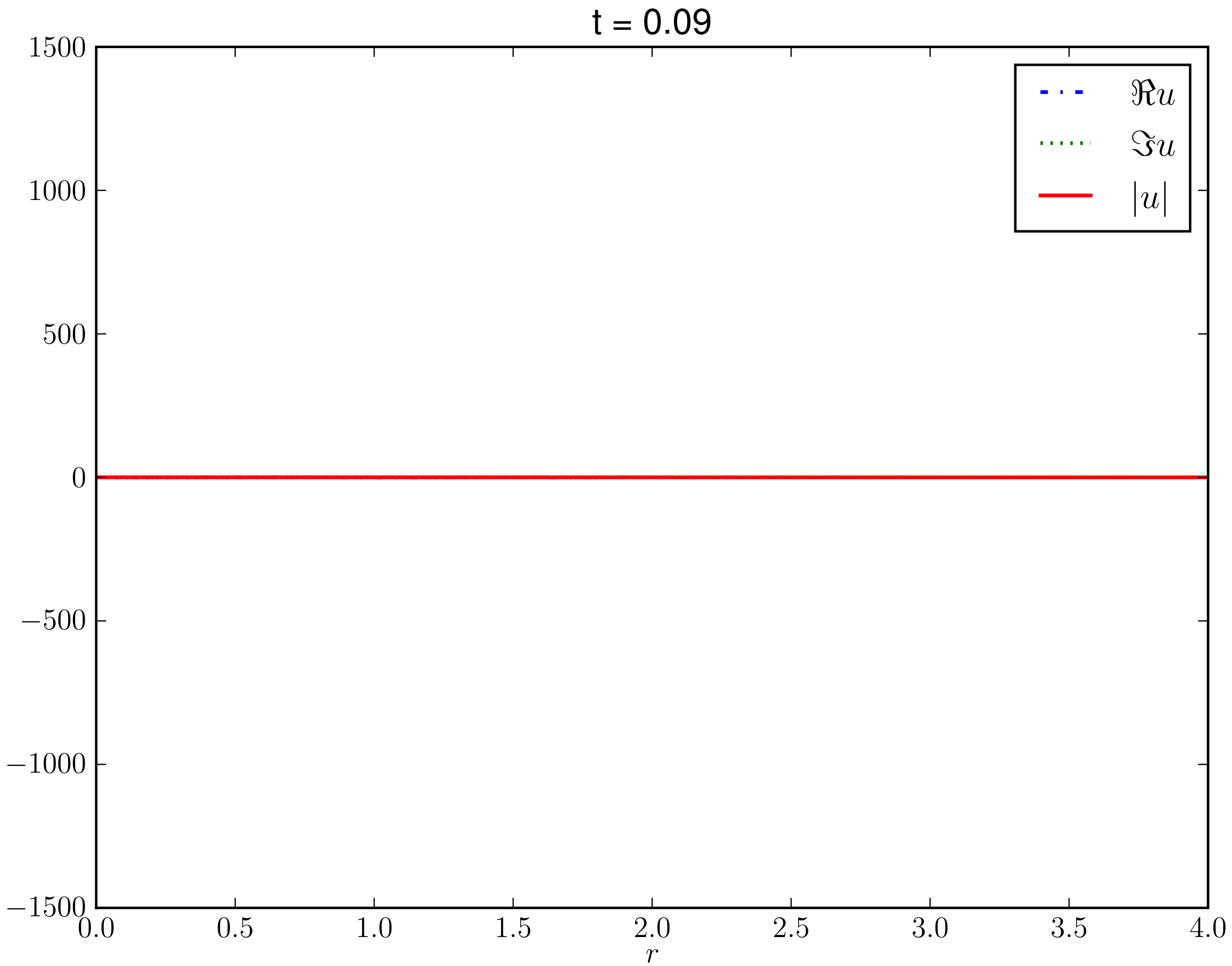

t = 0.07
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

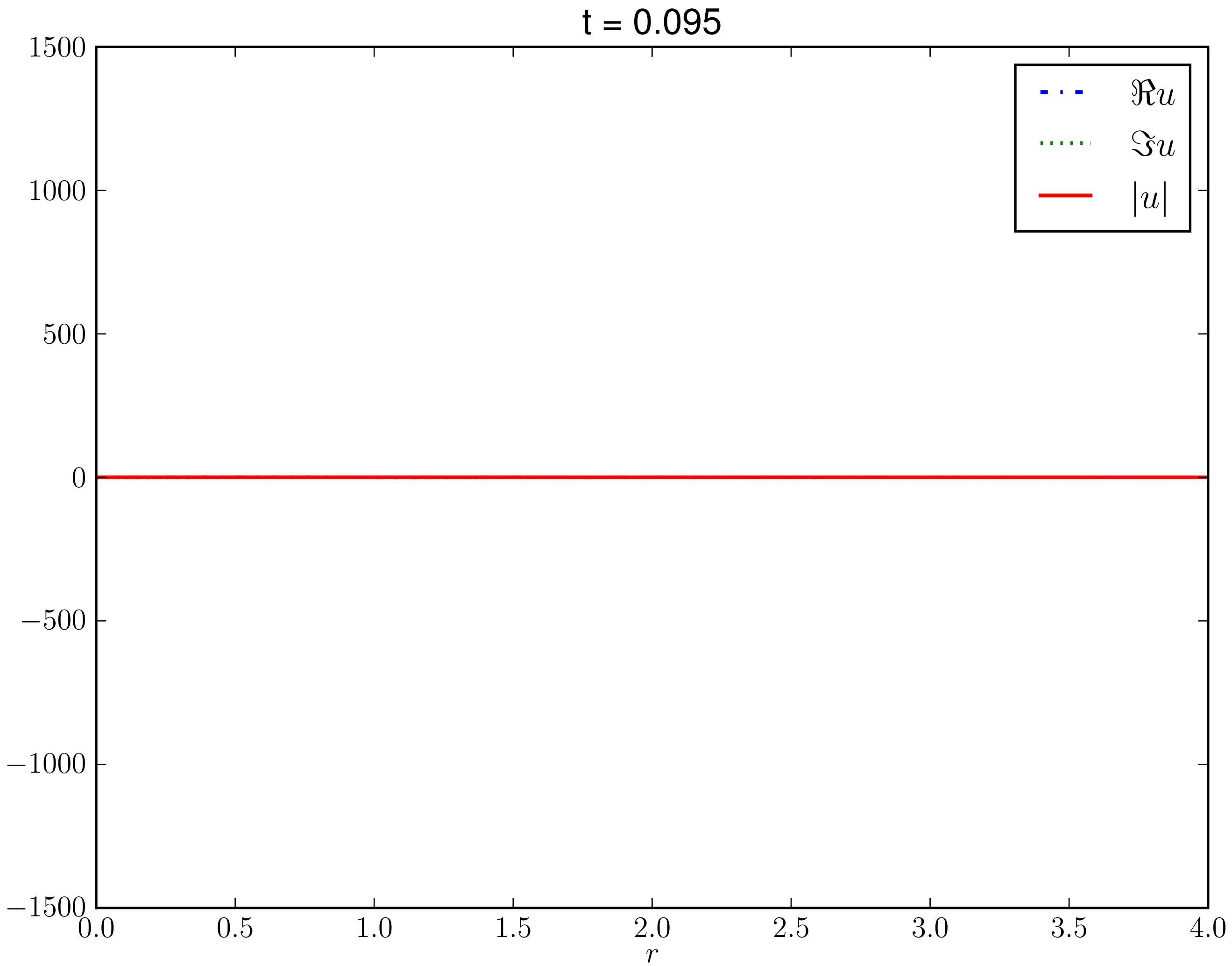

t = 0.075
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

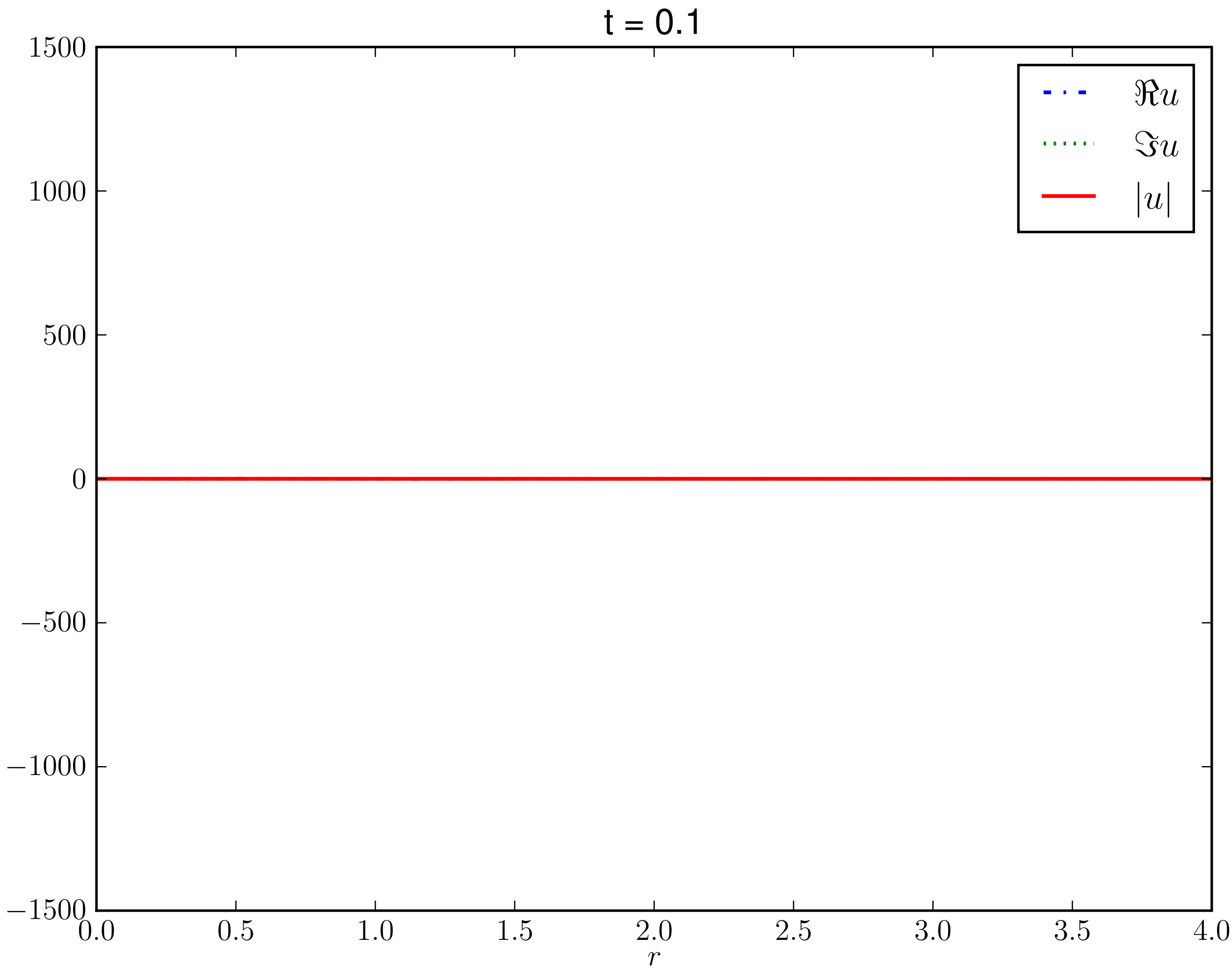


t = 0.08
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

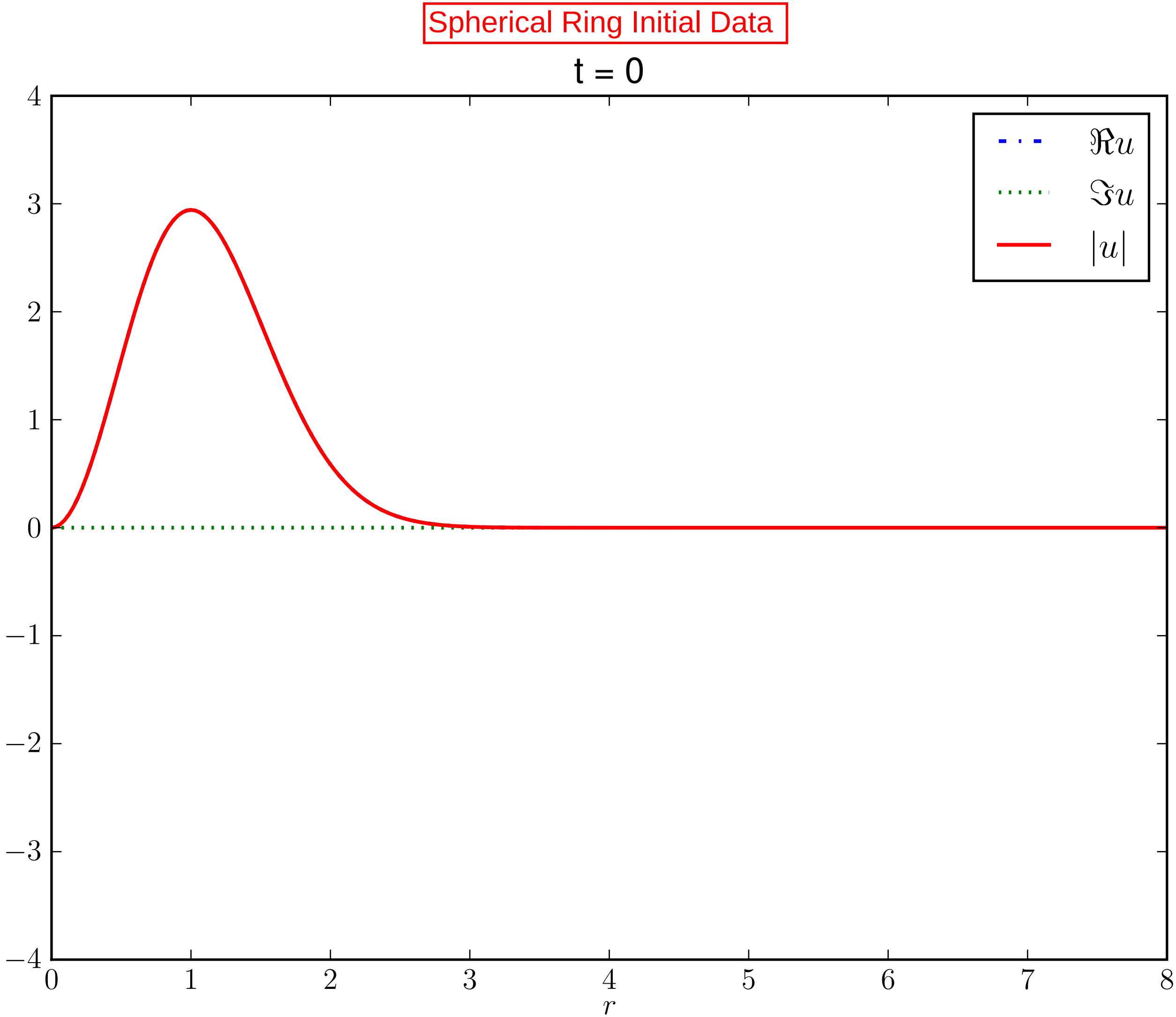

t = 0.085
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

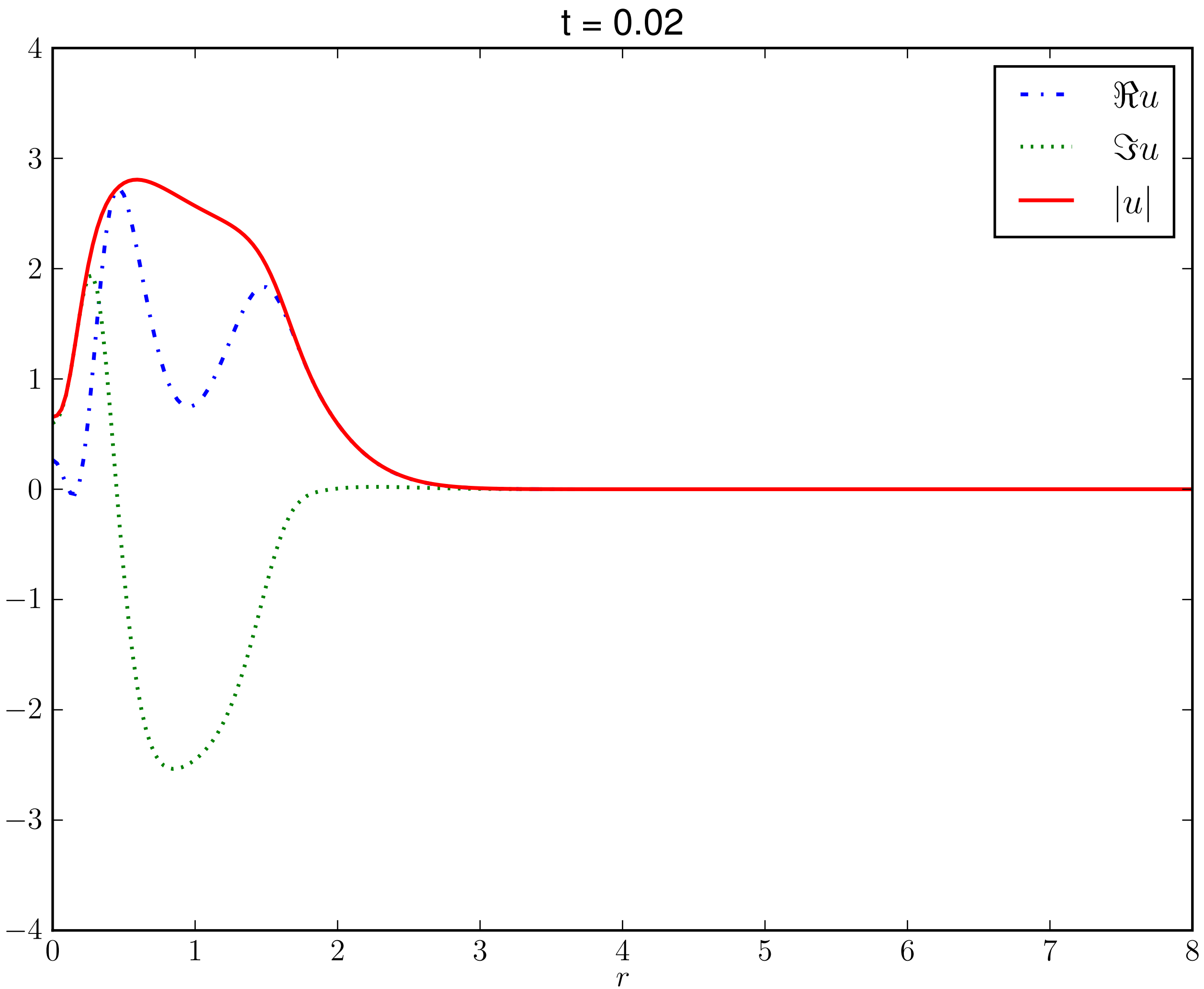

t = 0.09
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

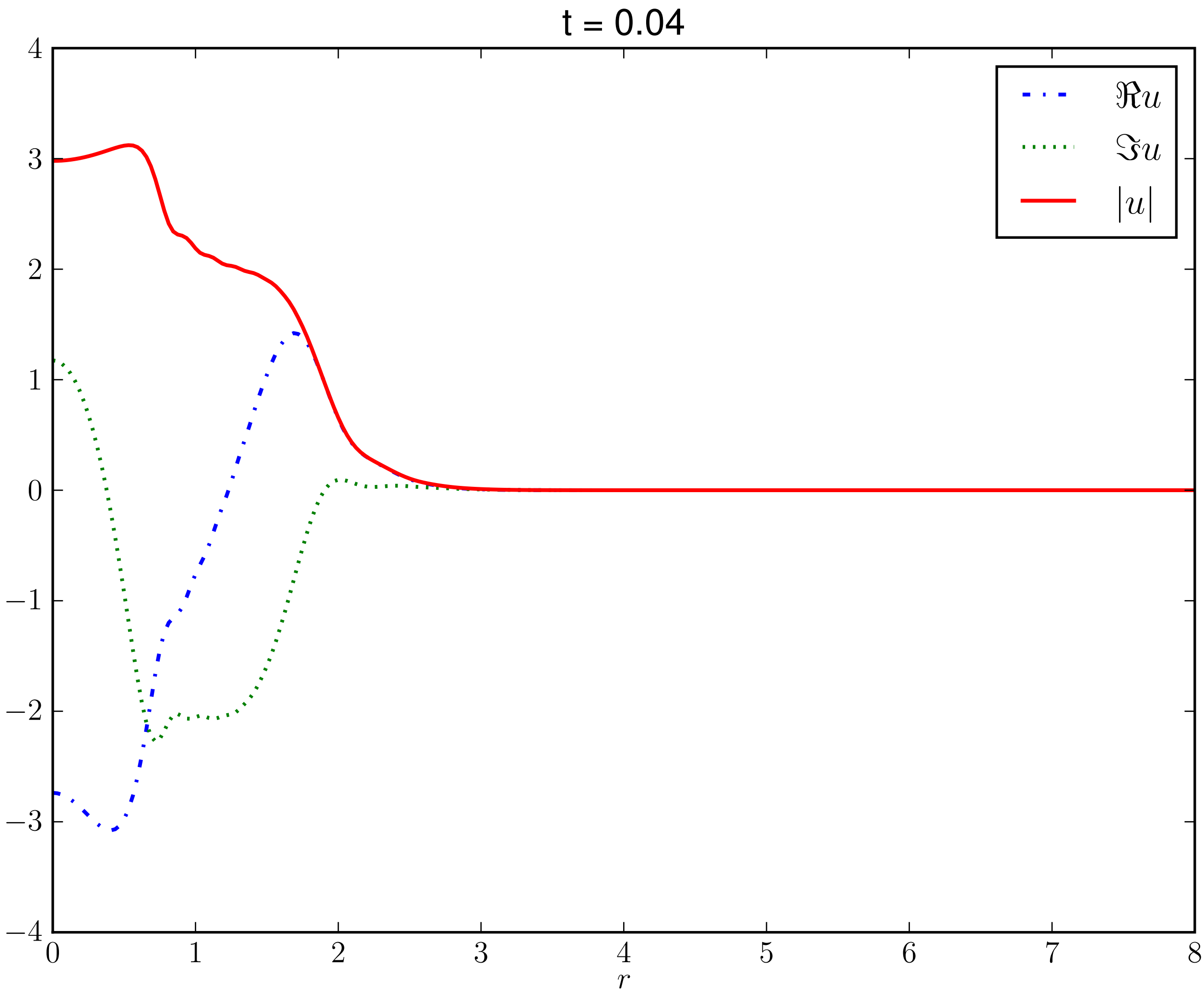

t = 0.095
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



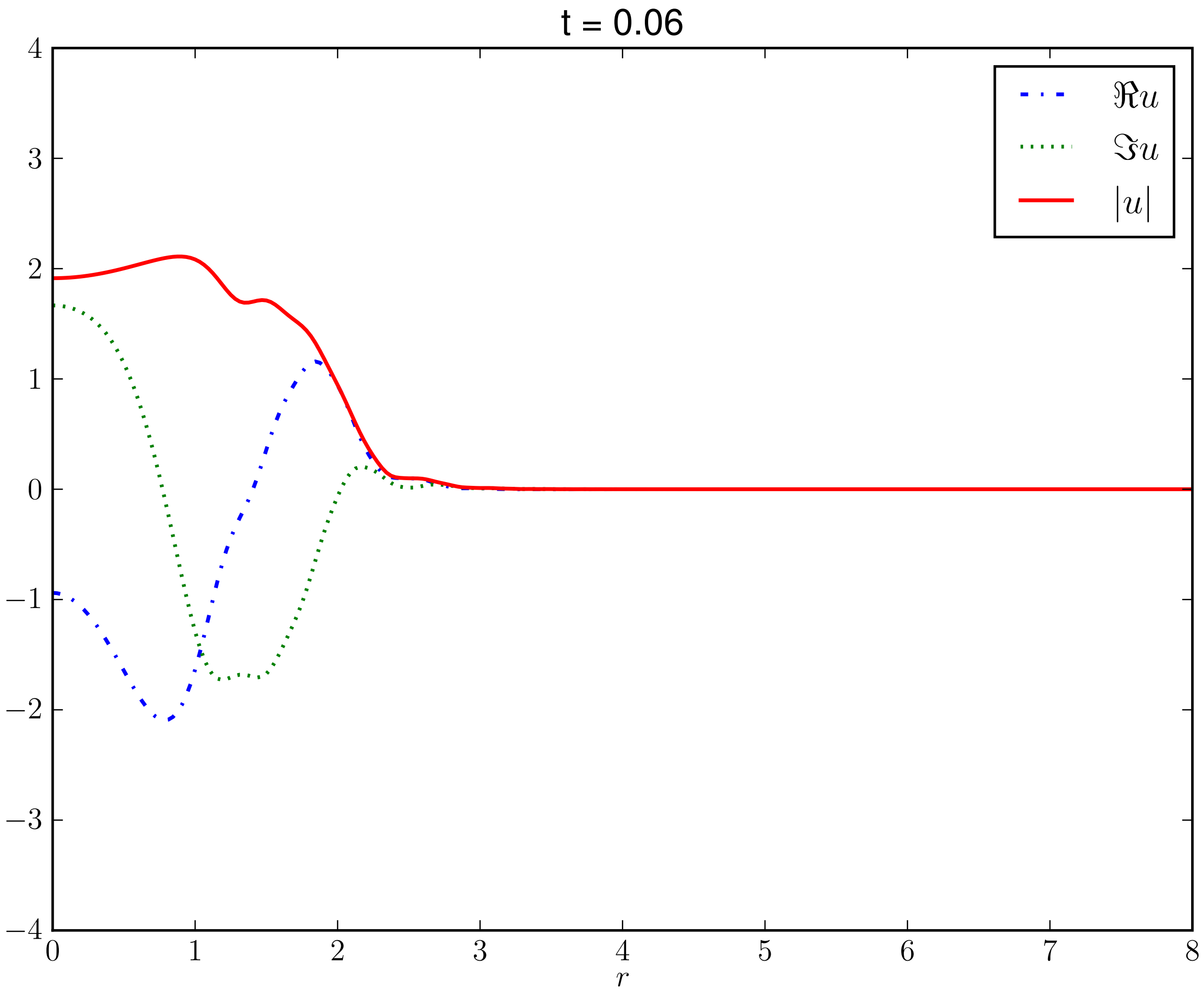
t = 0.1
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

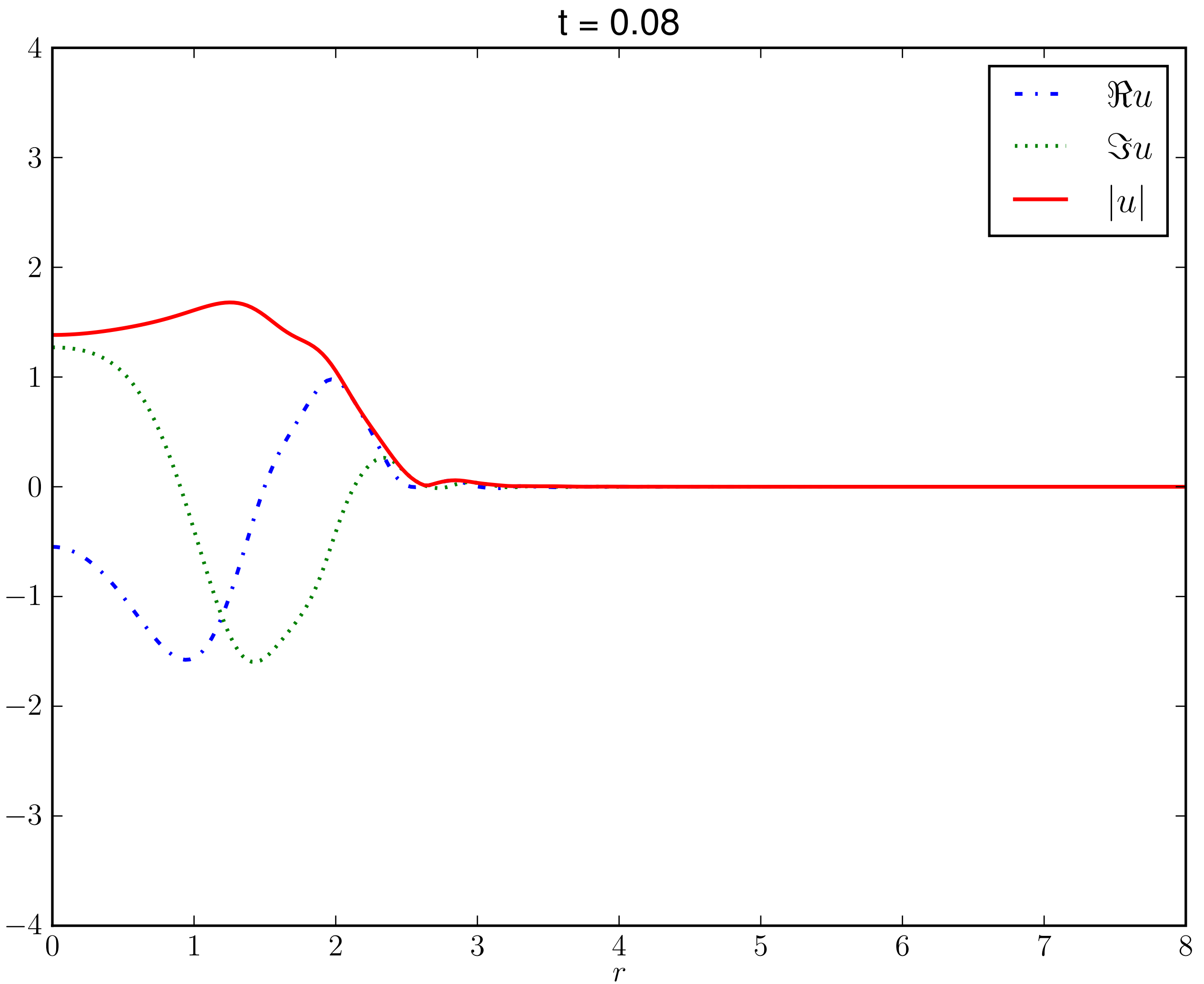

Spherical Ring Initial Data
t = 0
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r


t = 0.02
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

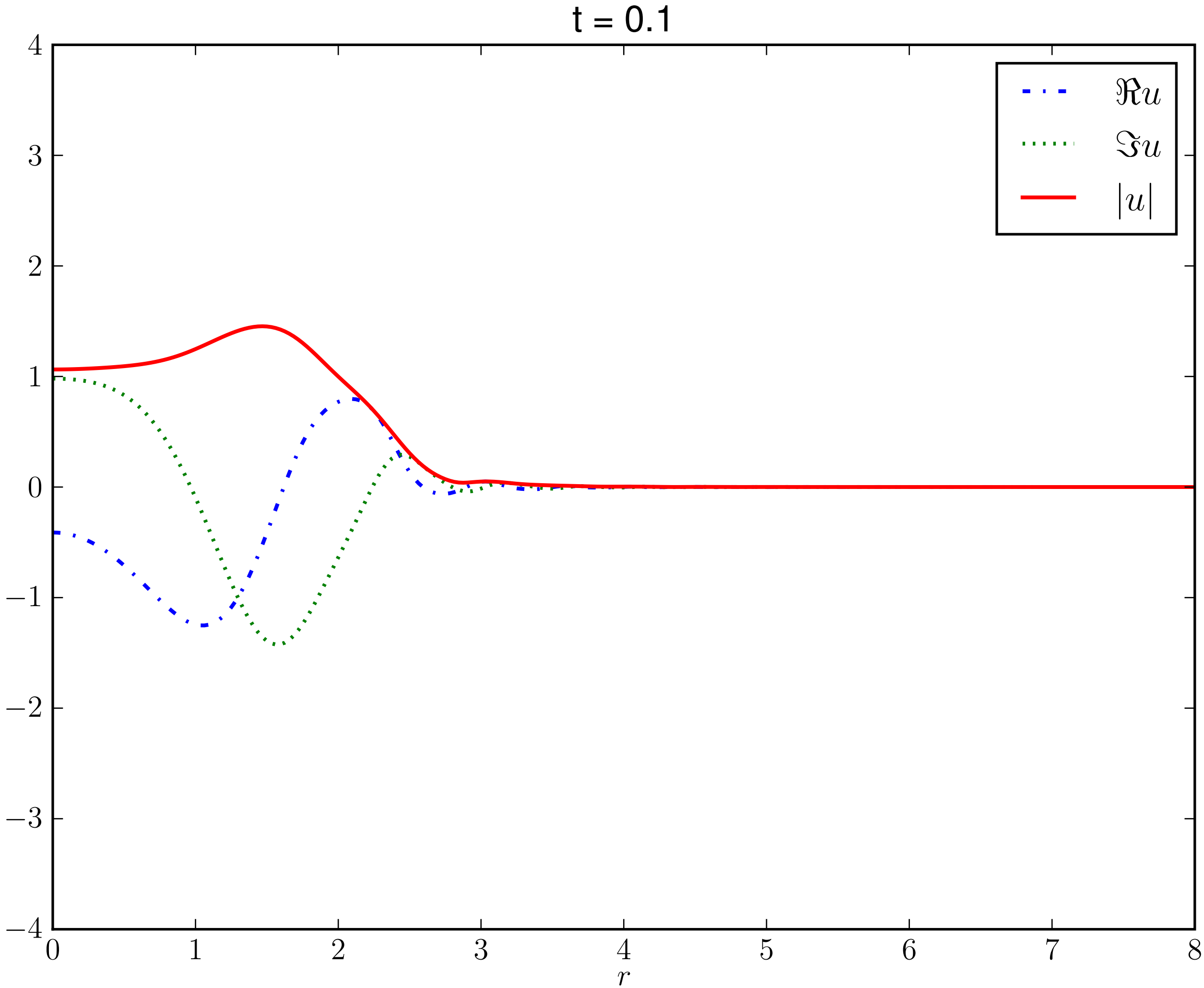


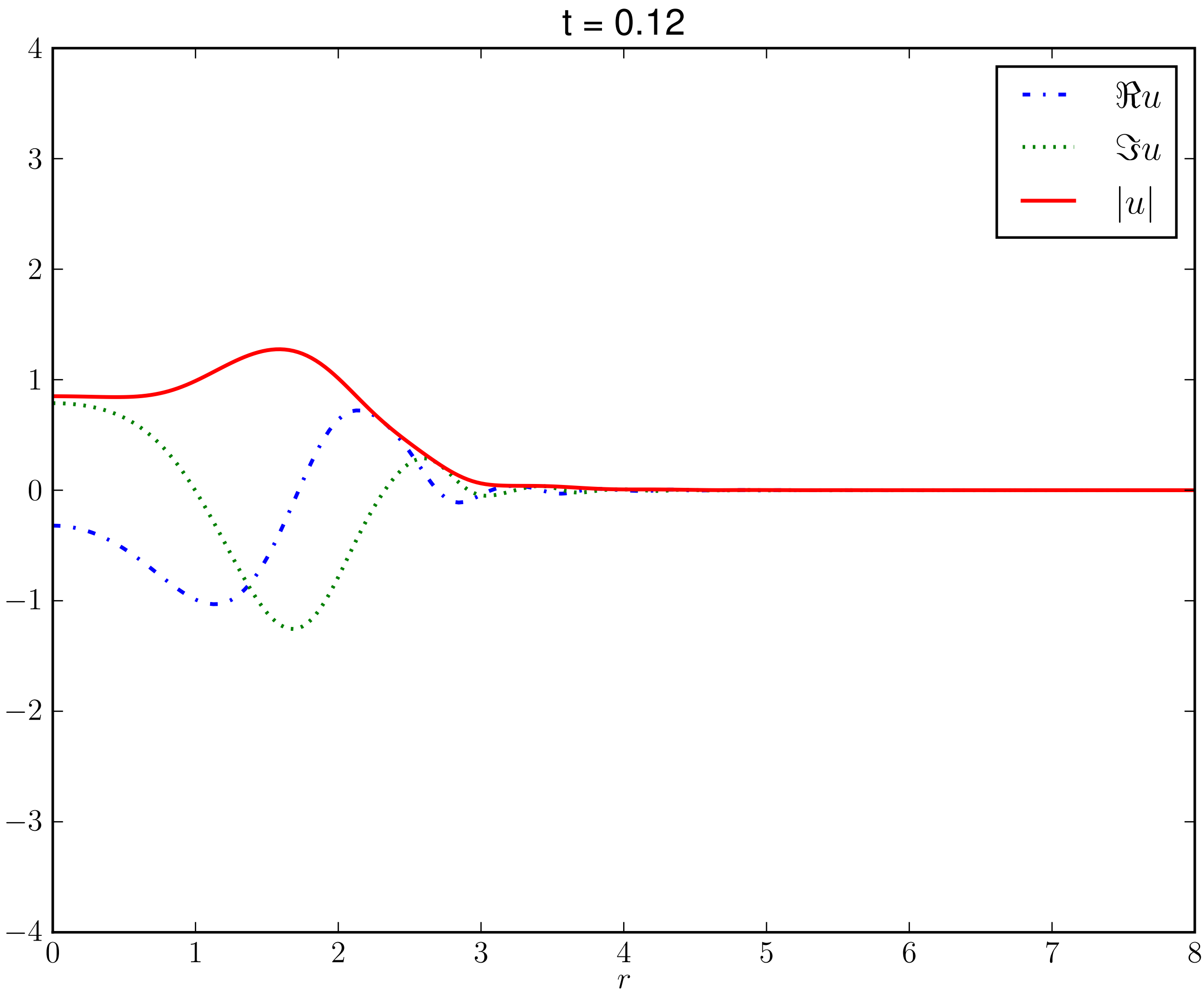
t = 0.04
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

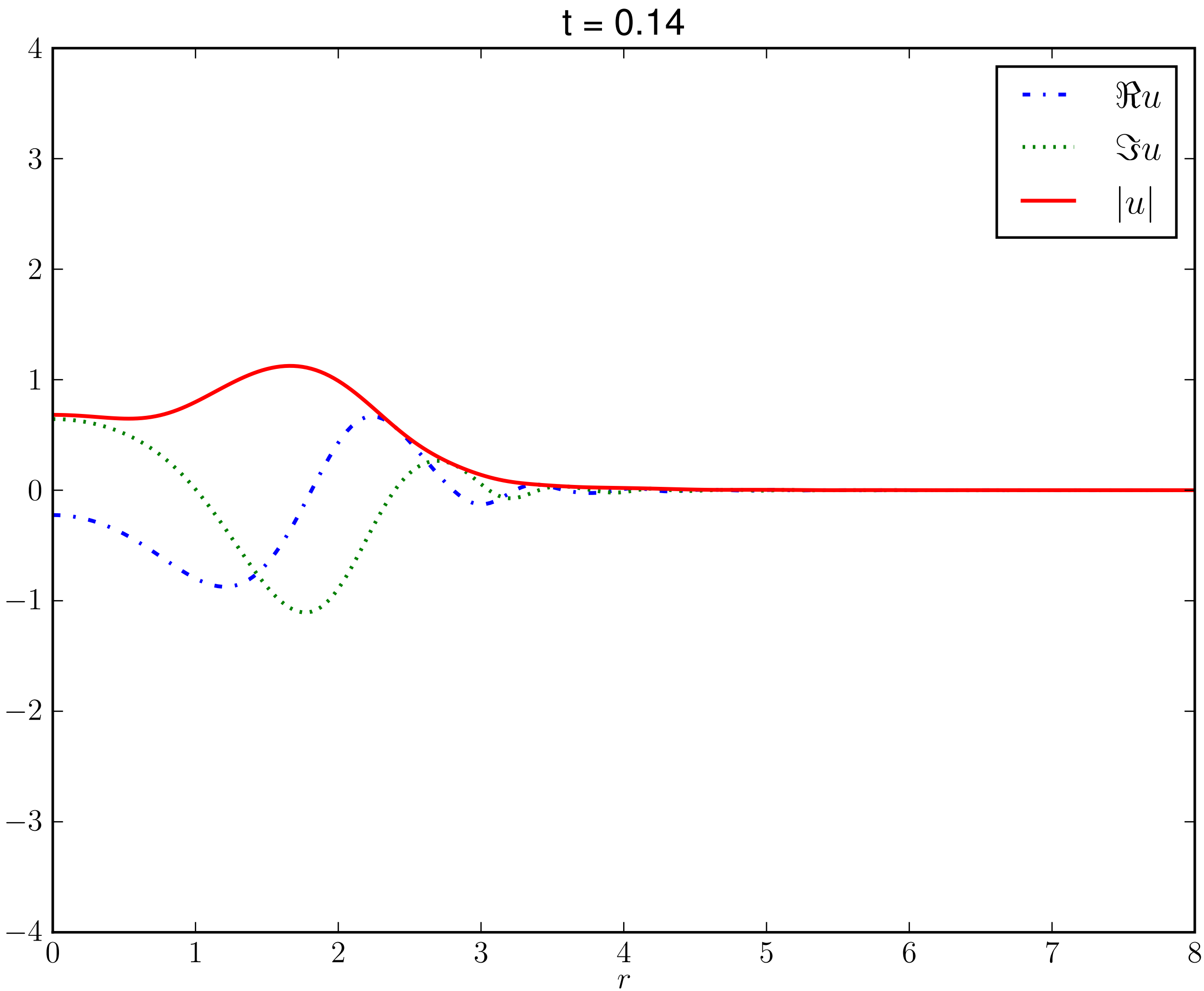

t = 0.06
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r



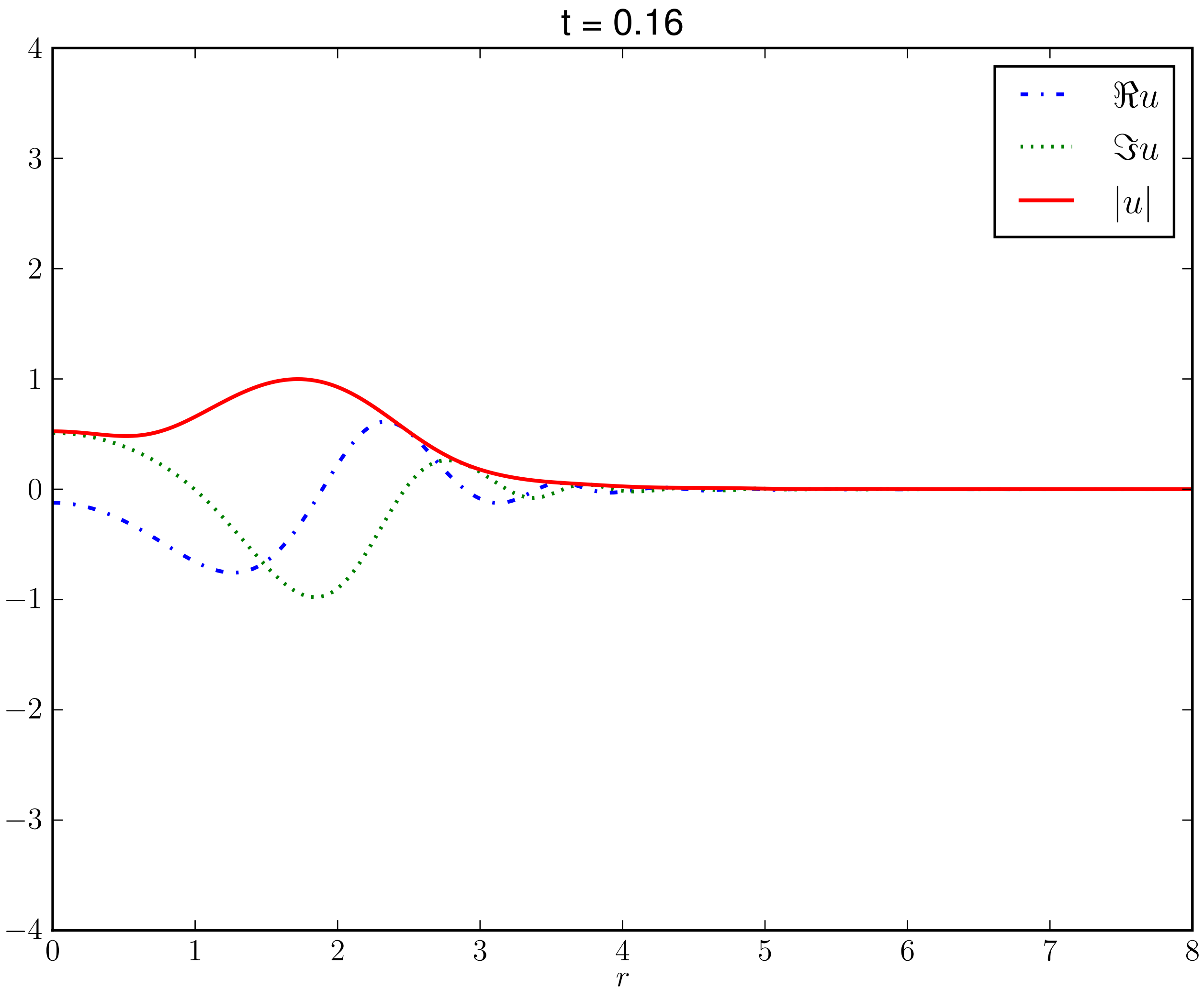
t = 0.08
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

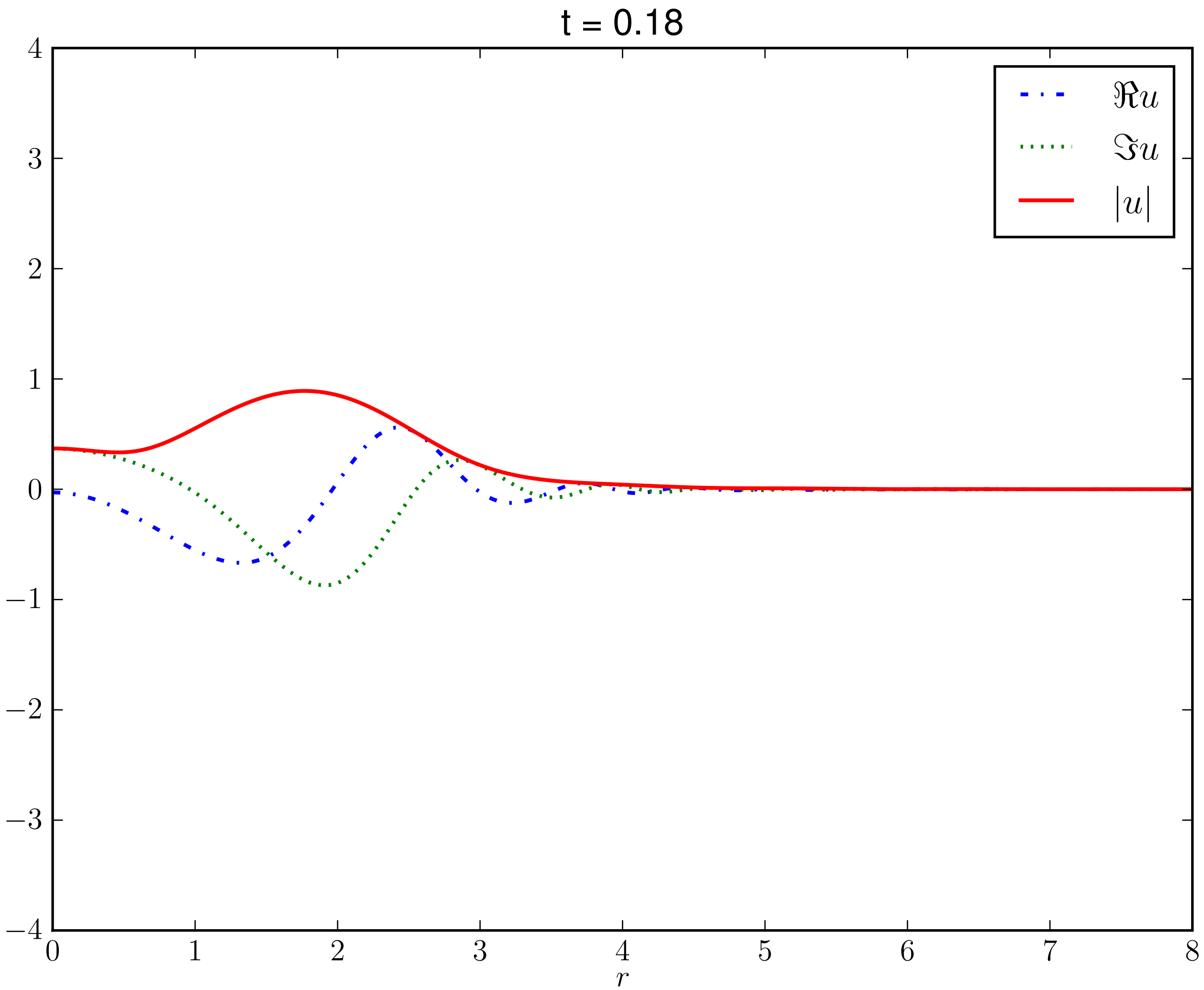

t = 0.1
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r


t = 0.12
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

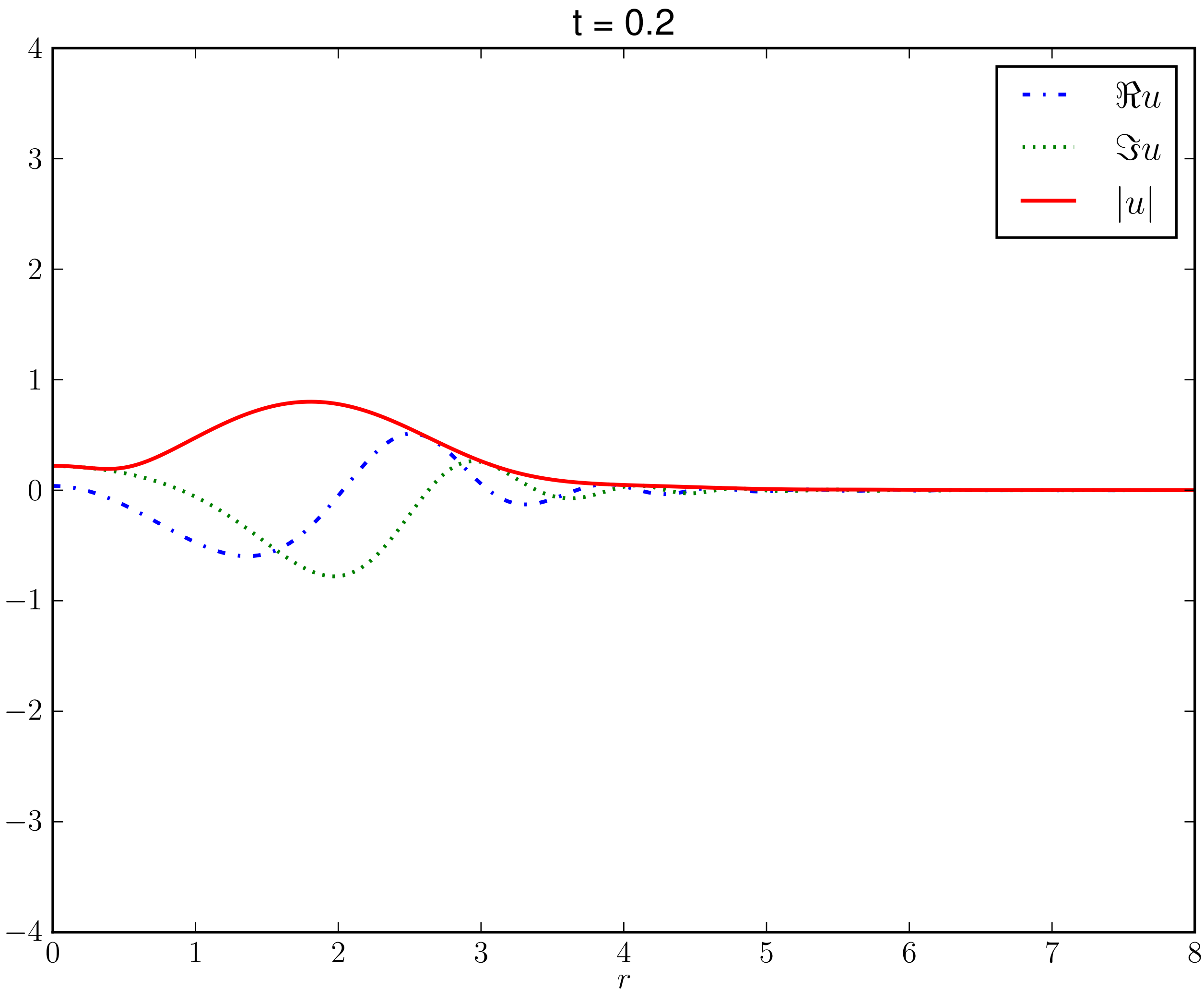


t = 0.14
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

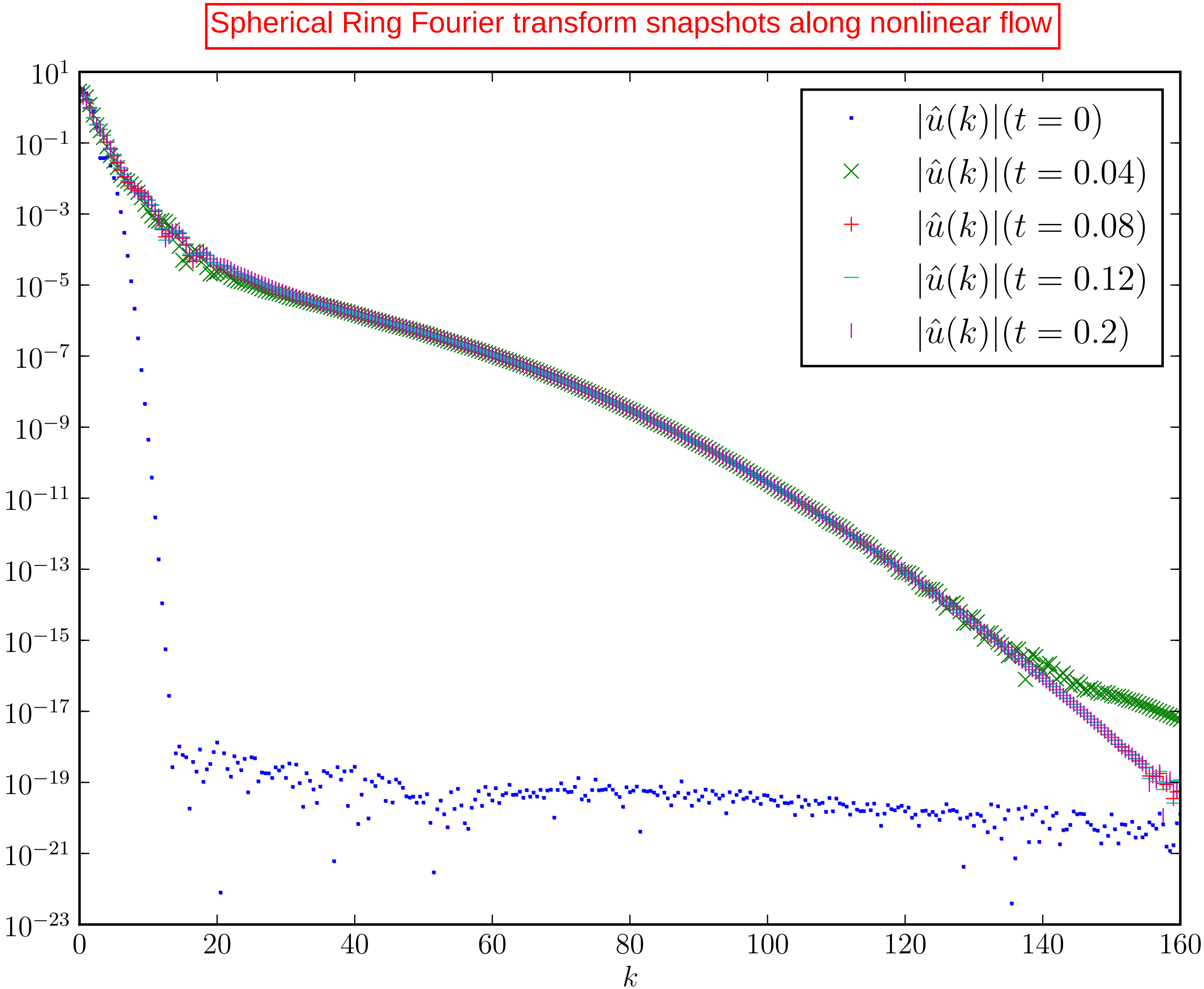

t = 0.16
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r
$\Re u$
$\Im u$
$|u|$



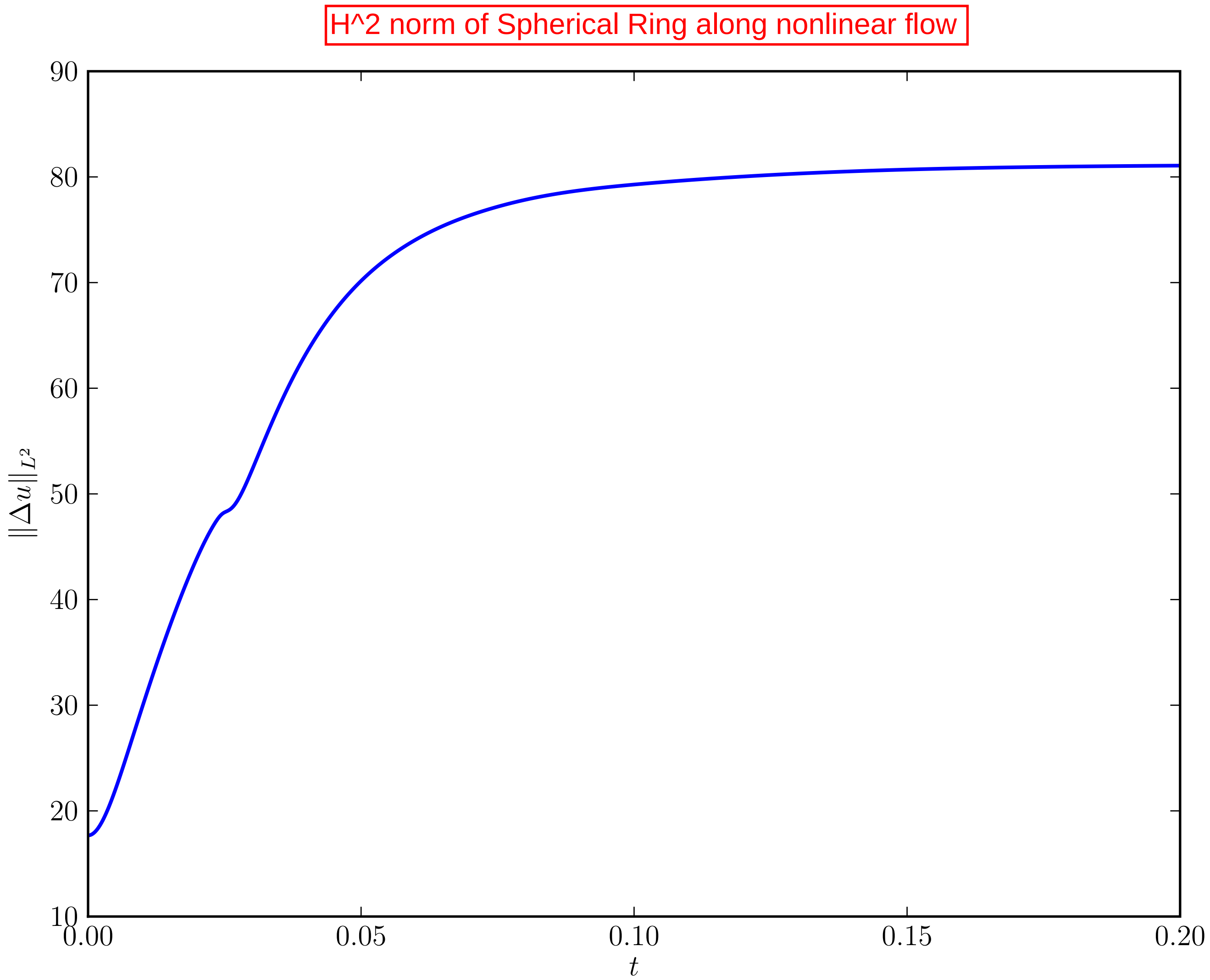
t = 0.18
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

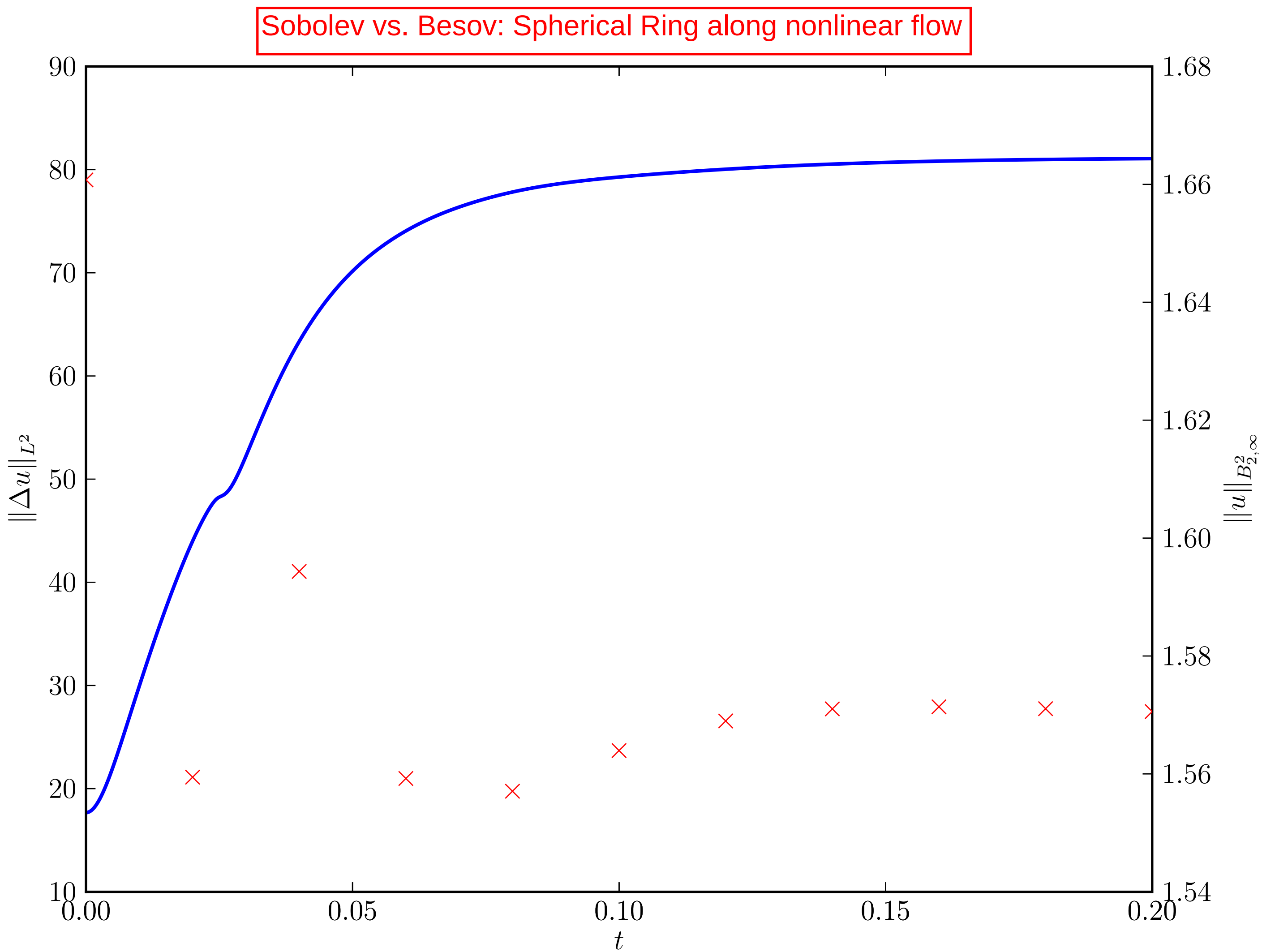


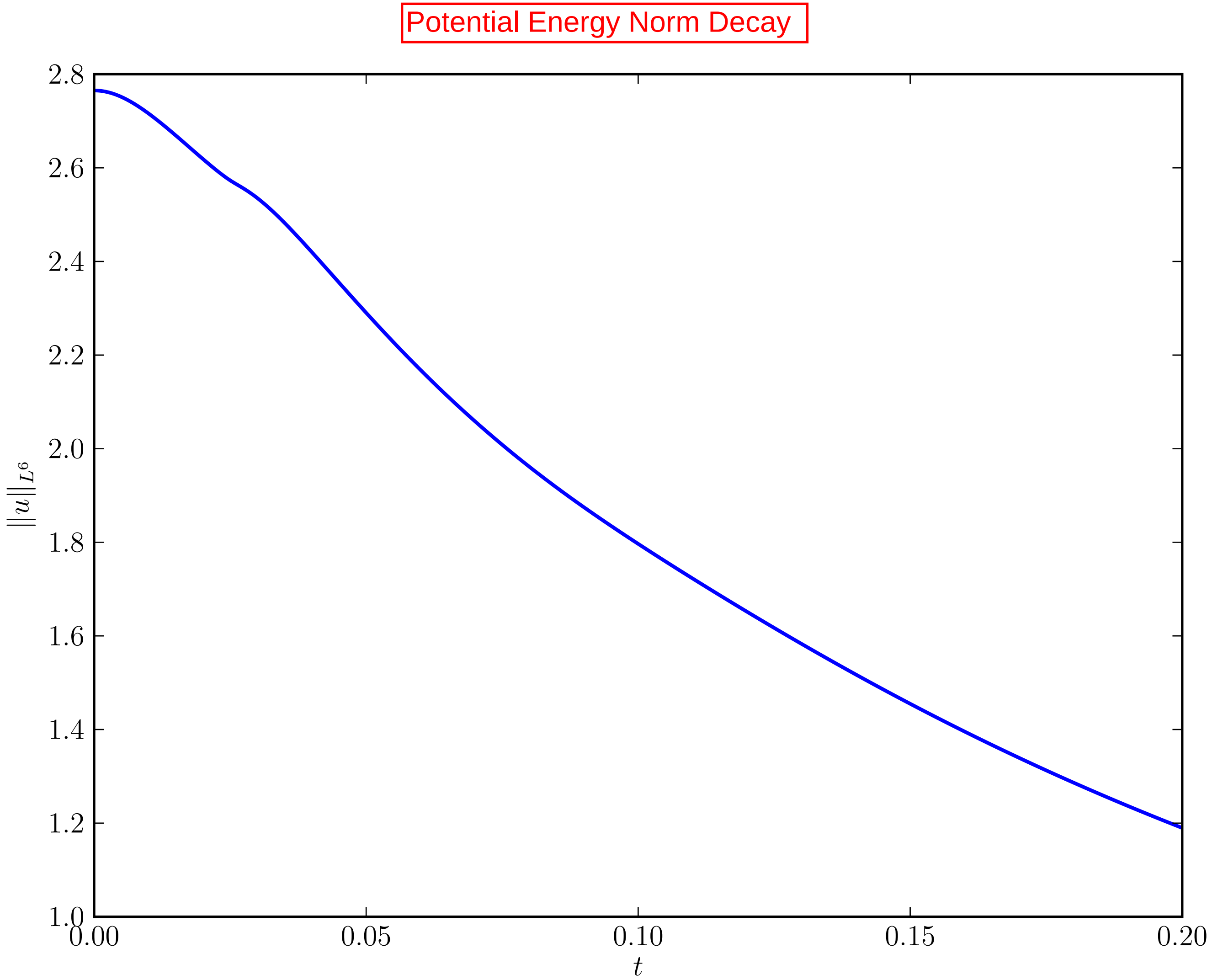
t = 0.2
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r



Spherical Ring Fourier transform snapshots along nonlinear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.04)$
$|\hat{u}(k)|(t=0.08)$
$|\hat{u}(k)|(t=0.12)$
$|\hat{u}(k)|(t=0.2)$
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
10^{-17}
10^{-19}
10^{-21}
10^{-23}
0
20
40
60
80
100
120
140
160
k


H^2 norm of Spherical Ring along nonlinear flow
90
80
70
60
50
40
30
20
10
$\|\Delta u\|_{L^2}$
0.00
0.05
0.10
0.15
0.20
t


Sobolev vs. Besov: Spherical Ring along nonlinear flow
$\|\Delta u\|_{L^2}$
$\|u\|_{B^2_{2,\infty}}$
t
10
20
30
40
50
60
70
80
90
1.54
1.56
1.58
1.60
1.62
1.64
1.66
1.68
0.00
0.05
0.10
0.15
0.20



Potential Energy Norm Decay
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
$\|u\|_{L^6}$
0.00
0.05
0.10
0.15
0.20
t

2. GWP of Cubic NLS on $\mathbb{R}^2$

We consider the initial value problems:

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^{\pm}(\mathbb{R}^2))$$

The $+$ case is called defocusing; $-$ is focusing. $NLS_3^{\pm}$ is ubiquitous in physics. The solution has a dilation symmetry

$$u^{\lambda}(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

which is invariant in $L^2(\mathbb{R}^2)$. This problem is L^2-critical.

Time Invariant Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t)\nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx \pm \frac{1}{2}|u(t)|^4 dx.$$

- Mass is L^2; Momentum is close to $H^{1/2}$; Energy involves H^1.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Frequency Localizations?

LOCAL-IN-TIME THEORY FOR $NLS_3^{\pm}(\mathbb{R}^2)$

- $\forall\ u_0 \in L^2(\mathbb{R}^2)\ \exists\ T_{lwp}(u_0)$ determined by

$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$

  $\exists$ unique $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$ solving $NLS_3^{+}(\mathbb{R}^2)$.
- $\forall\ u_0 \in H^s(\mathbb{R}^2), s > 0,\ T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists: $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.
- Define the maximal forward existence time $T^*(u_0)$ by

$$\|u\|_{L^4_{tx}([0,T^*-\delta]\times\mathbb{R}^2)} < \infty$$

  for all $\delta > 0$ but diverges to ∞ as $\delta \searrow 0$.
- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

Qualitative Aspects of Small Data Theory

- Robust, open set in L^2.
- Asymptotically linear behavior.
- Smallness brutally controls solution via fixed point argument.
- What is the boundary of small data scattering portion of phase space L^2?

$\boxed{\text{Phase Space Basin}}$

Global-in-time theory?

What is the ultimate fate of the local-in-time solutions?

L^2-critical Scattering Conjecture:

$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Moreover, $\exists\ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \|e^{\pm it\Delta}u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Same statement for focusing $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known for large radial data [Killip-Tao-Visan 07].

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{1}{3}$	resonant cut & I-Morawetz	[C-Roy08]
$s > 0$?		

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

Theorem (C-Roy) $NLS_3^+(\mathbb{R}^2)$ GWP in $H^s, s > 1/3$.

Overview of Proof

Let $u_0 \in H^s(\mathbb{R}^2),\ 0 \leq s \leq \frac{2}{5}$. Eventually, we require $\frac{1}{3} < s$.

- Task: Construct $u_0 \longmapsto u(t)\ \forall\ t \in [0, T],\ T$ fixed large.
- Equivalent Task: Construct $u_\lambda(\tau)\ \forall\ \tau \in [0, \lambda^2 T]$ where

$$u_\lambda(\tau, y) = \frac{1}{\lambda} u(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}).$$

We reserve the right to choose $\lambda > 0$ later.

I-Method Setup

- Define a spatial smoothing operator $I_N : H^s \to H^1$ via

$$\widehat{I_N f}(\xi) = m(\frac{\xi}{N})\widehat{f}(\xi)$$

where the smooth monotone Fourier multiplier m is defined

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < 1 \\ \xi^{s-1} & \text{for } |\xi| > 2. \end{cases}$$

- Rough solution induces finite energy reference evolution $I_N u$.
- Energy based control on $I_N u$ globalize u.

Modified Energy; Choice of λ

$$H[I_N u] = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}|I_N u|^4 dx.$$

$$\|u\|_{H^s}^2 \leq H[I_N u] + \|u_0\|_{L^2}^2.$$

Scaling and parameter dependence of $H[I_N u]$ gives

$$H[I_N u] \leq C(u_0)\frac{N^{2(1-s)}}{\lambda^{2s}}.$$

Choose $\lambda = \lambda(N) = CN^{\frac{1-s}{s}} \implies$

$$H[I_N u_\lambda] \leq \frac{1}{100}.$$

(Drop λ subscript; Recall target $[0, \lambda^2 T]$. Eventually $N = N(T)$.)

First Layer Decomposition; Morawetz Input

- We first construct $u(t) \ \forall \ t \in J_1 = [0, N^3] \subset [0, \lambda^2 T]$.
- Decompose $J_1 = \cup_{i=1}^{\mathcal{I}} I_i$ where the $\{I_i\}$ satisfy $I_i \cap I_j = \phi$ and

$$\|u\|_{L^4_{t \in I_i, x}}^4 \sim \frac{1}{100}.$$

  In principle, the number $\mathcal{I}$ of such intervals could be HUGE.
- Morawetz input: On any J where u exists, [CGTz] proved

$$\|Iu\|_{L^4_{t \in J, x}}^4 \leq C_0 |J|^{1/3}.$$

- Thus, taking $J = J_1$, we find (provided u exists on all of J_1),

$$\mathcal{I} \lesssim N.$$

I-method Input

Using resonant decomposition, [CKSTT] constructed $\widetilde{E}[u]$:

- Proximity to $H[Iu(t)]$ at each time t:

$$|H[Iu(t)] - \widetilde{E}[u(t)]| \lesssim N^{-1+}(H[Iu(t)])^2.$$

- Almost Conservation Law:

$$\mathrm{osc}_{I_1} \widetilde{E}[u(\cdot)] := \sup_{I_1} \widetilde{E}[u(\cdot)] - \inf_{I_1} \widetilde{E}[u(\cdot)] \leq C_0 N^{-2+}.$$

(Corresponding estimate for $H[I_N u]$ had $N^{-3/2+}$.)

Bookkeeping; Double Layer Construction

- As t traverses I_1:
  - $H[Iu(t)]$ stays with N^{-1} of $\widetilde{E}[u(t)]$;
  - $\widetilde{E}[u(t)]$ increments by at most CN^{-2+}.
- As t traverses $I_1 \cup I_2$:
  - $H[Iu(t)]$ stays with N^{-1} of $\widetilde{E}[u(t)]$;
  - $\operatorname{osc}_{I_1 \cup I_2} \widetilde{E}[u(\cdot)] \leq 2CN^{-2+}$.
- Morawetz control gives $\mathcal{I} \lesssim N$ so

$$\operatorname{osc}_{J_1 = \bigcup_{i=1}^{\mathcal{I}}} \widetilde{E}[u(\cdot)] \leq C\mathcal{I}N^{-2+} \lesssim N^{-1+}.$$

- Let $J_2 = [N^3, 2N^3] \subset [0, \lambda^2 T], J_3 = [3N^3, 4N^3], J_4, \ldots$. Process can be iterated N^{-1} times before $\widetilde{E}$ doubles.
- We need $N^{4-} > \lambda^2 T \iff N^{-\frac{2}{s}+6} > T \implies s > \frac{1}{3}$ suffices.

3. Elliptic-Elliptic Davey-Stewartson Blowup

This section describes work of **G. Richards** (Toronto student).

The Davey-Stewartson system is

$$\begin{cases} i\partial_t u + \sigma u_{xx} + u_{yy} = \pm|u|^2 u + \phi_x u \\ \alpha\phi_{xx} + \phi_{yy} + \gamma(|u|^2)_x = 0 \\ u(0,x) = u_0(x). \end{cases} \qquad (DS^{\pm}_{\sigma,\alpha;\gamma}(\mathbb{R}^2))$$

- DS arises in models of the ocean.
- Parameters $\sigma, \alpha \in \mathbb{R},\ \gamma \geq 0$.
- When $\sigma > 0$ and $\alpha > 0$, system is called **elliptic-elliptic**.

Set $\sigma = \alpha = 1$.

Solving Elliptic Equation; Collapsing System

Using Fourier transform, elliptic equation for ϕ is reexpressed

$$\widehat{\phi_x} = -\gamma \frac{\xi_1^2}{\xi_1^2 + \xi_2^2} \widehat{|u|^2}(\xi) = -\gamma \widehat{\mathcal{B}|u|^2}(\xi).$$

Substituting into the Schrödinger equation for u yields

$$\begin{cases} i\partial_t u + u_{xx} + u_{yy} + \mathcal{L}(|u|^2)u = 0 \\ u(0,x) = u_0(x) \end{cases} \qquad (DS^{\pm}_{e,e}(\mathbb{R}^2))$$

where $\mathcal{L} = \pm\mathbb{I} + \gamma\mathcal{B}$.

$DS^{\pm}_{e,e}$ is similar to NLS^{-}_{3}

- Conservation of mass, momentum, energy
- L^2 critical
- Pseudoconformal invariance
- Similar LWP theory
  - $L^4_{t,x}$ maximality critereon
  - subcritical scaling of local existence time

(See [Ghidaglia-Saut])

In fact, the analogy between $DS^{\pm}_{e,e}$ and NLS^{-}_{3} goes deeper.

H^1 THEORY FOR $DS^{\pm}_{e,e}$

- $E[u] = \int \frac{1}{2}|\nabla u|^2 - \frac{1}{4}\mathcal{L}(|u|^2)|u|^2 dx$ conserved.
- Weinstein Inequality [Papanicolau-Sulem-Sulem-Wang 94]:

$$\int \mathcal{L}(|u|^2)|u|^2 dx \leq C_{opt}\|\nabla u\|^2_{L^2}\|u\|^2_{L^2}$$

where $C_{opt} = \frac{2}{\|R\|^2_{L^2}}$ for some $R \in H^1, R > 0$. Uniqueness??

- Soliton: $u(t,x) = e^{it}R(x)$ solves $DS^{\pm}_{e,e}$.
- PC(soliton) explicit blowup. (Not observed numerically.)
- Virial identity, energy criteria for blowup. [Ghidaglia-Saut 90]
- Log log blowups?? (Numerically and formally expected.)

Theorem (G. Richards) H^1 mass concentration

Let $H^1 \ni u_0 \longmapsto u$ solve $DS^{\pm}_{e,e}$ which blows up as $t \nearrow T^* < \infty$.
Fix any $\lambda(t) > 0$ such that $\lambda(t)\|\nabla u(t)\|_{L^2} \to \infty$ as $t \nearrow T^*$.
Then $\exists\ x(t) \in \mathbb{R}^2$ such that

$$\liminf_{t \nearrow T^*} \int_{|x-x(t)|<\lambda(t)} |u(t,x)|^2 dx \geq \frac{2}{C_{opt}}.$$

Remarks:

- Analog of [Merle-Tsutsumi], [Nawa] results for NLS_3^-.
- Proof based on profile decomposition from [Hmidi-Keraani].

Theorem (G. Richards) L^2 mass concentration

Let $L^2 \ni u_0 \longmapsto u$ solve $DS^{\pm}_{e,e}$ which blows up as $t \nearrow T^* < \infty$. Then

$$\limsup_{t \nearrow T^*} \sup_{\text{parabolic squares } Q} \int_Q |u(t,x)|^2 dx \geq \eta(\|u_0\|_{L^2}).$$

<u>Remarks</u>:

- Parabolic squares have sidelength$(Q) < (T^* - t)^{1/2}$.
- Analog of [Bourgain], [Merle-Vega] mass concentration result.
- Proof essentially the same; based on linear refinements.

4. Rough Blowup Solutions of $NLS_3^-(\mathbb{R}^2)$

Known Maximal-in-Time Solution Scenarios

1. Soliton solutions exist: $u(t,x) = e^{it}R(x)$
   - $Q(x)$ ground state; also excited states.
   - non-scattering; Strichartz S^0 norms diverge global-in-time.
   - a priori H^1 control if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [Weinstein]
2. $\{\text{radial}\} \cap L^2 \ni u_0 \longmapsto u$ scatters if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [KTV]
3. $\mathcal{PC}$ transformation + solitons $\implies$ explicit (fast) $\frac{1}{t}$-blowups.
   - $\mathcal{PC}$ is a Stricharz S^0 isometry.
   - There exists an enlarged class of $\frac{1}{t}$-blowups [Bourgain-Wang].
   - Stability?

Open Question

4. Virial Blowup Solutions
   - Obstructive argument
   - Qualitative properties?

Ground State

- H^1-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$ [Weinstein]:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u,\ T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- Q is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

Pseudoconformal Symmetry

- Pseudoconformal transformation:

$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$

- $\mathcal{PC}$ is L^2-critical *NLS* solution symmetry:
  Suppose $0 < t_1 < t_2 < \infty$. If

$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$

  then

$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$

  solves

$$i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v.$$

- $\mathcal{PC}$ is an L^2-Strichartz isometry:
  If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then

$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}},$$

  is an explicit blowup solution.

- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in S is conically concentrated into a point.

- Minimal mass H^1 blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 |u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

NLS Blowup Dynamic?

Question: What are the dynamical properties of $NLS_3^-(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation

$\boxed{\text{What Happens?}}$

log log BLOWUP REGIME

- Numerical/Persuasive arguments [LPSS] led to:
  - Prediction of blowups with log log speed:

$$\|u(t)\|_{H^1} \sim \sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}.$$

  - Prediction that such blowups are generic/stable/observed.
  - Identification of certain mechanisms forecasting log log.
- $NLS_5^-(\mathbb{R}^1)$ has log log blowup solutions. [Perelman]
- Detailed Description of log log regime in series by [MR].

Qualitative Aspects of $\log\log$ regime

- Robust, open set in H^1.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomona in critical space (L^2 instability?).
- Conjectured quantization properties?
- Boundary of $\log\log$ regime in phase space?

$\boxed{\text{Phase Space Basin}}$

Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- Small Excess Mass: $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.
- Negative Total Energy: $H[u_0] < 0$.

The associated solution $u_0 \longmapsto u$ explodes with $T^* < \infty$ and

- $\exists \ (\lambda(t), x(t), \gamma(t) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.
- Sharp log log speed law holds:

Low Regularity Emerges

$$\lambda(t)\sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$

- $u^* \notin H^s$ for $s > 0$; $u^* \notin L^p$ for $p > 2$. (Rough residual)

Theorem (Raphaël): H^1 Stability of $\log\log$

- Fact: $\mathcal{PC} + \log\log$ for $E < 0 \implies \exists \; \log\log$ with $E > 0$.
- H^1-Stability Theorem: The set of data with $u_0 \in H^1$ with small excess mass blowing up in $\log\log$ regime is open in H^1.
- Develops **bootstrap** approach to *constructing* $\log\log$.
- Further applications of Raphaël's bootstrap/stability:
  - Domains: [Planchon-R:Ω]
  - Singular $S^1 \subset \mathbb{R}^2$: [R:Ring]
  - Singular $S^{d-1} \subset \mathbb{R}^d$: [R-Szeftel:Spheres]
  - Singular $S^1 \subset \mathbb{R}^3$: [Zwiers: Codimension Two Ring]
  - Higher Codimensional Singular Sets?
  - Rough Blowups

Theorem (C-Raphaël): H^s Stability of $\log\log$

- Let $u_0 \in H^1$ evolve into the $\log\log$ regime.
- $\forall\ s > 0\ \exists\ \epsilon = \epsilon(s, u_0) > 0$ such that $\forall\ v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^-(\mathbb{R}^2)$ solution $v_0 \longmapsto v$ blows up in $\log\log$ regime.

Thus, the H^1 $\log\log$ blowup solutions constructed by [MR] are contained in an open superset of $\log\log$ blowups in H^s, $\forall\ s > 0$.

Remarks about the H^s stability of $\log\log$

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of H^s $\log\log$ blowups.
- The condition $s > 0$ is expected to be optimal.
  Small L^2 (but huge H^s) perturbation destroys rough residual mass ($u^* \notin H^s, \ \forall\ s > 0$) leading to fast $\frac{1}{t}$-blowup?
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via I-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled I-method?

Aspects of the [MR] Analysis

- Geometrical description of $\log\log$ blowup solutions.
  - Various profiles $Q, Q_b, \widetilde{Q}_b, \widetilde{Q}_{b(t)} + \zeta_{b(t)}$. (Obscure Notation)
  - Modulation parameters related to solution symmetries.
  - Three zones: blowup core, radiation, distant/decoupled.
- Virial/Coercivity constraints; Orthogonality conditions.
- A key role played by Energy conservation.

Geometrical Description

- Near T^*, $\log\log$ blowups satisfy **geometrical ansatz**

$$u(t,x) = \frac{1}{\lambda(t)}(Q_{b(t)} + \epsilon)\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}.$$

- Parameters $(\lambda(t), x(t), \gamma(t), b(t))$ solve ODEs forced by $F(\epsilon)$.
- ODEs emerge from geometrical ansatz, taking inner products with equation, imposing orthogonality conditions. (These choices change across the [MR] works.)

Energy Conservation in [MR] analysis

- Control of ϵ:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- Energy conservation and $\lambda \searrow 0 \implies$

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)