Exploding, Rough and Turbulent Solutions of Nonlinear Schrödinger Equations

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1 Introduction

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3 Blowup

4 Weak Turbulence
We consider the defocusing initial value problem:

\[
\begin{cases}
(i \partial_t + \Delta) u = \pm |u|^{p-1} u \\
u(0, x) = u_0(x).
\end{cases}
\]

The + case is called defocusing; − is focusing.

- $NLS_3^\pm$ is ubiquitous in physics. $NLS_p^\pm$ introduced to explore interplay between dispersion and strength of nonlinearity.
- $NLS_p^\pm$ is an infinite dimensional Hamiltonian System: study infinite dimensional dynamical behaviors!
- The **main question** about an evolution PDE: *What is the ultimate fate of solutions?* We want to understand the maximal-in-time behavior of the solutions.
- Conservation and invariance properties motivate the study of $NLS_p^\pm(\mathbb{R}^d)$ for low (and minimal) regularity initial data.
**Time Invariant Quantities**

Mass = \[ \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx. \]

Momentum = \[ 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) \, dx. \]

Energy = \[ H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 \, dx \pm \frac{2}{p+1} |u(t)|^{p+1} \, dx. \]

- Mass is \( L^2 \); Momentum is close to \( H^{1/2} \); Energy involves \( H^1 \).
- Dynamics on a sphere in \( L^2 \); focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved:
  e.g., \( \partial_t |u|^2 = \nabla \cdot 2\Im (\overline{u} \nabla u) \). Space/Frequency Localizations?
One solution $u$ generates parametrized family $\{u^\lambda\}_{\lambda > 0}$ of solutions:

$$u : [0, T) \times \mathbb{R}^d_x \rightarrow \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)$$

where

$$u^\lambda : [0, \lambda^2 T) \times \mathbb{R}^d_x \rightarrow \mathbb{C} \text{ solves } NLS^\pm_p(\mathbb{R}^d)$$

where

$$u^\lambda(\tau, y) = \lambda^{-2/(p-1)} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

Norms which are invariant under $u \longmapsto u_\lambda$ are critical.

Other Symmetries

- Phase, space, time translation solution symmetries $\Rightarrow$ mass, momentum, energy conservation laws.
- One solution spawns solution family by symmetry orbit.
In the $L^2$-based Sobolev scale,

$$\| D^s u^\lambda (t) \|_{L^2} = \lambda^{-\frac{2}{p-1} - s + \frac{d}{2}} \| D^s u(t) \|_{L^2}. $$

The critical Sobolev index for $NLS_{p}^{\pm}(\mathbb{R}^d)$ is

$$s_c := \frac{d}{2} - \frac{2}{p - 1}. $$

**Scaling/Conservation Criticality**

<table>
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<th>regime</th>
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<tr>
<td>$s_c &lt; 0$</td>
<td>mass subcritical</td>
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<tr>
<td>$s = 0$</td>
<td>mass critical</td>
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<tr>
<td>$0 &lt; s_c &lt; 1$</td>
<td>mass super/energy subcritical</td>
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<tr>
<td>$s_c = 1$</td>
<td>energy critical</td>
</tr>
<tr>
<td>$1 &lt; s_c &lt; d/2$</td>
<td>energy supercritical</td>
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LWP & Maximal-in-time Implications

Strichartz Estimates, Duhamel, Contraction: $NLS_3(\mathbb{R}^2)$ case.

- Optimal (Sobolev $H^s$) regularity: $s \geq s_c = 0$ [CW], [KPV].
- Maximality/Blowup Criteria: If $T^* < \infty$
  - Strichartz Divergence, e.g.
    $$\|u\|_{L^4([0,t] \times \mathbb{R}^2)} \text{ diverges as } t \nearrow T^*.$$

- Subcritical Scaling Lower Bound,
  $$\|u(t)\|_{H^s(\mathbb{R}^2)} \gtrsim \frac{1}{(T^* - t)^{s/2}}, \quad 0 < s.$$

- What blowup speeds are realized by NLS evolutions?
- Small Data Scattering Theory: $\exists \gamma_0 > 0$ such that
  $$\|u_0\|_{L^2} < \gamma_0 \implies u(t) \text{ global, asymptotically linear.}$$
Advances around **Fourier Restriction Phenomena** led to...

- LWP for spaces of initial data larger than $L^2$ [MVV], [B]. . .
- “Small” data scattering valid for certain large $L^2$ data.
- Further Implications of $T^* < \infty$:
  - Critical Norm (Mass) Concentration (along time sequence) [B].
  - Asymptotic Compactness Modulo Symmetries [MV].
- Links between rates of blowup quantities [B], [C-Roudenko].
Qualitative Aspects of Small Data Theory

- Robust, open set in $L^2$.
- Asymptotically linear behavior.
- Smallness brutally controls solution via fixed point argument.
- What is the boundary of small data scattering portion of phase space $L^2$?
Known Maximal-in-Time Solution Scenarios

1. Asymptotically linear (Scattering) solutions exist.

2. Soliton solutions exist: $u(t, x) = e^{it} R(x)$ (focusing case)
   - $Q(x)$ ground state; also excited states.
   - non-scattering; Strichartz norms diverge global-in-time.

3. Finite time blowup solutions are known, e.g. $NLS_3^-(\mathbb{R}^2)$:
   - $PC$ transformation + solitons $\implies$ explicit (fast) $\frac{1}{t}$-blowups.
   - There exists an enlarged class of $\frac{1}{t}$-blowups [BW].
   - Virial argument $\implies$ many blowup solutions.
   - Qualitative properties? Recent advances [MR]. “log log”

4. Weakly turbulent solutions of $NLS_3^+(\mathbb{T}^2)$. [CKSTT]
2. Critical Scattering

What is the ultimate fate of the local-in-time solutions?

**L^2-critical Defocusing Scattering Conjecture:**
\[ L^2 \ni u_0 \mapsto u \] solving \( NLS^+_3(\mathbb{R}^2) \) is global-in-time and
\[ \|u\|_{L^4_{t,x}} < A(u_0) < \infty. \]

Moreover, \( \exists u_\pm \in L^2(\mathbb{R}^2) \) such that
\[ \lim_{t \to \pm \infty} \|e^{\pm it \Delta} u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0. \]

Remarks:
- Known for small data \( \|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0 \).
- Known [Tao-Visan-Zhang 06] for \( NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d) \) for large radial data, \( d \geq 3 \). Same for \( d = 2 \) [Killip-Tao-Visan 07].
- GWP for \( L^2 \) data \( \iff \) Scattering for \( L^2 \) data. [Blue-C 06]
Critical Regularity Scattering Conjecture?

Consider defocusing case $NLS^+_p(\mathbb{R}^d)$ with critical Sobolev index

$$s_c = \frac{d}{2} - \frac{2}{p - 1}.$$

The critical (diagonal) Strichartz index is

$$q_c = \frac{(p - 1)(2 + d)}{2} \iff \frac{2}{q_c} + \frac{d}{q_c} = \frac{d}{2} - s_c.$$

**$H^{s_c}$-critical defocusing scattering conjecture:**

$H^{s_c}(\mathbb{R}^d) \ni u_0 \mapsto u$ solving $NLS^+_p(\mathbb{R}^d)$ is global-in-time and

$$\|u\|_{L_{t,x}^{q_c}} < A(u_0) < \infty.$$
Critical Regularity Scattering Conjecture?

Present status of the defocusing scattering conjecture

<table>
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<tr>
<th>criticality</th>
<th>general data</th>
<th>radial data</th>
<th>evidence</th>
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<tbody>
<tr>
<td>$s_c = 0$</td>
<td>???</td>
<td>[TVZ],[KTV]</td>
<td>GWP: $s_* &lt; s &lt; 1$</td>
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<tr>
<td>$0 &lt; s_c &lt; 1$</td>
<td>$s_c &lt; s_* &lt; s &lt; 1$</td>
<td>$s = s_c$??</td>
<td>$s_* &lt; s &lt; 1$</td>
</tr>
<tr>
<td>$s_c = 1$</td>
<td>[CKSTT],[RV],[V]</td>
<td>[B99],[G],[T]</td>
<td>$s_* &lt; s &lt; 1$</td>
</tr>
<tr>
<td>$1 &lt; s_c &lt;$</td>
<td>???</td>
<td>[KM] + bound</td>
<td>$s_* &lt; s &lt; 1$</td>
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- Scattering for $NLS_p^-$ under natural threshold? [HR]
- The existence (and value) of $s_*$ depends upon $p, d$.
- The work [B99] introduced **induction on energy idea**.
- Simplified/Abstracted **road map** to critical scattering. [KM]
3. **Blowup**

**Ground State**

- **$H^1$-GWP mass threshold for $NLS^-_3(\mathbb{R}^2)$ [W]:**

  \[ \|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, \ T^* = \infty, \]

  based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

  \[ \|u\|_{L^4}^4 \leq \left[ \frac{2}{\|Q\|_{L^2}^2} \right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2. \]

- $Q$ is the ground state solution to $-Q + \Delta Q = -Q^3$.

- The ground state soliton solution to $NLS^-_3(\mathbb{R}^2)$ is

  \[ u(t, x) = e^{it} Q(x). \]
Pseudoconformal Symmetry

- **Pseudoconformal transformation**:
  \[ \mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u \left( -\frac{1}{\tau}, \frac{y}{\tau} \right), \]

- \( \mathcal{PC} \) is \( L^2 \)-critical NLS solution symmetry:
  Suppose \( 0 < t_1 < t_2 < \infty \). If
  \[ u : [t_1, t_2] \times \mathbb{R}_x^2 \rightarrow \mathbb{C} \text{ solves } \text{NLS}^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d) \]
  then
  \[ \mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}_y^2 \rightarrow \mathbb{C} \]
  solves
  \[ i\partial_\tau v + \Delta_y v = \pm|v|^{4/d} v. \]

- \( \mathcal{PC} \) is an \( L^2 \)-Strichartz isometry:
  If \( \frac{2}{q} + \frac{d}{r} = \frac{d}{2} \) then
  \[ \| \mathcal{PC}[u] \|_{L^q_t L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \| u \|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)} \cdot \]
The *pseudoconformal* image of ground state soliton $e^{it} Q(x)$,

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + i},$$

is an explicit blowup solution.

$S$ has minimal mass:

$$\| S(-1) \|_{L_x^2} = \| Q \|_{L^2}.$$  

All mass in $S$ is conically concentrated into a point.

 Minimal mass $H^1$ blowup solution characterization:  
$u_0 \in H^1, \| u_0 \|_{L^2} = \| Q \|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [M]
Many non-explicit Blowup Solutions

Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t, x) dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi} \nabla \phi) dx.$$ 

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta \Delta a)|\phi|^2 + 4a_{jk} \Re(\overline{\phi_j} \phi_k) - a_{jj}|u|^4 dx.$$ 

Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx = 16H[u_0].$$

$H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 dx < \infty$ blows up.

How do these solutions blow up?
**Question:** What are the dynamical properties of $NLS_3^{-}(\mathbb{R}^2)$ blowup solutions?

- maximality criteria; critical norm behavior
- asymptotic compactness; profile decompositions
- conservation structure; virial ideas; parameter modulation
Numerical/Persuasive arguments [LPSS] led to:

Prediction of blowups with log log speed:

\[ \| u(t) \|_{H^1} \sim \sqrt{\frac{\log |\log(T^*-t)|}{T^*-t}} \gg \frac{1}{\sqrt{T^*-t}}. \]

Prediction that such blowups are generic/stable/observed.
Identification of certain mechanisms forecasting log log.

\( NLS_5^{-}(\mathbb{R}^1) \) has log log blowup solutions. [P]

Detailed Description of log log regime in series by [MR].
Qualitative Aspects of log log regime

- Robust, open set in $H^1$.  
- Asymptotically nonlinear with subtle interaction.  
- Delicate phenomena in critical space ($L^2$ instability?).  
- Conjectured quantization properties?  
- Boundary of log log regime in phase space?
Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- **Small Excess Mass:** $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.  
- **Negative Total Energy:** $H[u_0] < 0$.

The associated solution $u_0 \mapsto u$ explodes with $T^* < \infty$ and

- $\exists (\lambda(t), x(t), \gamma(t) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$  

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.  
- Sharp log log speed law holds:

$$\lambda(t) \sqrt{\frac{\log \left| \log (T^* - t) \right|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$  

- $u^* \not\in H^s$ for $s > 0$; $u^* \not\in L^p$ for $p > 2$. (Rough residual)
**Theorem (Raphaël):** \( H^1 \) Stability of \( \log \log \)

- **Fact:** \( PC + \log \log \) for \( E < 0 \) \( \iff \exists \) \( \log \log \) with \( E > 0 \).
- **\( H^1 \)-Stability Theorem:** The set of data with \( u_0 \in H^1 \) with small excess mass blowing up in \( \log \log \) regime is open in \( H^1 \).
- Develops **bootstrap** approach to *constructing* \( \log \log \).
- Further Bootstrap/stability applications [PR:Ω], [R:Ring].
Theorem (C-Raphaël): $H^s$ Stability of log log

- Let $u_0 \in H^1$ evolve into the log log regime.
- $\forall s > 0 \exists \epsilon = \epsilon(s, u_0) > 0$ such that $\forall v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^-(\mathbb{R}^2)$ solution $v_0 \mapsto v$ blows up in log log regime.

Thus, the $H^1$ log log blowup solutions constructed by [MR] are contained in an open superset of log log blowups in $H^s$, $\forall s > 0$. 
Remarks about the $H^s$ stability of log log

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of $H^s$ log log blowups.
- The condition $s > 0$ is expected to be optimal.
  Small $L^2$ (but huge $H^s$) perturbation destroys rough residual mass ($u^* \notin H^s$, $\forall s > 0$) leading to fast $\frac{1}{t}$-blowup? (Zwiers)
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via $I$-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled $I$-method?
**Energy Conservation in [MR] Analysis**

- Control of $\epsilon$:
  \[ \int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|. \]

- Energy conservation and $\lambda \downarrow 0 \implies$
  \[ \int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|. \]

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)

- Systematically replace $E(u)$ by $E(I_N u)$. 

3. Weak Turbulence

[CKSTT: joint work with Keel, Staffilani, Takaoka and Tao]

We consider the defocusing initial value problem:

\[
\begin{align*}
(-i \partial_t + \Delta)u &= |u|^2 u \\
u(0, x) &= u_0(x), \text{ where } x \in \mathbb{T}^2.
\end{align*}
\]  

\( \text{(NLS}(\mathbb{T}^2)) \)

Smooth solution \( u(x, t) \) exists globally and

- **Mass**  
  \[ \text{Mass} = M(u) = \|u(t)\|^2 = M(0) \]

- **Energy**  
  \[ \text{Energy} = E(u) = \int \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(x, t)|^4 \, dx = E(0) \]

We want to understand the shape of \( |\hat{u}(t, \xi)| \). The conservation laws impose \( L^2 \)-moment constraints on this object.
Notion of Weak Turbulence

Definition

Weak turbulence is the phenomenon of global-in-time defocusing solutions shifting their mass toward increasingly high frequencies. This shift is also called a forward cascade.

- A way to measure weak turbulence is to study

\[ \|u(t)\|_{H^s}^2 = \int |\hat{u}(t, \xi)|^2 |\xi|^{2s} d\xi \]

and prove that it grows for large times \( t \).

- Turbulence is incompatible with scattering and integrability.

- Finite time blowup behavior is not weak turbulence.
**Incompatible with Scattering & Integrability**

- **Scattering:** \( \forall \) global solution \( u(t, x) \in H^s \) \( \exists u_0^+ \in H^s \) such that,
  \[
  \lim_{t \to +\infty} \| u(t, x) - e^{it\Delta} u_0^+(x) \|_{H^s} = 0.
  \]

  Note: \( \| e^{it\Delta} u_0^+ \|_{H^s} = \| u_0^+ \|_{H^s} \implies \| u(t) \|_{H^s} \) is bounded. 
  Proofs rely on Morawetz-type (global dispersive) estimate.

- **Complete Integrability:** The 1d equation
  \[
  (i\partial_t + \partial_x^2)u = |u|^2 u
  \]
  has infinitely many conservation laws. Combining them in the right way one gets that \( \| u(t) \|_{H^s} \leq C_s \) for all times.
Glassey’s virial identity shows corresponding focusing problem

\[
\begin{aligned}
(-i\partial_t + \Delta)u &= -|u|^2u \\
u(0, x) &= u_0(x), \text{ where } x \in \mathbb{R}^2.
\end{aligned}
\] (NLS\(^-\)(\(\mathbb{R}^2\))

has many finite time blowup solutions.

The associated energy has a changed sign:

\[
E(u) = \int \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{4} |u(x, t)|^4 \, dx.
\]

Blowup solutions explode in \(H^1\) in finite time.
**Past Results (defocusing case)**

- **Bourgain**: (late 90’s)
  For the periodic IVP $\textit{NLS}(\mathbb{T}^2)$ one can prove
  \[ \|u(t)\|_{H^s}^2 \leq C_s |t|^{4s}. \]

  The idea is to improve the local estimate for $t \in [-1, 1]$
  \[ \|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1 \]
  \((\implies \|u(t)\|_{H^s} \lesssim C|t| \text{ upper bounds})\) to obtain
  \[ \|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1, \]
  for some $\delta > 0$. This iterates to give
  \[ \|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}. \]

- **Improvements**: Staffilani, Colliander-Delort-Kenig-Staffilani.
Past Results

- **Bourgain**: (late 90’s)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x, t)$ to the modified wave equation
  
  $$(\partial_{tt} - \tilde{\Delta})u = u^p$$
  
  such that $\|u(t)\|_{H^s} \sim |t|^m$.

- **Physics**: Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ **DO NOT** exhibit weak turbulence. ∃ solutions to dispersive equations on $\mathbb{T}^d$ that exhibit weak turbulence. In particular for NLS($\mathbb{T}^2$) there exists $u(x, t)$ s. t.

$$\|u(t)\|_{H^s}^2 \to \infty \text{ as } t \to \infty.$$
Main Result

We consider the defocusing initial value problem:

\[
\begin{cases}
  (-i\partial_t + \Delta) u = |u|^2 u \\
  u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2.
\end{cases}
\]  \quad \text{\textit{\(NLS(\mathbb{T}^2)\)}}

**Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)**

Let \( s > 1, \ k \gg 1 \) and \( 0 < \sigma < 1 \) be given. Then there exists a global smooth solution \( u(x, t) \) and \( T > 0 \) such that

\[ \|u_0\|_{H^s} \leq \sigma \]

and

\[ \|u(t)\|_{H^s}^2 \geq K. \]
2. Overview of Proof
Preliminary reductions

- **Gauge Freedom:**
  If \( u \) solves NLS then \( v(t, x) = e^{-i2Gt} u(t, x) \) solves
  \[
  \begin{cases}
  i\partial_t v + \Delta v = (2G + |v|^2)v \\
  v(0, x) = v_0(x), \quad x \in \mathbb{T}^2.
  \end{cases}
  \quad (NLS_G)
  
- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,
  \[
  v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2t)}.
  \]

\[
\begin{cases}
  i\partial_t a_n = 2Ga_n + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\
  n_1 - n_2 + n_3 = n \\
  a_n(0) = \hat{u}_0(n),
\end{cases}
\quad \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2.
\quad (F NLS_G)
Preliminary reductions

- Diagonal decomposition of sum:

\[
\sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
= \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
+ \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

\[
+ \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n
\]

- Choice of $G$:

\[
G = -\| u_0 \|_{L^2}^2.
\]
\section*{Resonant truncation}

\begin{itemize}
\item \textit{NLS} dynamic is recast as

\begin{equation*}
-i \partial_t a_n = -a_n |a_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma(n)} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i \omega_4 t}. \quad (\mathcal{F} NLS)
\end{equation*}

where

\begin{equation*}
\Gamma(n) = \{ n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n \}.
\end{equation*}

\item \textit{The resonant truncation} of \( \mathcal{F} NLS \) is

\begin{equation*}
-i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b}_{n_2} b_{n_3}. \quad (R\mathcal{F} NLS)
\end{equation*}

\end{itemize}
A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$

A finite dimensional resonant truncation of $\mathcal{F}NLS$ is

$$-i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} \overline{b}_{n_2} b_{n_3}. \ (R\mathcal{F}NLS_\Lambda)$$

\forall \text{ resonant-closed finite } \Lambda \subset \mathbb{Z}^2 \ R\mathcal{F}NLS_\Lambda \text{ is an ODE.}

If $\text{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}NLS$-evolution $a_n(0) \mapsto a_n(t)$ is nicely approximated by $R\mathcal{F}NLS_\Lambda$-ODE $a_n(0) \mapsto b_n(t)$.

Given $\epsilon, s, K$, build $\Lambda$ so that $R\mathcal{F}NLS_\Lambda$ has weak turbulence.
Imagine a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ with properties. Define a nuclear family to be a rectangle $(n_1, n_2, n_3, n_4)$ where the frequencies $n_1, n_3$ (the 'parents') live in generation $\Lambda_j$ and $n_2, n_4$ ('children') live in generation $\Lambda_{j+1}$.

- $\forall 1 \leq j < M$ and $\forall n_1 \in \Lambda_j \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- $\forall 1 \leq j < M$ and $\forall n_2 \in \Lambda_{j+1} \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- The sibling of a frequency is never its spouse.
- Besides nuclear families, $\Lambda$ contains no other rectangles.
- The function $n \mapsto a_n(0)$ is constant on each generation $\Lambda_j$. 

Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...
Cartoon Construction of $\Lambda$
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Cartoon Construction of $\Lambda$
The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $RFNLS_\Lambda$ simplifies to the toy model ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\bar{b}_j(t)[b_j(t)^2 - b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 (\sum_{n \in \Lambda_j} |n|^{2s}).$$

We also want $\Lambda$ to satisfy Wide Diaspora Property

$$\sum_{n \in \Lambda_M} |n|^{2s} \gg \sum_{n \in \Lambda_1} |n|^{2s}.$$
Properties of the Toy Model ODE

- Solution of the Toy Model is a vector flow $t \rightarrow b(t) \in \mathbb{C}^M$
  \[ b(t) = (b_1(t), \ldots, b_M(t)) \in \mathbb{C}^M; b_j = 0 \ \forall \ j \leq 0, j \geq M + 1. \]

- Local Well-Posedness; Let $S(t)$ denote associated flowmap.

- Mass Conservation: $|b(t)|^2 = |b(0)|^2 \implies$

- Toy Model ODE is Globally Well-Posed.

- Invariance of the sphere: $\Sigma = \{ x \in \mathbb{C}^M : |x|^2 = 1 \}$
  \[ S(t)\Sigma = \Sigma. \]
Properties of the Toy Model ODE

- **Support Conservation:**

\[
\partial_t |b_j|^2 = 2 \text{Re}(\overline{b_j} \partial_t b_j) \\
= 4 \text{Re}(i b_j^2 [b_{j-1}^2 - b_{j+1}^2]) \\
\leq C |b_j|^2.
\]

Thus, if \(b_j(0) = 0\) then \(b_j(t) = 0\) for all \(t\).

- **Invariance of coordinate tori:**

\[\mathbb{T}_j = \{(b_1, \ldots, b_M \in \Sigma): |b_j| = 1, b_k = 0 \ \forall \ k \neq j\}\]

Mass Conservation \(\implies S(T)\mathbb{T}_j = \mathbb{T}_j\).
Dynamics on the invariant tori is easy:

\[b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \ \forall \ k \neq j.\]
Consider $M = 2$. Then $ODE$ is of the form

$$\begin{align*}
\partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1} b_2^2 \\
\partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2} b_1^2.
\end{align*}$$

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

$$\begin{align*}
b_1(t) &= \frac{e^{-it}}{\sqrt{1 + e^{2\sqrt{3}t}}} \omega, \\
b_2(t) &= \frac{e^{-it}}{\sqrt{1 + e^{-2\sqrt{3}t}}} \omega^2.
\end{align*}$$

- As $t \to -\infty$, $(b_1(t), b_2(t)) \to (e^{-it} \omega, 0) \in \mathbb{T}_1$.
- As $t \to +\infty$, $(b_1(t), b_2(t)) \to (0, e^{-it} \omega^2) \in \mathbb{T}_2$. 
Explicit Slider Solution
Two Explicit Solution Families
Concatenated Sliders: Idea of Proof
**Theorem**

Let $M \geq 6$. Given $\epsilon > 0$ there exist $x_3$ within $\epsilon$ of $\mathbb{T}_3$ and $x_{M-2}$ within $\epsilon$ of $\mathbb{T}_{M-2}$ and a time $t$ such that

$$S(t)x_3 = x_{M-2}.$$ 

**Remark**

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time $t_0$ and then arbitrarily concentrated at mode $j = M - 2$ at later time $t$. 
The task is to construct a finite set $\Lambda \subset \mathbb{Z}^2$ satisfying the properties that led to the Toy Model ODE. We do this in two steps:

1. Build combinatorial model of $\Lambda$ called $\Sigma \subset \mathbb{C}^{M-1}$.

2. Build a map $f : \mathbb{C}^{M-1} \rightarrow \mathbb{R}^2$ which gives

   $$f(\Sigma) = \Lambda \subset \mathbb{Z}^2$$

satisfying the properties.
Construction of Combinatorial Model $\Sigma$

- Standard Unit Square: $S = \{0, 1, 1+i, i\} \subset \mathbb{C}$, $S = S_1 \cup S_2$
  where $S_1 = \{1, i\}$ and $S_2 = \{0, 1+i\}$

- $\mathbb{Z}^2 \equiv \mathbb{Z}[i]; (n_1, n_2) \equiv n_1 + in_2$
We define
\[ \Sigma_j = \{(z_1, z_2, \ldots, z_{M-1}) : z_1, \ldots, z_{j-1} \in S_2, z_j, \ldots, z_{M-1} \in S_1\} \]
with the properties
- \[ \Sigma_j = S_2^{j-1} \times S_1^{M-j} \subset \mathbb{C}^{M-1} \]
- \[ |\Sigma_j| = 2^{M-1} \]

Next, we define
\[ \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M. \]

- \[ |\Sigma| = M2^{M-1}. \]
- \( \Sigma_j \) is called a generation.
Consider the set \( F = \{F_0, F_1, F_{1+i}, F_i\} \subset \Sigma \) defined by

\[
F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n)
\]

with \( z_1, \ldots, z_{j-1} \in S_2 \) and \( z_{j+1}, \ldots, z_n \in S_2 \) and \( w \in S \).

- The elements \( F_0, F_{1+i} \in \Sigma_{j+1} \) are called *children*.
- The elements \( F_1, F_i \) are called *parents*.
- The four element set \( F \) is called a *combinatorial nuclear family* connecting the generations \( \Sigma_j \) and \( \Sigma_{j+1} \).

\( \forall j \exists 2^{M-2} \) combinatorial nuclear families connecting generations \( \Sigma_j \) and \( \Sigma_{j+1} \).

The set \( \Sigma \) satisfies

- Existence and uniqueness of spouse and children (of sibling and parents).
- Sibling is never also a spouse.
Construction of the Placement Function

We need to map $\Sigma \subset \mathbb{C}^{M-1}$ into the frequency lattice $\mathbb{Z}^2$.

- We first define $f_1 : \Sigma_1 \to \mathbb{C}$.
- $\forall 1 \leq j \leq M$ and each combinatorial nuclear family $F$ connecting generations $\Sigma_j$ and $\Sigma_{j+1}$, we associate an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.
- Given $f_1$ and the angles of all the families, we define placement functions $f_j : \Sigma_j \to \mathbb{C}$ recursively by the rule:

Suppose $f_j : \Sigma_j \to \mathbb{C}$ has been defined. We define $f_{j+1} : \Sigma_{j+1} \to \mathbb{C}$:

$$f_{j+1}(F_{1+i}) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) + \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

$$f_{j+1}(F_0) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) - \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

for all combinatorial nuclear families connecting $\Sigma_j$ to $\Sigma_{j+1}$. 
**Theorem: Good Placement Function**

Let \( M \geq 2, s > 1 \), and let \( N \) be a sufficiently large integer (depending on \( M \)). \( \exists \) an initial placement function \( f_1 : \Sigma_1 \to \mathbb{C} \) and choices of angles \( \theta(F) \) for each nuclear family \( F \) (and thus an associated complete placement function \( f : \Sigma \to \mathbb{C} \)) with the following properties:

- **(Non-degeneracy)** The function \( f \) is injective.
- **(Integrality)** We have \( f(\Sigma) \subset \mathbb{Z}[i] \).
- **(Magnitude)** We have \( C(M)^{-1}N \leq |f(x)| \leq C(M)N \) for all \( x \in \Sigma \).
- **(Closure/Faithfulness)** If \( x_1, x_2, x_3 \) are distinct elements of \( \Sigma \) are such that \( f(x_1), f(x_2), f(x_3) \) form a right-angled triangle, then \( x_1, x_2, x_3 \) belong to a combinatorial nuclear family.
- **(Wide Diaspora/Norm Explosion)** We have

\[
\sum_{n \in f(\Sigma_M)} |n|^{2s} > \frac{1}{2} 2^{(s-1)(M-1)} \sum_{n \in f(\Sigma_1)} |n|^{2s}.
\]