Exploding, Rough and Turbulent Solutions of Nonlinear Schrdinger Equations

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Nonlinear Schrödinger Initial Value Problem

We consider the defocusing initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^{p-1}u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_p^{\pm}(\mathbb{R}^d))$$

The $+$ case is called defocusing; $-$ is focusing.

- $NLS_3^{\pm}$ is ubiquitous in physics. $NLS_p^{\pm}$ introduced to explore interplay between dispersion and strength of nonlinearity.
- $NLS_p^{\pm}$ is an infinite dimensional Hamiltonian System: study infinite dimensional dynamical behaviors!
- The **main question** about an evolution PDE: What is the ultimate fate of solutions? We want to understand the maximal-in-time behavior of the solutions.
- Conservation and invariance properties motivate the study of $NLS_p^{\pm}(\mathbb{R}^d)$ for low (and minimal) regularity initial data.

Time Invariant Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t)\nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx \pm \frac{2}{p+1}|u(t)|^{p+1} dx.$$

- Mass is L^2; Momentum is close to $H^{1/2}$; Energy involves H^1.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Space/Frequency Localizations?

Dilation Invariance and Critical Regularity

One solution u generates parametrized family $\{u^\lambda\}_{\lambda>0}$ of solutions:

$$u : [0, T) \times \mathbb{R}^d_x \to \mathbb{C} \text{ solves } NLS^{\pm}_p(\mathbb{R}^d)$$

$$\Updownarrow$$

$$u^\lambda : [0, \lambda^2 T) \times \mathbb{R}^d_x \to \mathbb{C} \text{ solves } NLS^{\pm}_p(\mathbb{R}^d)$$

where

$$u^\lambda(\tau, y) = \lambda^{-2/(p-1)} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

Norms which are invariant under $u \longmapsto u_\lambda$ are *critical*.

Other Symmetries

- Phase, space, time translation solution symmetries $\Longrightarrow$ mass, momentum, energy conservation laws.
- One solution spawns solution family by symmetry orbit.

Dilation Invariance and Critical Regularity

In the L^2-based Sobolev scale,

$$\|D^s u^\lambda(t)\|_{L^2} = \lambda^{-\frac{2}{p-1}-s+\frac{d}{2}} \|D^s u(t)\|_{L^2}.$$

The critical Sobolev index for $NLS_p^{\pm}(\mathbb{R}^d)$ is

$$s_c := \frac{d}{2} - \frac{2}{p-1}.$$

Scaling/Conservation Criticality

scaling	**regime**
$s_c < 0$	mass subcritical
$s = 0$	mass critical
$0 < s_c < 1$	mass super/energy subcritical
$s_c = 1$	energy critical
$1 < s_c < d/2$	energy supercritical

LWP & Maximal-in-time Implications

Strichartz Estimates, Duhamel, Contraction: $NLS_3(\mathbb{R}^2)$ case.

- Optimal (Sobolev H^s) regularity: $s \geq s_c = 0$ [CW], [KPV].
- Maximality/Blowup Criteria: If $T^* < \infty$
  - Strichartz Divergence, e.g.

$$\|u\|_{L^4([0,t)\times\mathbb{R}^2)} \text{ diverges as } t \nearrow T^*.$$

  - Subcritical Scaling Lower Bound,

$$\|u(t)\|_{H^s(\mathbb{R}^2)} \gtrsim \frac{1}{(T^* - t)^{s/2}}, \ 0 < s.$$

  - What blowup speeds are realized by NLS evolutions?
- Small Data Scattering Theory: $\exists\ \gamma_0 > 0$ such that

$$\|u_0\|_{L^2} < \gamma_0 \implies u(t) \text{ global, asymptotically linear.}$$

Strichartz Refinements

Advances around Fourier Restriction Phenomena led to...

- LWP for spaces of initial data larger than L^2 [MVV], [B]. . . .
- "Small" data scattering valid for certain large L^2 data.
- Further Implications of $T^* < \infty$:
  - Critical Norm (Mass) Concentration (along time sequence) [B].
  - Asymptotic Compactness Modulo Symmetries [MV].
- Links between rates of blowup quantities [B], [C-Roudenko].

Qualitative Aspects of Small Data Theory

- Robust, open set in L^2.
- Asymptotically linear behavior.
- Smallness brutally controls solution via fixed point argument.
- What is the boundary of small data scattering portion of phase space L^2?

Known Maximal-in-Time Solution Scenarios

1. Asymptotically linear (Scattering) solutions exist.
2. Soliton solutions exist: $u(t,x) = e^{it}R(x)$ (focusing case)
   - $Q(x)$ ground state; also excited states.
   - non-scattering; Strichartz norms diverge global-in-time.
3. Finite time blowup solutions are known, e.g. $NLS_3^-(\mathbb{R}^2)$:
   - $\mathcal{PC}$ transformation + solitons $\implies$ explicit (fast) $\frac{1}{t}$-blowups.
   - There exists an enlarged class of $\frac{1}{t}$-blowups [BW].
   - Virial argument $\implies$ many blowup solutions.
   - Qualitative properties? Recent advances [MR]. "log log"
4. Weakly turbulent solutions of $NLS_3^+(\mathbb{T}^2)$. [CKSTT]

2. Critical Scattering

What is the ultimate fate of the local-in-time solutions?

L^2-critical Defocusing Scattering Conjecture:

$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Moreover, $\exists\ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \|e^{\pm it\Delta}u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known [Tao-Visan-Zhang 06] for $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ for large radial data, $d \geq 3$. Same for $d = 2$ [Killip-Tao-Visan 07].
- GWP for L^2 data $\iff$ Scattering for L^2 data. [Blue-C 06]

Critical Regularity Scattering Conjecture?

Consider defocusing case $NLS_p^+(\mathbb{R}^d)$ with critical Sobolev index

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

The critical (diagonal) Strichartz index is

$$q_c = \frac{(p-1)(2+d)}{2} \iff \frac{2}{q_c} + \frac{d}{q_c} = \frac{d}{2} - s_c.$$

H^{s_c}-critical defocusing scattering conjecture:

$H^{s_c}(\mathbb{R}^d) \ni u_0 \longmapsto u$ solving $NLS_p^+(\mathbb{R}^d)$ is global-in-time and

$$\|u\|_{L_{t,x}^{q_c}} < A(u_0) < \infty.$$

Critical Regularity Scattering Conjecture?

Present status of the defocusing scattering conjecture

criticality	**general data**	**radial data**	**evidence**
$s_c = 0$	???	[TVZ],[KTV]	GWP: $s_* < s < 1$
$0 < s_c < 1$	✓ : $s_c < s_* < s < 1$	$s = s_c$??	✓: extra smooth
$s_c = 1$	[CKSTT],[RV],[V]	[B99],[G], [T]	✓: Resolved!
$1 < s_c <$	????	[KM] + bound	Numerics

- Scattering for NLS_p^- under natural threshold? [HR]
- The existence (and value) of s_* depends upon p, d.
- The work [B99] introduced **induction on energy idea**.
- Simplified/Abstracted *road map* to critical scattering. [KM]

3. Blowup

Ground State

- H^1-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$ [W]:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u,\, T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- Q is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

Pseudoconformal Symmetry

- Pseudoconformal transformation:

$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$

- $\mathcal{PC}$ is L^2-critical NLS solution symmetry:
  Suppose $0 < t_1 < t_2 < \infty$. If

$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$

  then

$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$

  solves

$$i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v.$$

- $\mathcal{PC}$ is an L^2-Strichartz isometry:
  If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then

$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t}+\frac{i}{t}},$$

  is an explicit blowup solution.
- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in S is conically concentrated into a point.
- Minimal mass H^1 blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [M]

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

NLS Blowup Dynamic?

Question: What are the dynamical properties of $NLS_3^-(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation

$\log\log$ BLOWUP REGIME

- Numerical/Persuasive arguments [LPSS] led to:
  - Prediction of blowups with $\log\log$ speed:

$$\|u(t)\|_{H^1} \sim \sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}.$$

  - Prediction that such blowups are generic/stable/observed.
  - Identification of certain mechanisms forecasting $\log\log$.
- $NLS_5^-(\mathbb{R}^1)$ has $\log\log$ blowup solutions. [P]
- Detailed Description of $\log\log$ regime in series by [MR].

Qualitative Aspects of $\log\log$ regime

- Robust, open set in H^1.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomona in critical space (L^2 instability?).
- Conjectured quantization properties?
- Boundary of $\log\log$ regime in phase space?

Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- Small Excess Mass: $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.
- Negative Total Energy: $H[u_0] < 0$.

The associated solution $u_0 \longmapsto u$ explodes with $T^* < \infty$ and

- $\exists\ (\lambda(t), x(t), \gamma(t) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.
- Sharp log log speed law holds:

$$\lambda(t)\sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$

- $u^* \notin H^s$ for $s > 0$; $u^* \notin L^p$ for $p > 2$. (Rough residual)

Theorem (Raphaël): H^1 Stability of $\log\log$

- Fact: $\mathcal{PC}$ + $\log\log$ for $E < 0 \implies \exists$ $\log\log$ with $E > 0$.
- H^1-Stability Theorem: The set of data with $u_0 \in H^1$ with small excess mass blowing up in $\log\log$ regime is open in H^1.
- Develops **bootstrap** approach to *constructing* $\log\log$.
- Further Bootstrap/stability applications [PR:Ω], [R:Ring].

Theorem (C-Raphaël): H^s Stability of $\log\log$

- Let $u_0 \in H^1$ evolve into the $\log\log$ regime.
- $\forall\ s > 0\ \exists\ \epsilon = \epsilon(s, u_0) > 0$ such that $\forall\ v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^-(\mathbb{R}^2)$ solution $v_0 \longmapsto v$ blows up in $\log\log$ regime.

Thus, the H^1 $\log\log$ blowup solutions constructed by [MR] are contained in an open superset of $\log\log$ blowups in H^s, $\forall\ s > 0$.

Remarks about the H^s stability of $\log\log$

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of H^s $\log\log$ blowups.
- The condition $s > 0$ is expected to be optimal.
  Small L^2 (but huge H^s) perturbation destroys rough residual mass ($u^* \notin H^s$, $\forall\ s > 0$) leading to fast $\frac{1}{t}$-blowup? (Zwiers)
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via I-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled I-method?

Energy Conservation in [MR] analysis

- Control of ϵ:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- Energy conservation and $\lambda \searrow 0 \implies$

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)
- Systematically replace $E(u)$ by $E(I_N u)$.

3. Weak Turbulence

[CKSTT: joint work with **Keel, Staffilani, Takaoka and Tao**]
We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x,t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$
$$\text{Energy} = E(u) = \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4 \, dx = E(0)$$

We want to understand the shape of $|\hat{u}(t,\xi)|$. The conservation laws impose L^2-moment constraints on this object.

Notion of Weak Turbulence

Definition

Weak turbulence is the phenomenon of global-in-time defocusing solutions shifting their mass toward increasingly high frequencies.

This shift is also called a forward cascade.

- A way to measure weak turbulence is to study

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,\xi)|^2 |\xi|^{2s} d\xi$$

  and prove that it grows for large times t.
- Turbulence is incompatible with scattering and integrability.
- Finite time blowup behavior is not weak turbulence.

Incompatible with Scattering & Integrability

- Scattering: $\forall$ global solution $u(t,x) \in H^s$ $\exists$ $u_0^+ \in H^s$ such that,
$$\lim_{t\to+\infty} \|u(t,x) - e^{it\Delta}u_0^+(x)\|_{H^s} = 0.$$
Note: $\|e^{it\Delta}u_0^+\|_{H^s} = \|u_0^+\|_{H^s} \implies \|u(t)\|_{H^s}$ is bounded.
Proofs rely on Morawetz-type (global dispersive) estimate.
- Complete Integrability: The 1d equation
$$(i\partial_t + \partial_x^2)u = |u|^2 u$$
has infinitely many conservation laws. Combining them in the right way one gets that $\|u(t)\|_{H^s} \leq C_s$ for all times.

Distinctions from finite time blowup setting

- Glassey's virial identity shows corresponding focusing problem

$$\begin{cases} (-i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{R}^2. \end{cases} \qquad (NLS^-(\mathbb{R}^2))$$

  has many finite time blowup solutions.
- The associated energy has a changed sign:

$$E(u) = \int \frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{4}|u(x,t)|^4 \, dx.$$

- Blowup solutions explode in H^1 in finite time.

Past Results (defocusing case)

- Bourgain: (late 90's)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove

$$\|u(t)\|_{H^s}^2 \leq C_s |t|^{4s}.$$

  The idea is to improve the local estimate for $t \in [-1, 1]$

$$\|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1$$

  ($\Longrightarrow$ $\|u(t)\|_{H^s} \lesssim C^{|t|}$ upper bounds) to obtain

$$\|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,$$

  for some $\delta > 0$. This iterates to give

$$\|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}.$$

- Improvements: Staffilani, Colliander-Delort-Kenig-Staffilani.

Past Results

- Bourgain: (late 90's)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x,t)$ to the modified wave equation
  $$(\partial_{tt} - \tilde{\Delta})u = u^p$$
  such that $\|u(t)\|_{H^s} \sim |t|^m$.
- Physics: Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ DO NOT exhibit weak turbulence. $\exists$ solutions to dispersive equations on $\mathbb{T}^d$ that exhibit weak turbulence. In particular for $NLS(\mathbb{T}^2)$ there exists $u(x,t)$ s. t.

$$\|u(t)\|_{H^s}^2 \to \infty \text{ as } t \to \infty.$$

Main Result

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

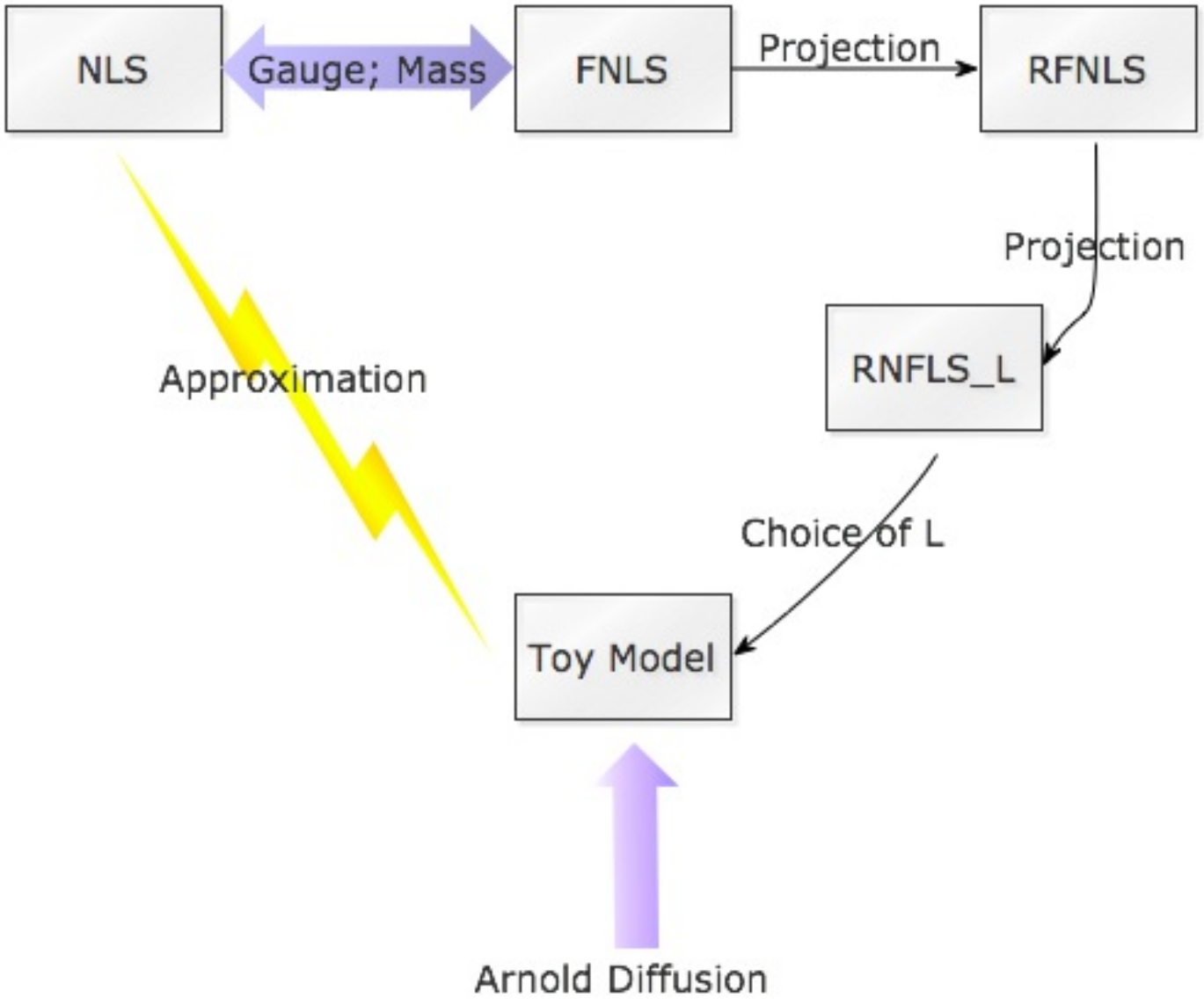
Let $s > 1$, $k \gg 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x,t)$ and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$\|u(t)\|_{H^s}^2 \geq K.$$

2. Overview of Proof



Preliminary reductions

- **Gauge Freedom:**
  If u solves NLS then $v(t,x) = e^{-i2Gt}u(t,x)$ solves

$$\begin{cases} i\partial_t v + \Delta v = (2G + |v|^2)v \\ v(0,x) = v_0(x), \qquad x \in \mathbb{T}^2. \end{cases} \tag{NLS_G}$$

- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)}.$$

$$\begin{cases} i\partial_t a_n = 2Ga_n + \displaystyle\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\ a_n(0) = \widehat{u_0}(n), \qquad n \in \mathbb{Z}^2. \end{cases} \tag{$\mathcal{F}NLS_G$}$$

Preliminary reductions

- **Diagonal decomposition of sum:**

$$\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} = \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n \neq n_1, n_3}} + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_1}}$$

$$+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_3}} - \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_1 = n_3}}$$

- **Choice of G:**

$$G = -\|u_0\|_{L^2}^2.$$

Resonant truncation

- *NLS* dynamic is recast as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a}_{n_2}a_{n_3}e^{i\omega_4 t}. \quad (\mathcal{F}NLS)$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.$$

-

$$\begin{aligned}\Gamma_{res}(n) &= \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}. \\ &= \{ \text{ Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n_4) \text{ is a rectangle } \}\end{aligned}$$

- The *resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma_{res}(n)} b_{n_1}\overline{b}_{n_2}b_{n_3}. \quad (R\mathcal{F}NLS)$$

Finite dimensional resonant truncation

- A set $\Lambda \subset \mathbb{Z}^2$ is *closed under resonant interactions* if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$

- A *finite dimensional resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma_{res}(n) \cap \Lambda^3} b_{n_1}\overline{b}_{n_2}b_{n_3}. \quad (R\mathcal{F}NLS_\Lambda)$$

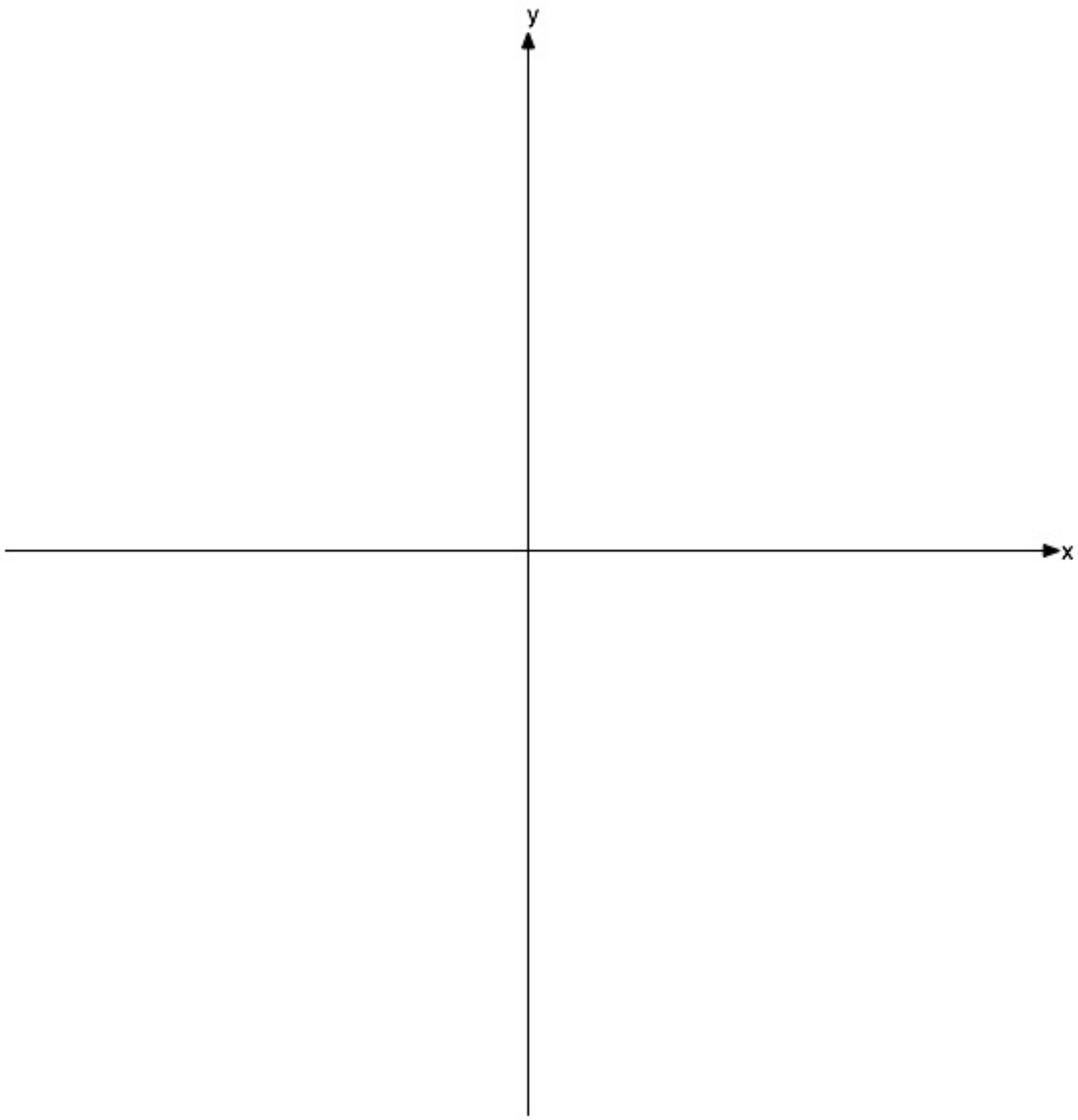
- $\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $R\mathcal{F}NLS_\Lambda$ is an ODE.
- If $\mathrm{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}NLS$-evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $R\mathcal{F}NLS_\Lambda$-ODE $a_n(0) \longmapsto b_n(t)$.
- Given ϵ, s, K, build Λ so that $R\mathcal{F}NLS_\Lambda$ has weak turbulence.

Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...

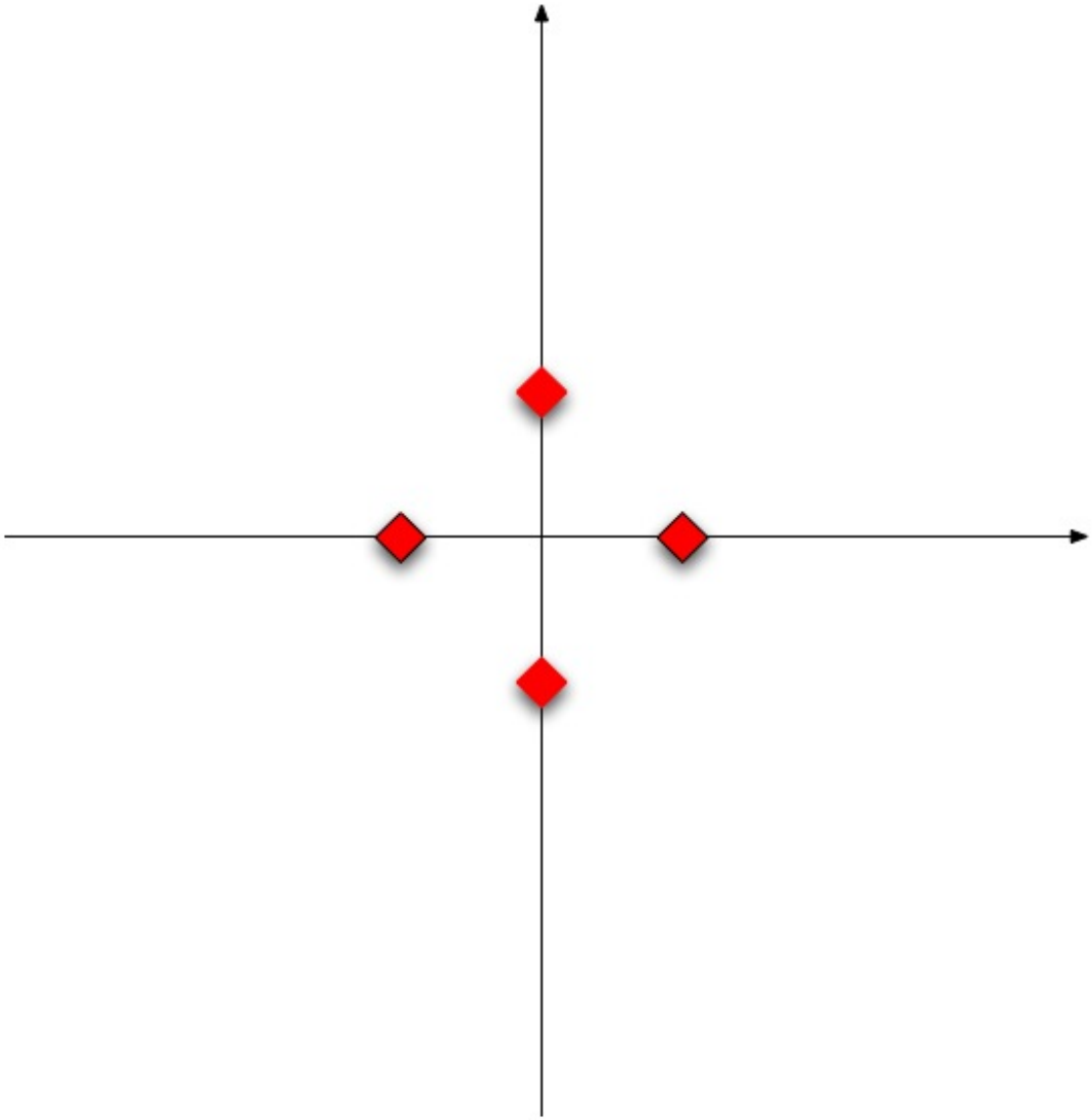
Imagine a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ with properties.
Define a nuclear family to be a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the 'parents') live in generation Λ_j and n_2, n_4 ('children') live in generation Λ_{j+1}.

- $\forall\ 1 \leq j < M$ and $\forall\ n_1 \in \Lambda_j\ \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- $\forall\ 1 \leq j < M$ and $\forall\ n_2 \in \Lambda_{j+1}\ \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- The sibling of a frequency is never its spouse.
- Besides nuclear families, Λ contains no other rectangles.
- The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.

Cartoon Construction of Λ



Cartoon Construction of Λ



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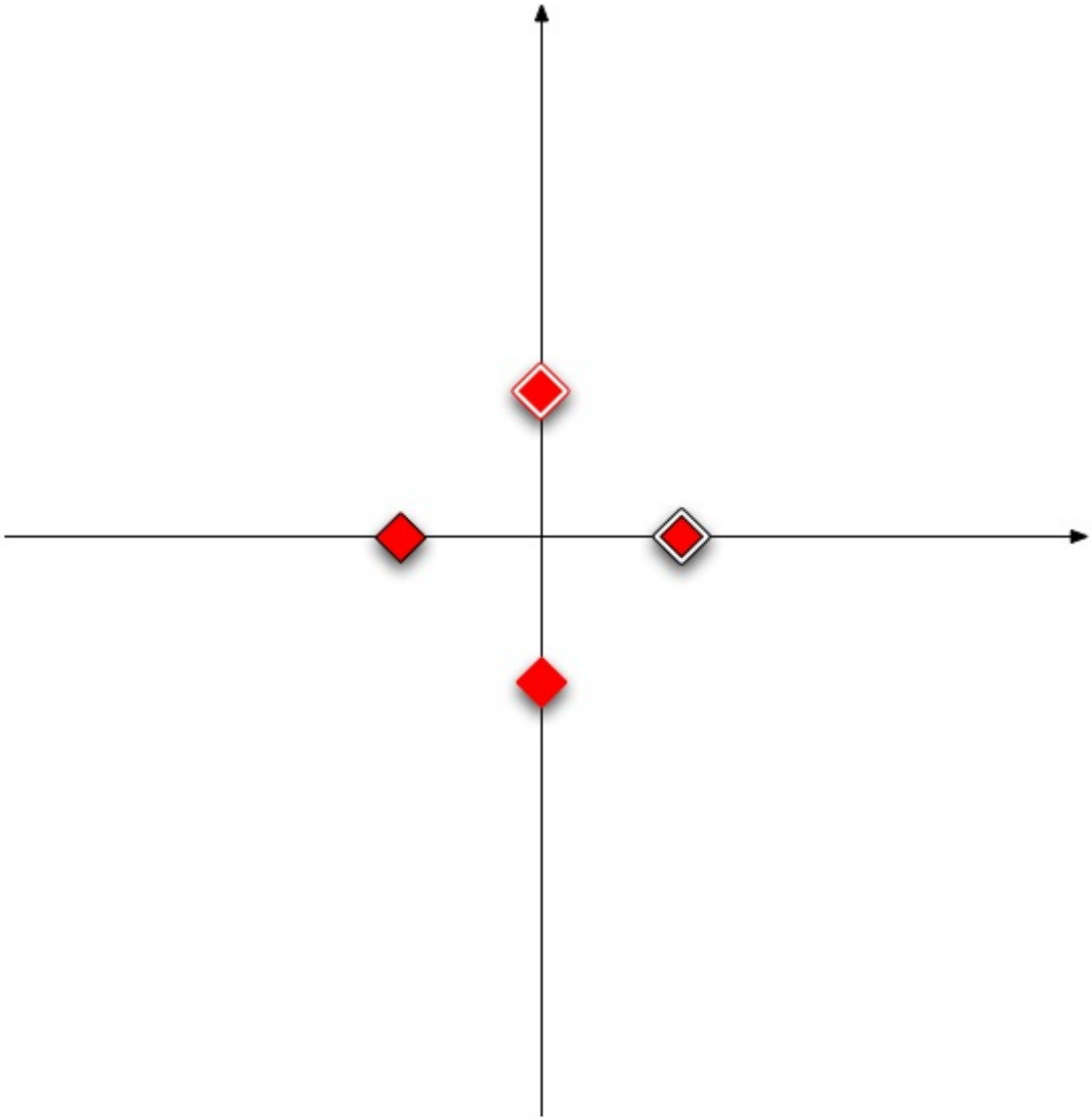

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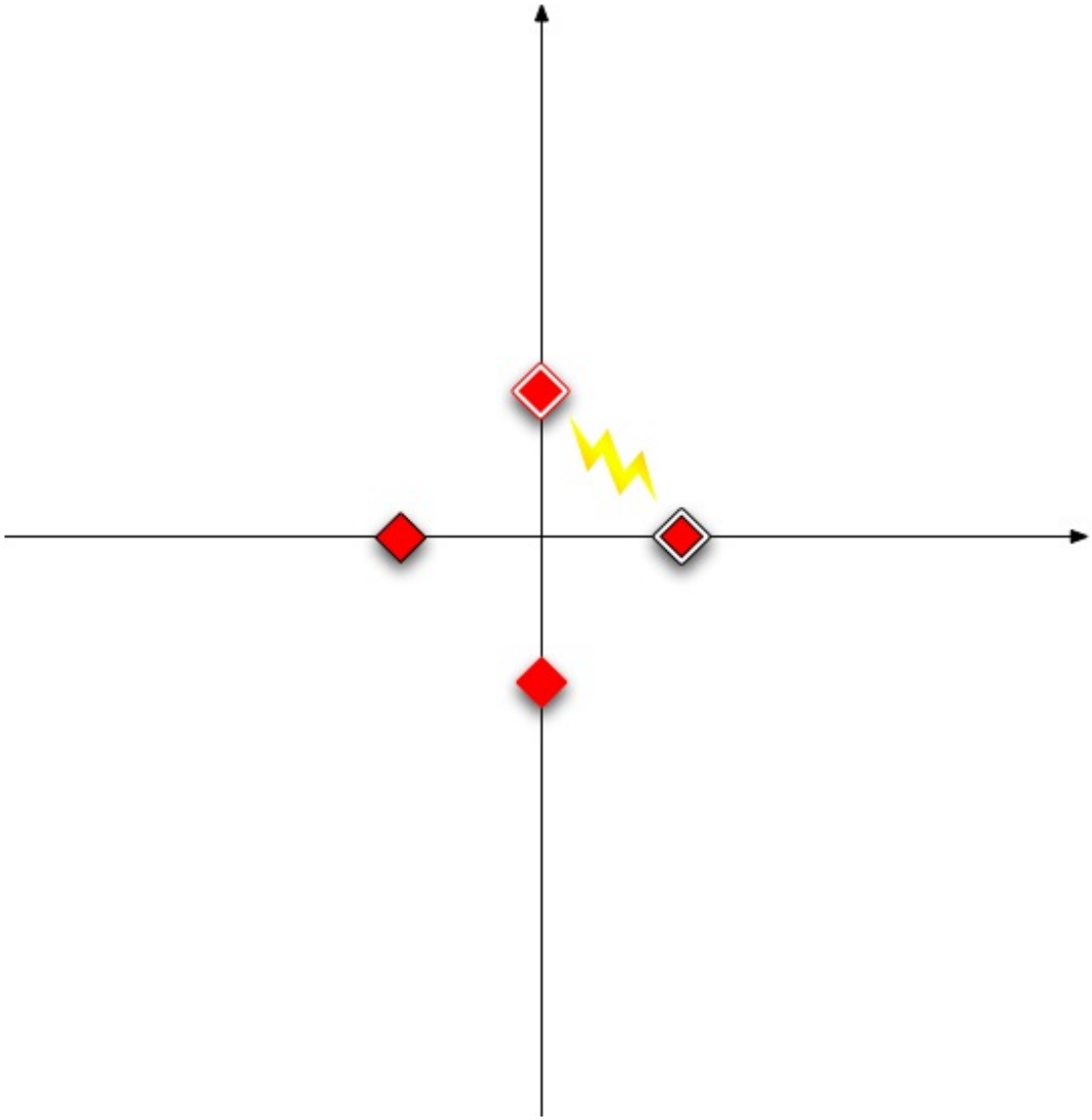

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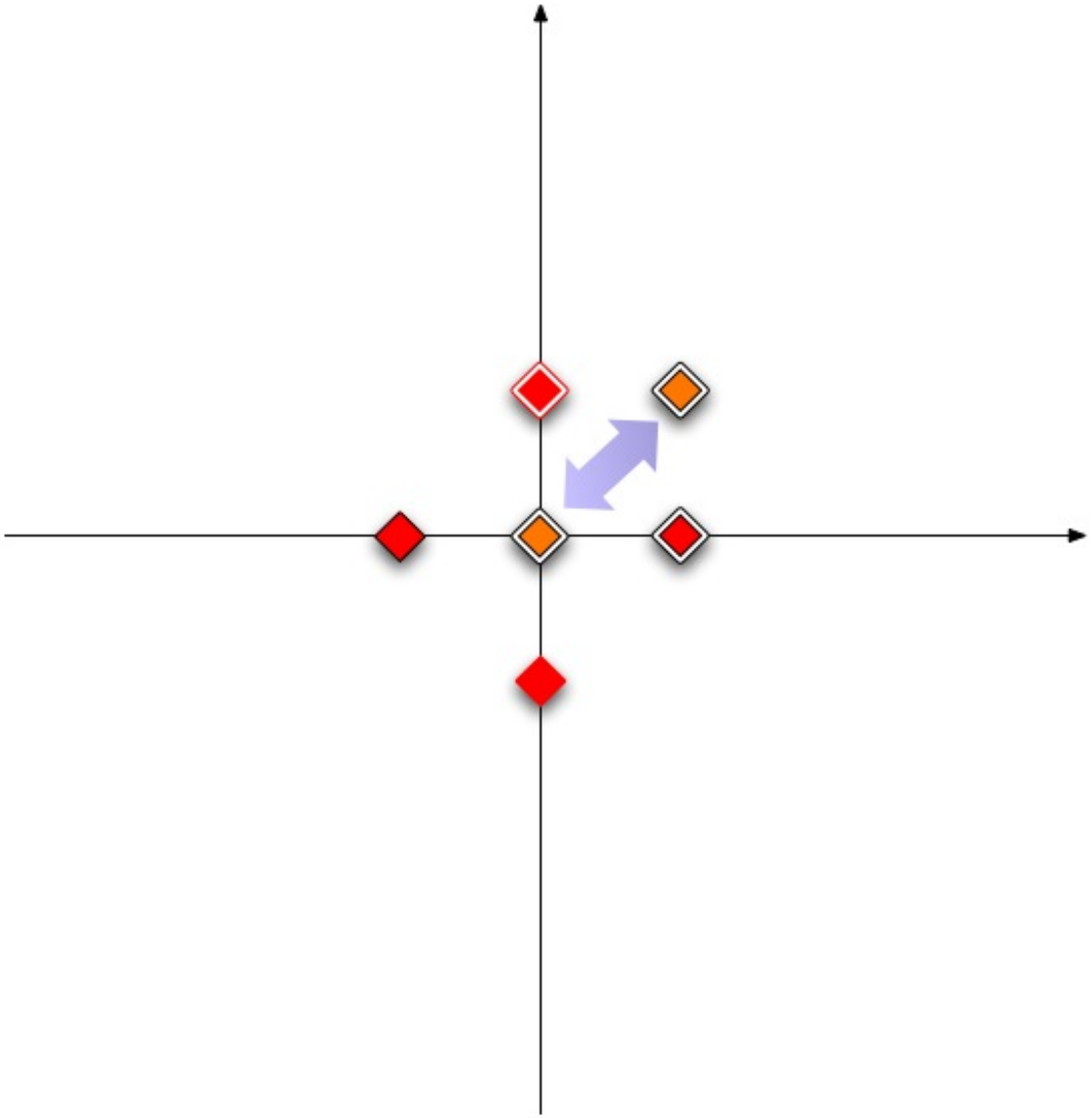

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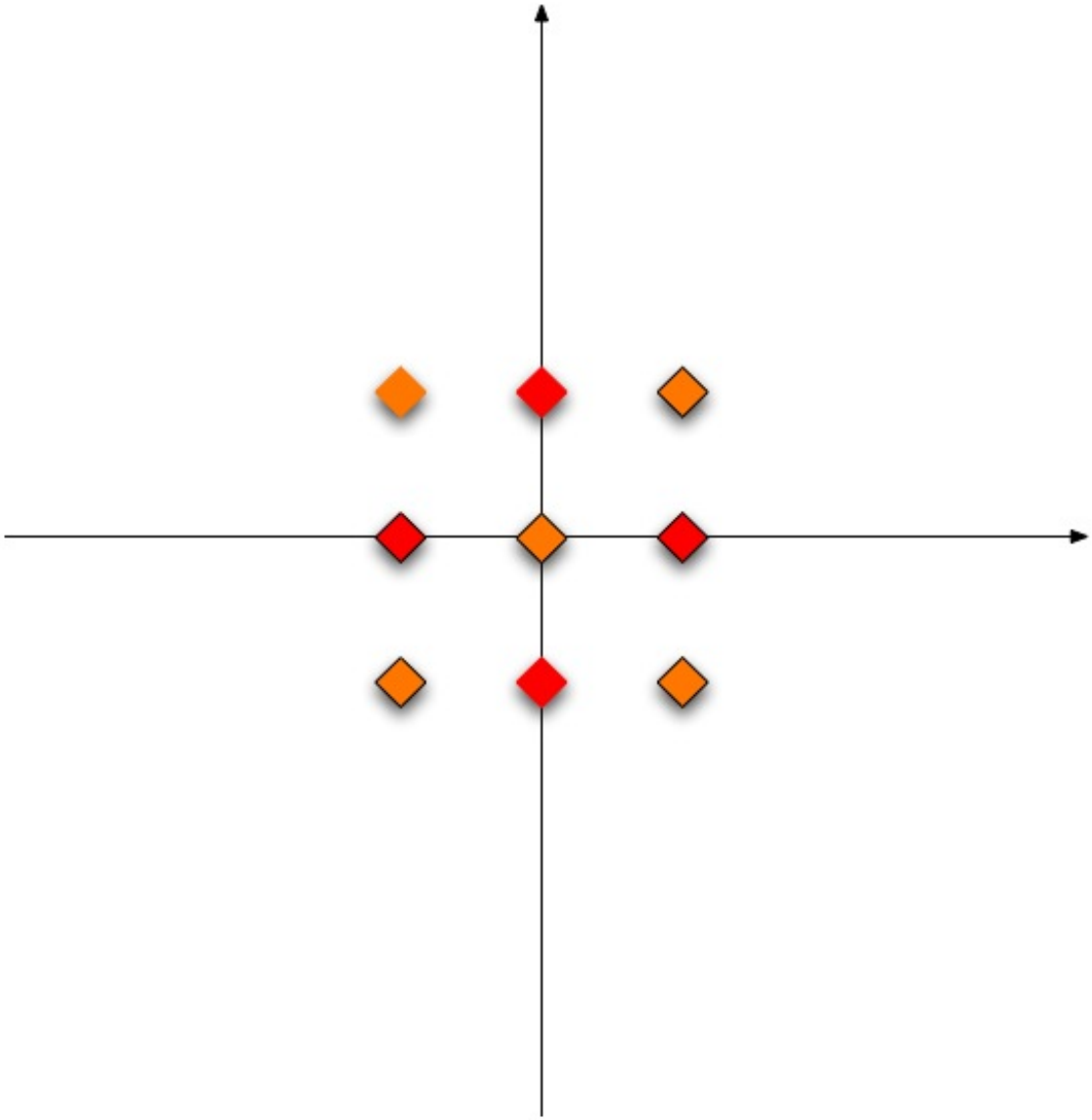

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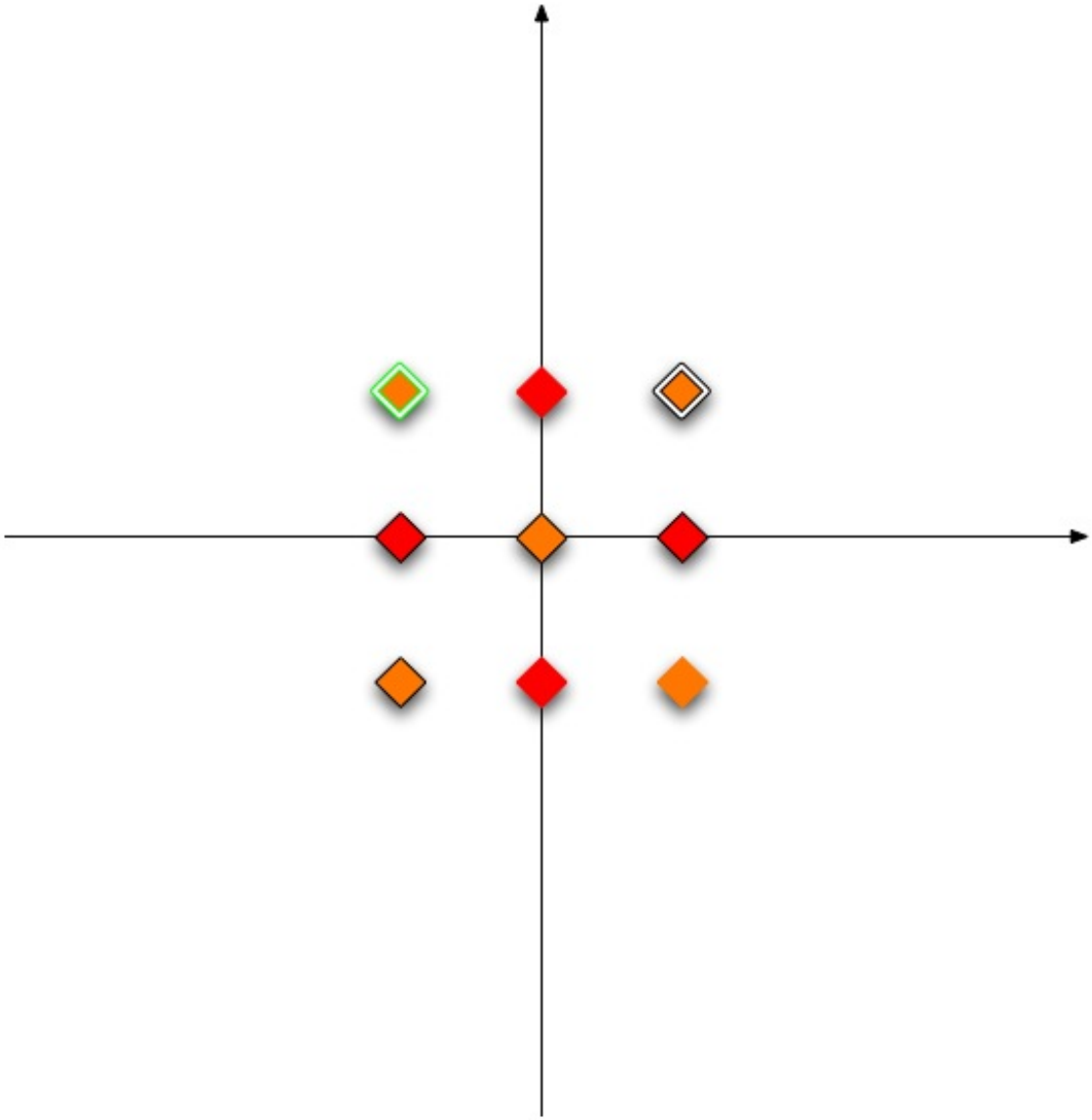

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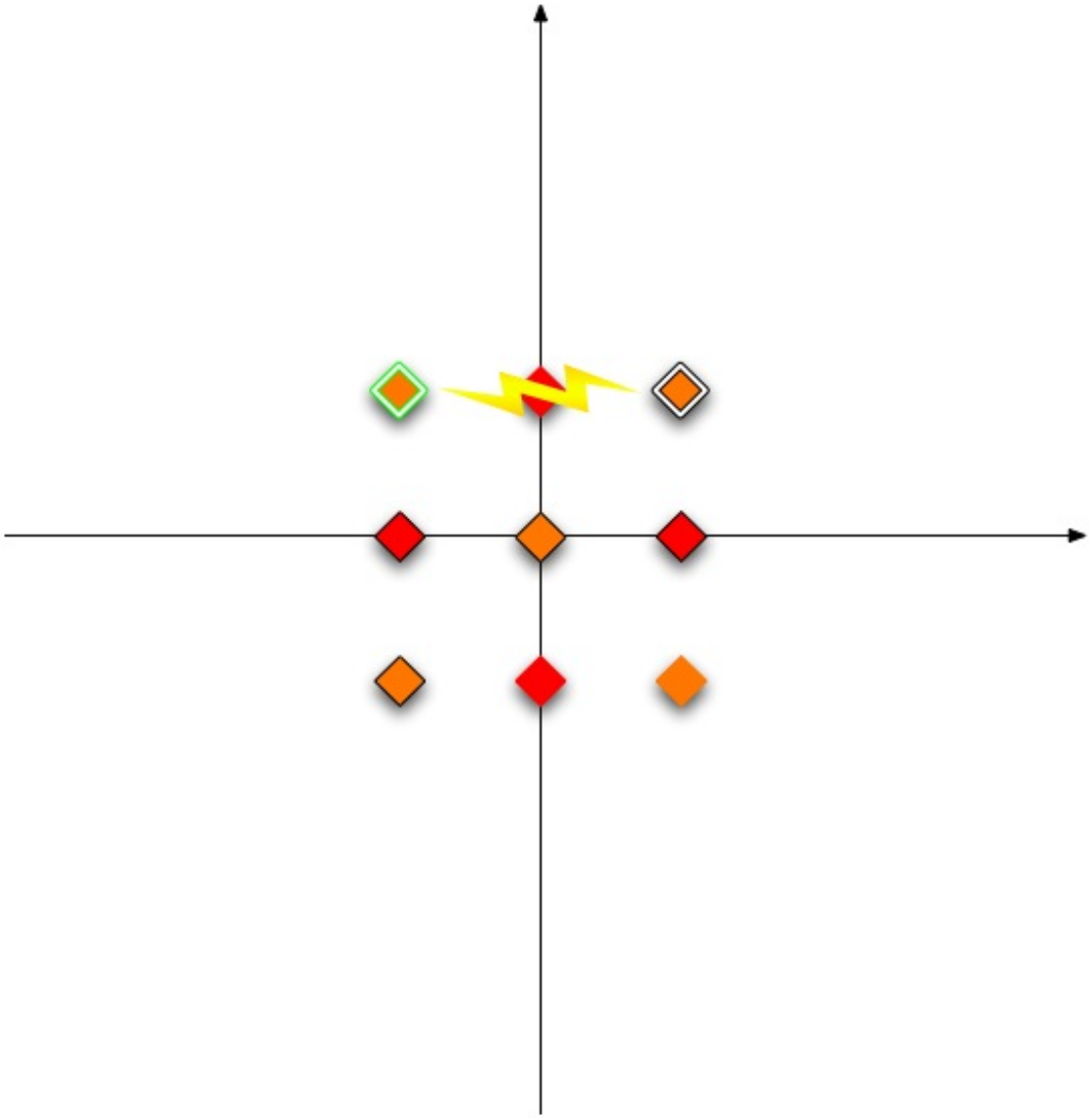

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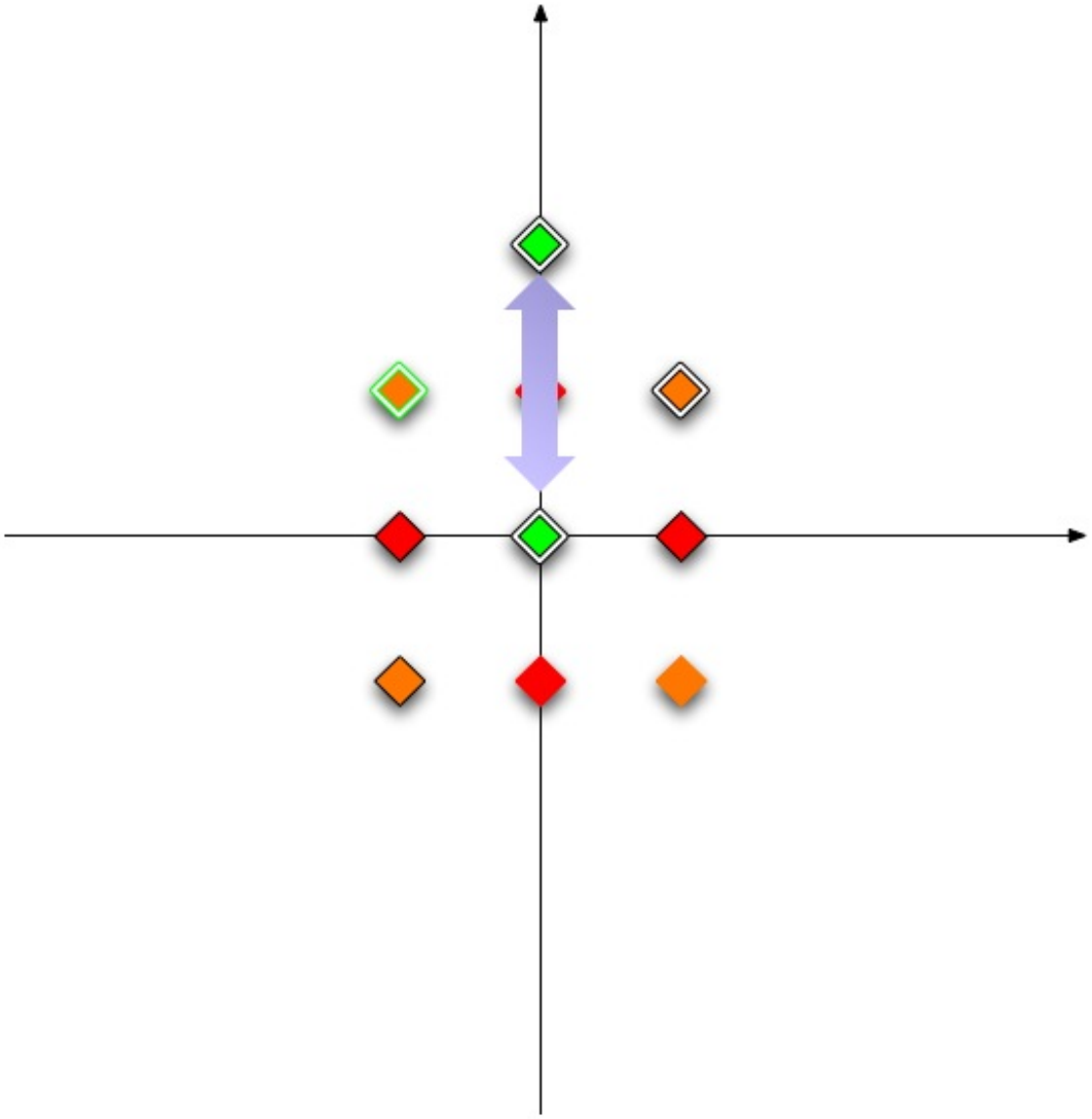

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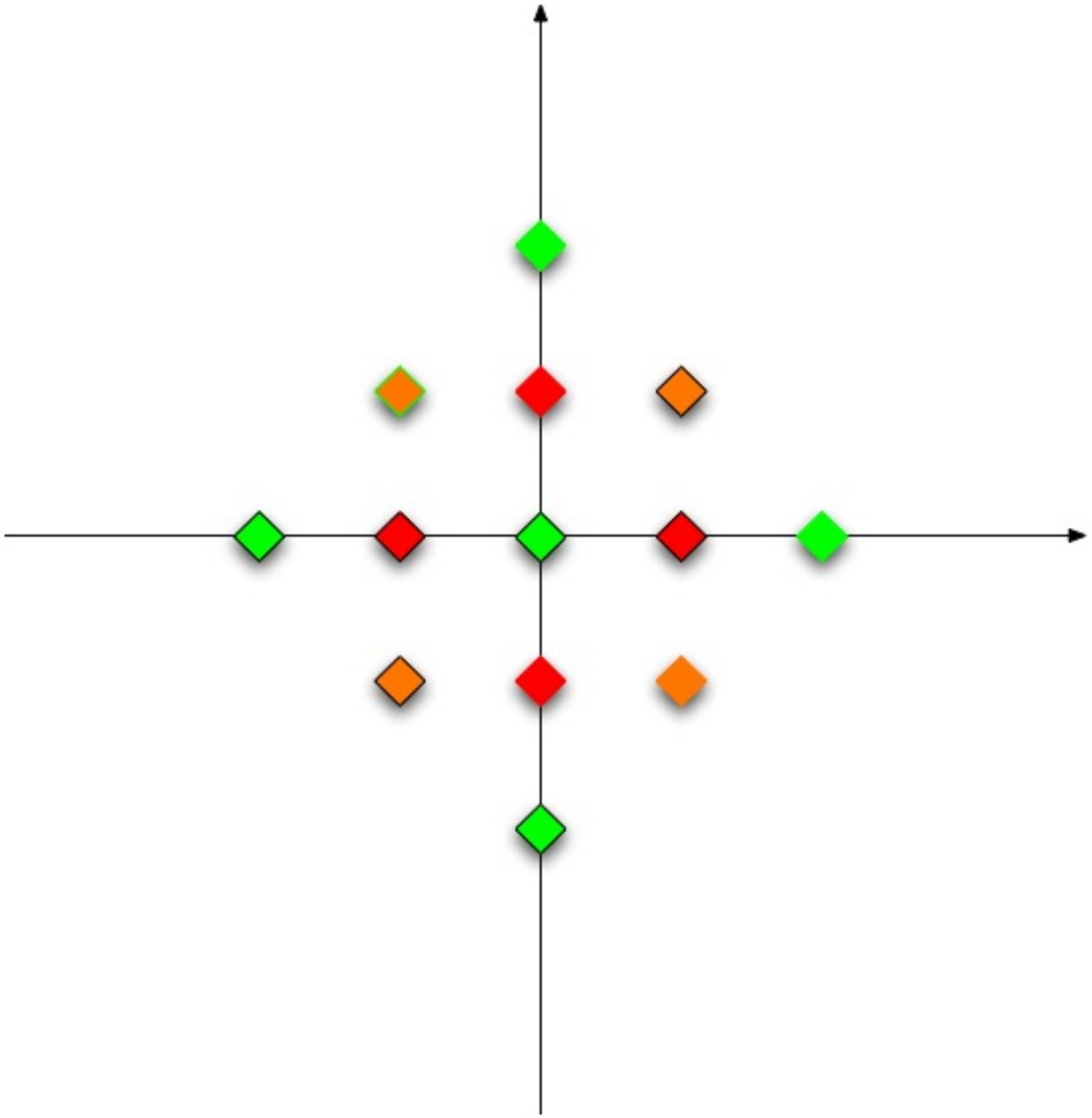

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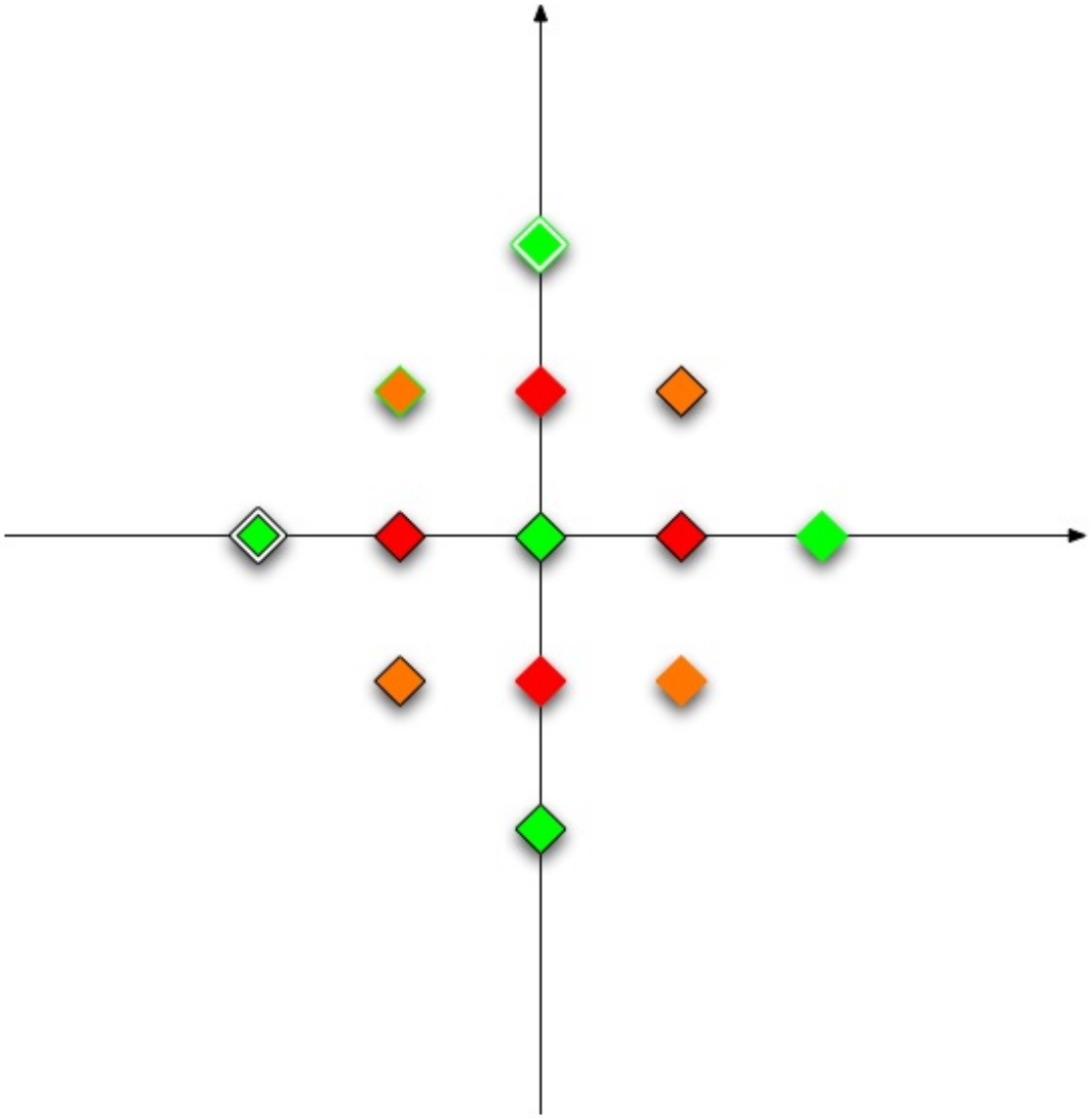

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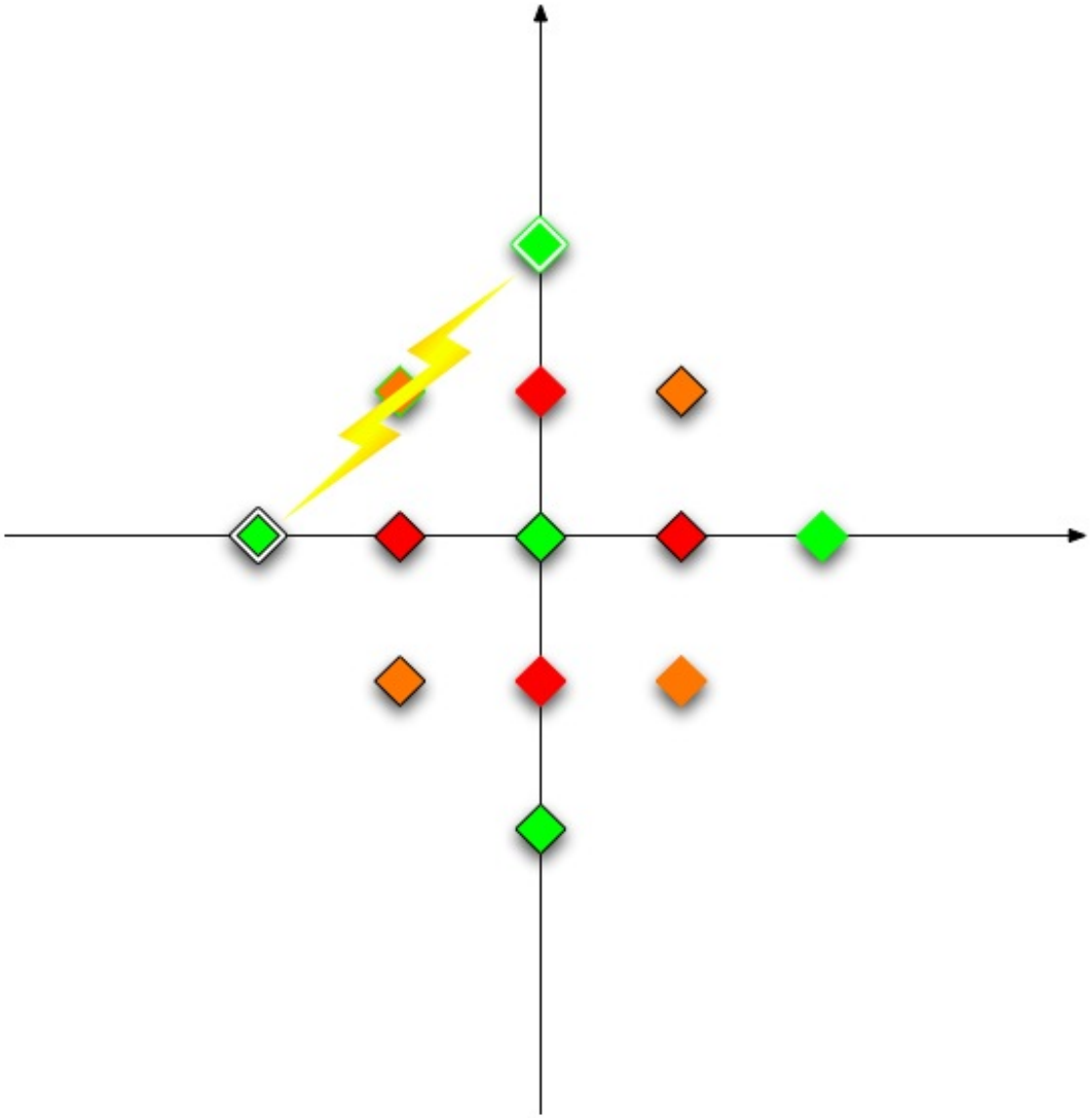

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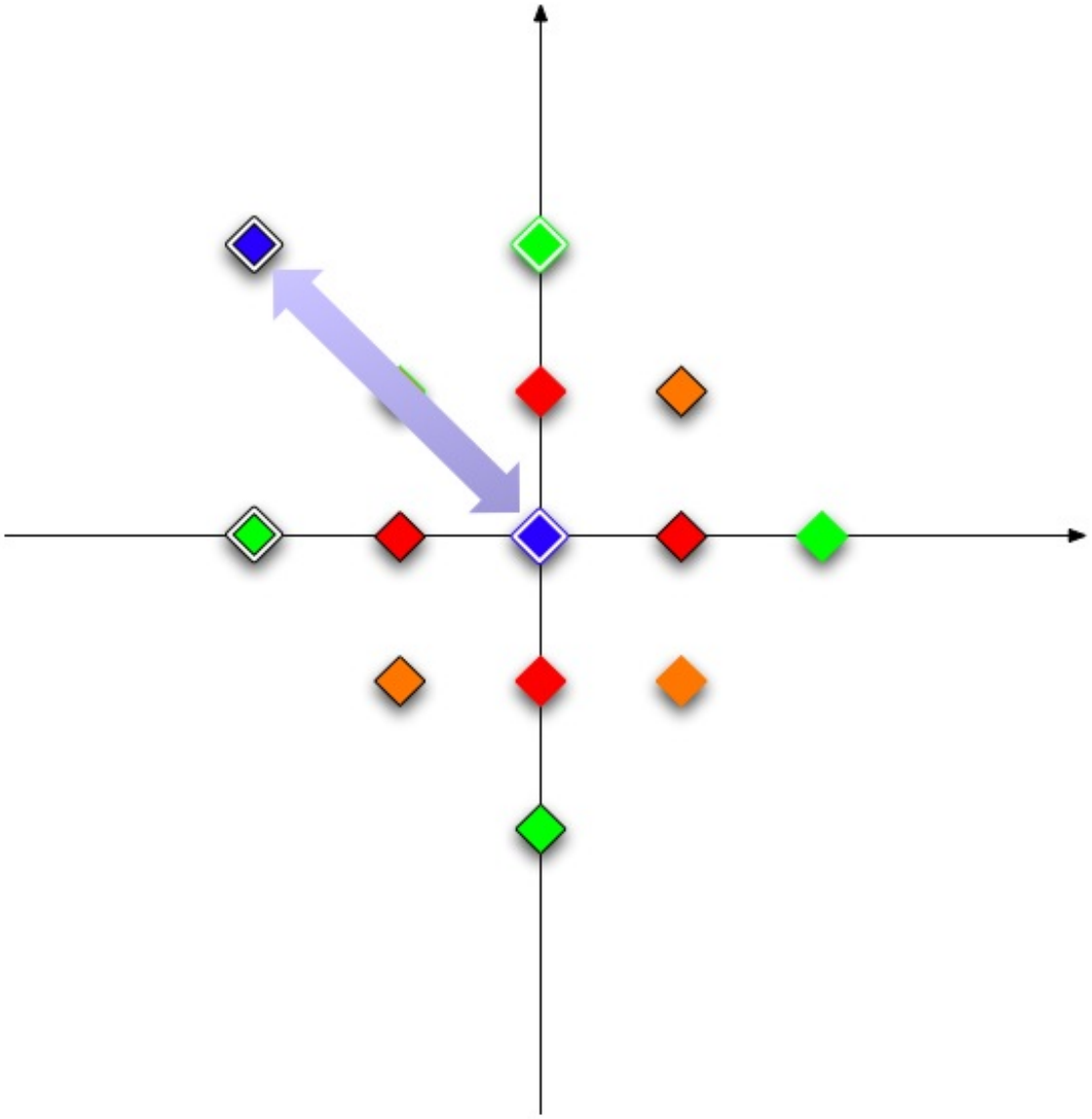

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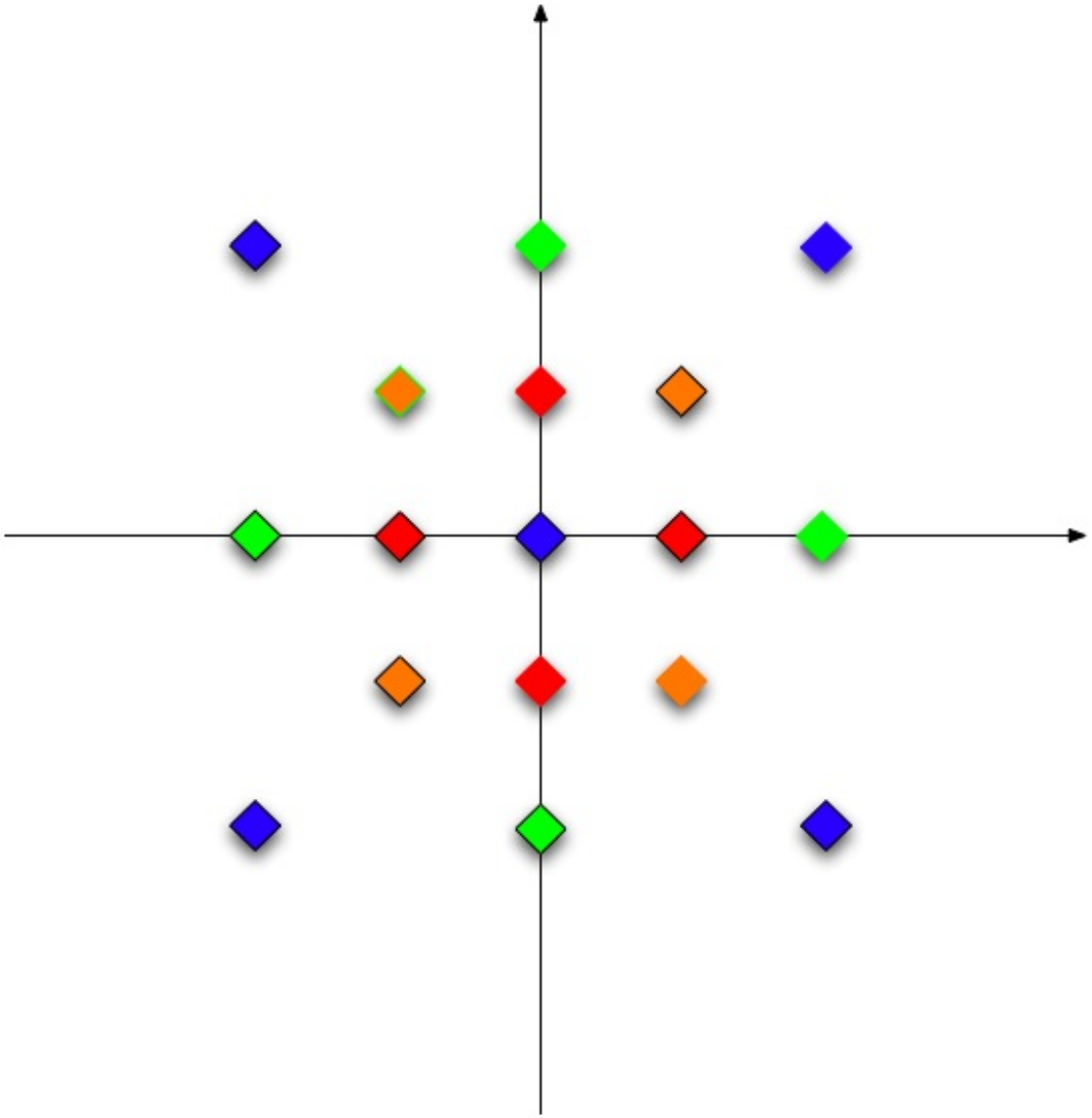
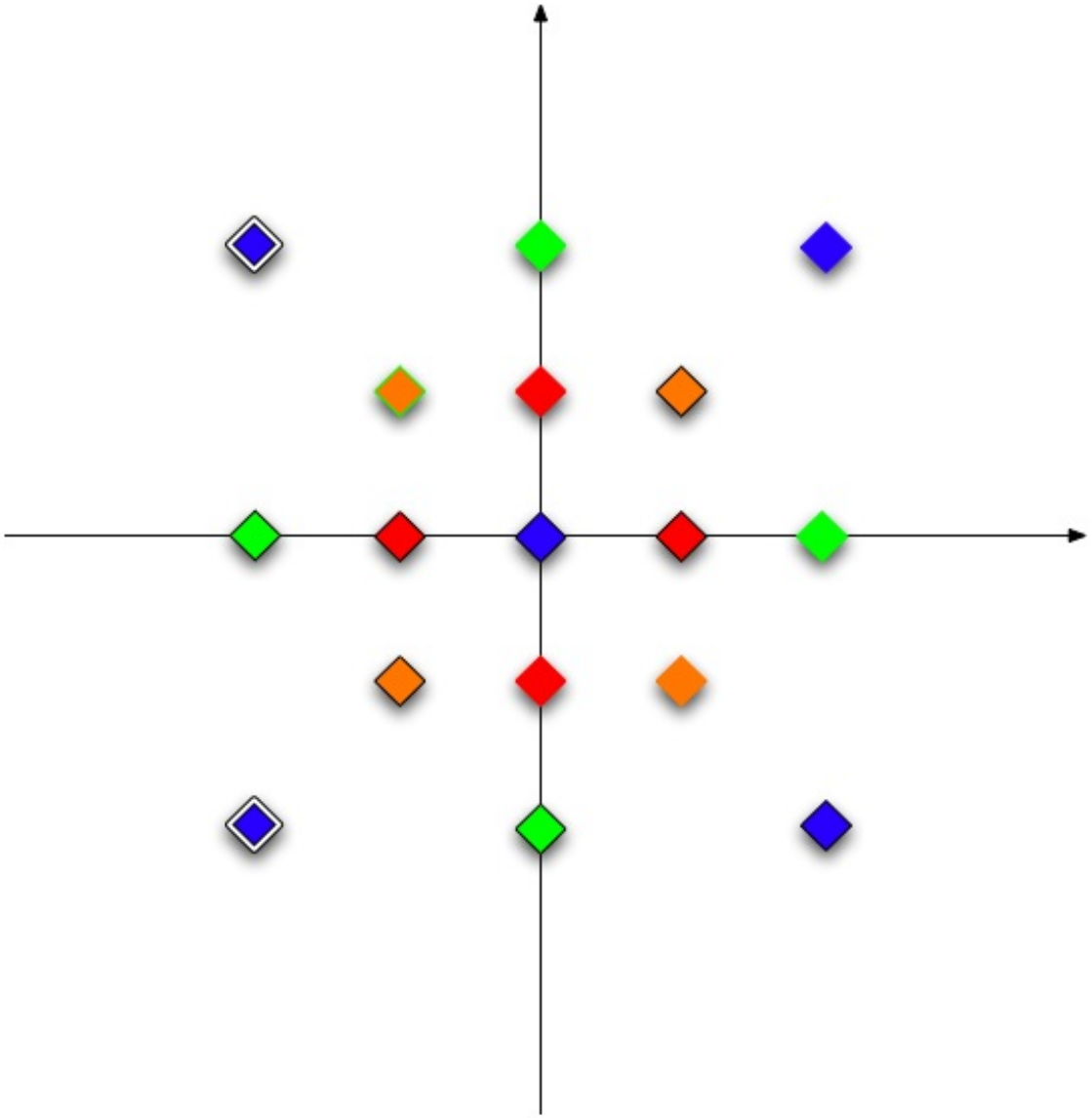
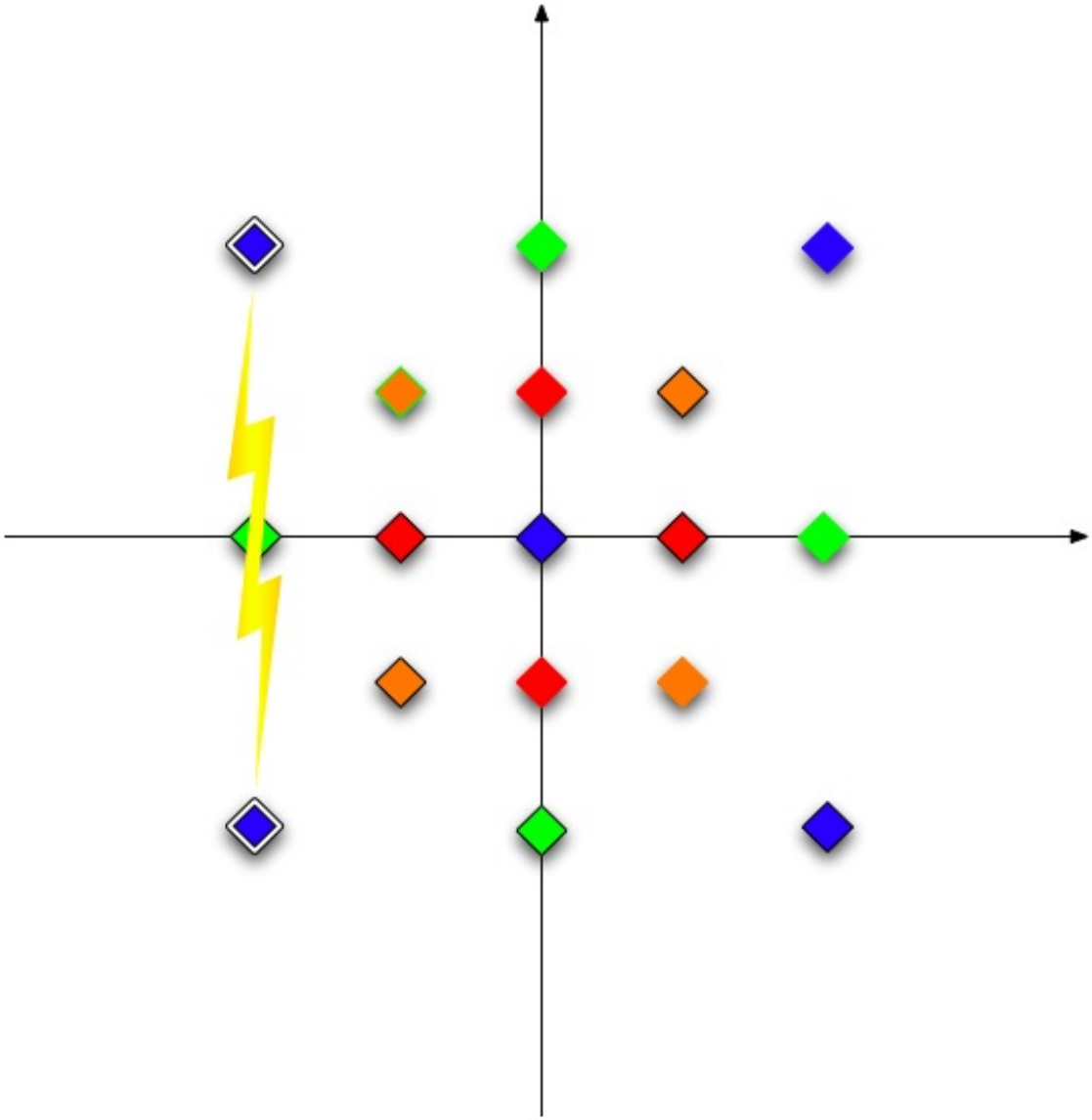


Cartoon Construction of Λ



Cartoon Construction of Λ



The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $R\mathcal{F}NLS_\Lambda$ simplifies to the toy model ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\overline{b}_j(t)[b_j(t)^2 - b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 (\sum_{n\in\Lambda_j} |n|^{2s}).$$

We also want Λ to satisfy Wide Diaspora Property

$$\sum_{n\in\Lambda_M} |n|^{2s} \gg \sum_{n\in\Lambda_1} |n|^{2s}.$$

Properties of the Toy Model *ODE*

- Solution of the Toy Model is a vector flow $t \to b(t) \in \mathbb{C}^M$

$$b(t) = (b_1(t), \ldots, b_M(t)) \in \mathbb{C}^M; b_j = 0 \ \forall \ j \leq 0, j \geq M+1.$$

- Local Well-Posedness; Let $S(t)$ denote associated flowmap.
- Mass Conservation: $|b(t)|^2 = |b(0)|^2 \implies$

  - Toy Model ODE is Globally Well-Posed.
  - Invariance of the sphere: $\Sigma = \{x \in \mathbb{C}^M : |x|^2 = 1\}$

$$S(t)\Sigma = \Sigma.$$

Properties of the Toy Model *ODE*

- Support Conservation:

$$\begin{aligned}\partial_t |b_j|^2 &= 2Re(\overline{b_j}\partial_t b_j) \\ &= 4Re(i\overline{b_j}^2[b_{j-1}^2 - b_{j+1}^2]) \\ &\leq C|b_j|^2.\end{aligned}$$

  Thus, if $b_j(0) = 0$ then $b_j(t) = 0$ for all t.

- Invariance of coordinate tori:

$$\mathbb{T}_j = \{(b_1, \ldots, b_M \in \Sigma) : |b_j| = 1, b_k = 0 \ \forall \ k \neq j\}$$

  Mass Conservation $\implies S(T)\mathbb{T}_j = \mathbb{T}_j$.
  Dynamics on the invariant tori is easy:

$$b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \ \forall \ k \neq j.$$

Explicit Slider Solution

Consider $M = 2$. Then *ODE* is of the form

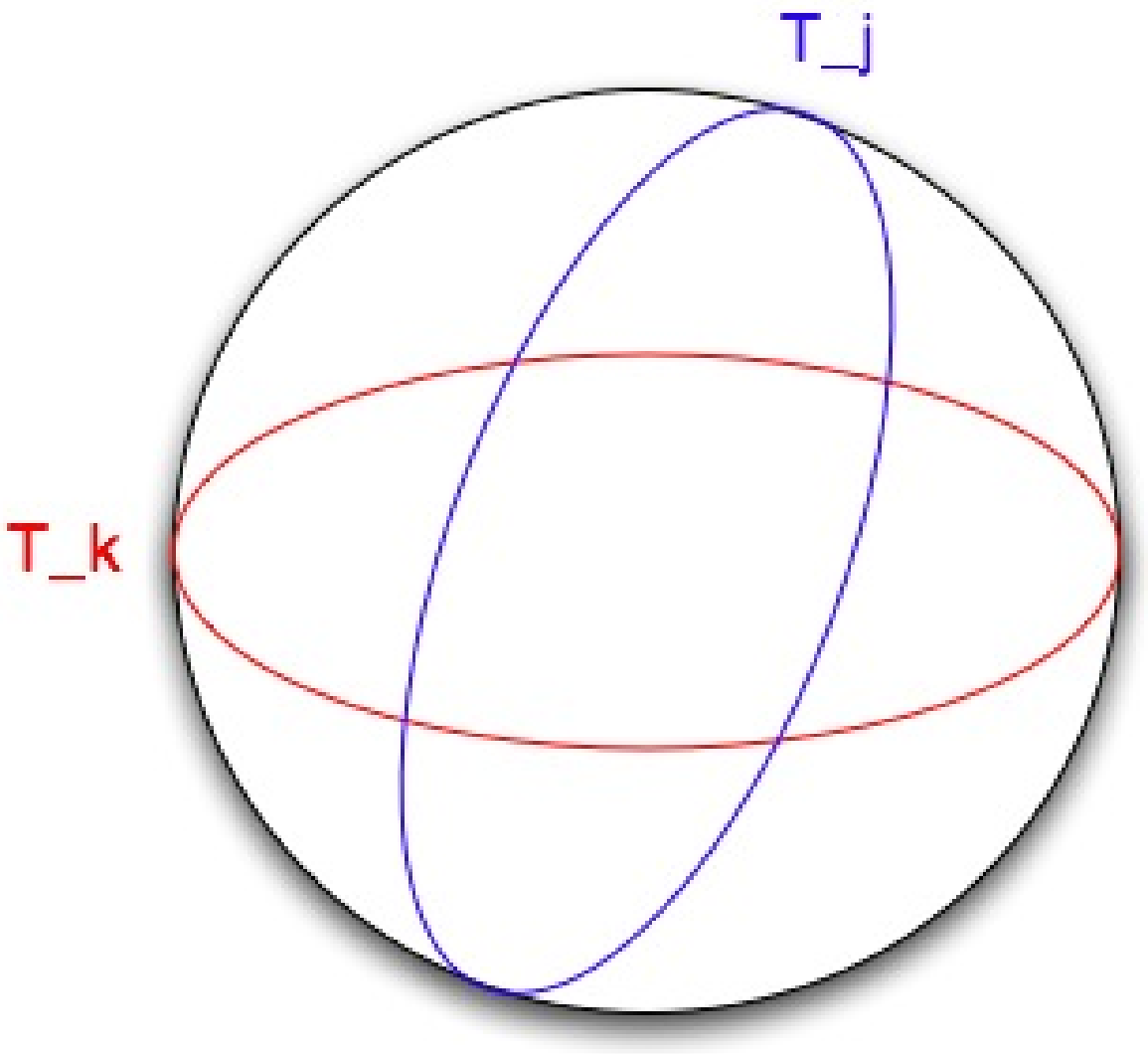
$$\begin{aligned} \partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1} b_2^2 \\ \partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2} b_1^2. \end{aligned}$$

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

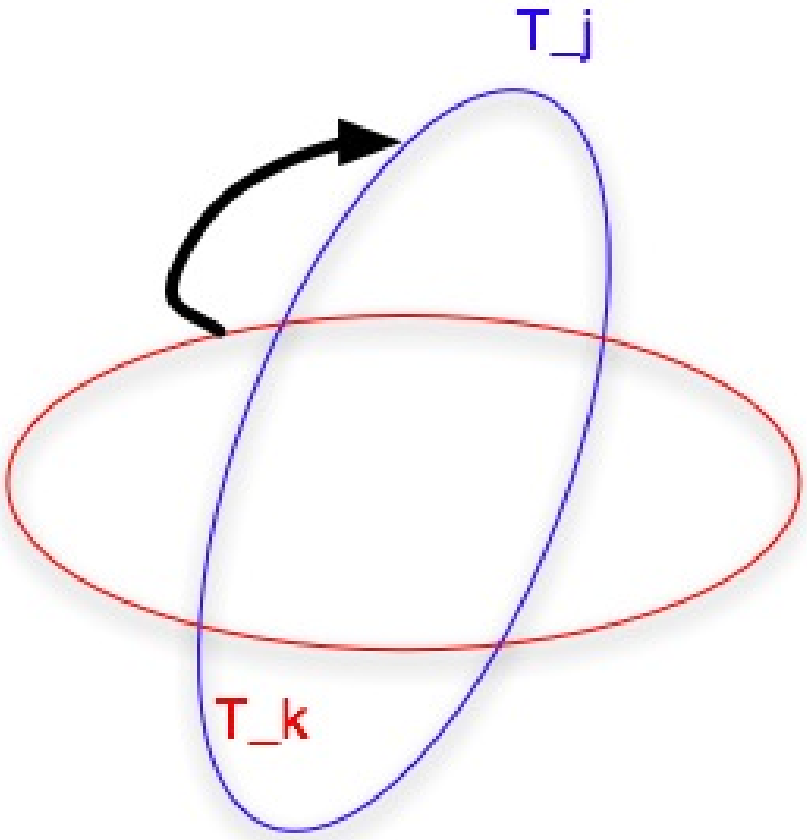
$$b_1(t) = \frac{e^{-it}}{\sqrt{1+e^{2\sqrt{3}t}}}\omega \ \ , b_2(t) = \frac{e^{-it}}{\sqrt{1+e^{-2\sqrt{3}t}}}\omega^2.$$

- As $t \to -\infty, (b_1(t), b_2(t)) \to (e^{-it}\omega, 0) \in \mathbb{T}_1$.
- As $t \to +\infty, (b_1(t), b_2(t)) \to (0, e^{-it}\omega^2) \in \mathbb{T}_2$.

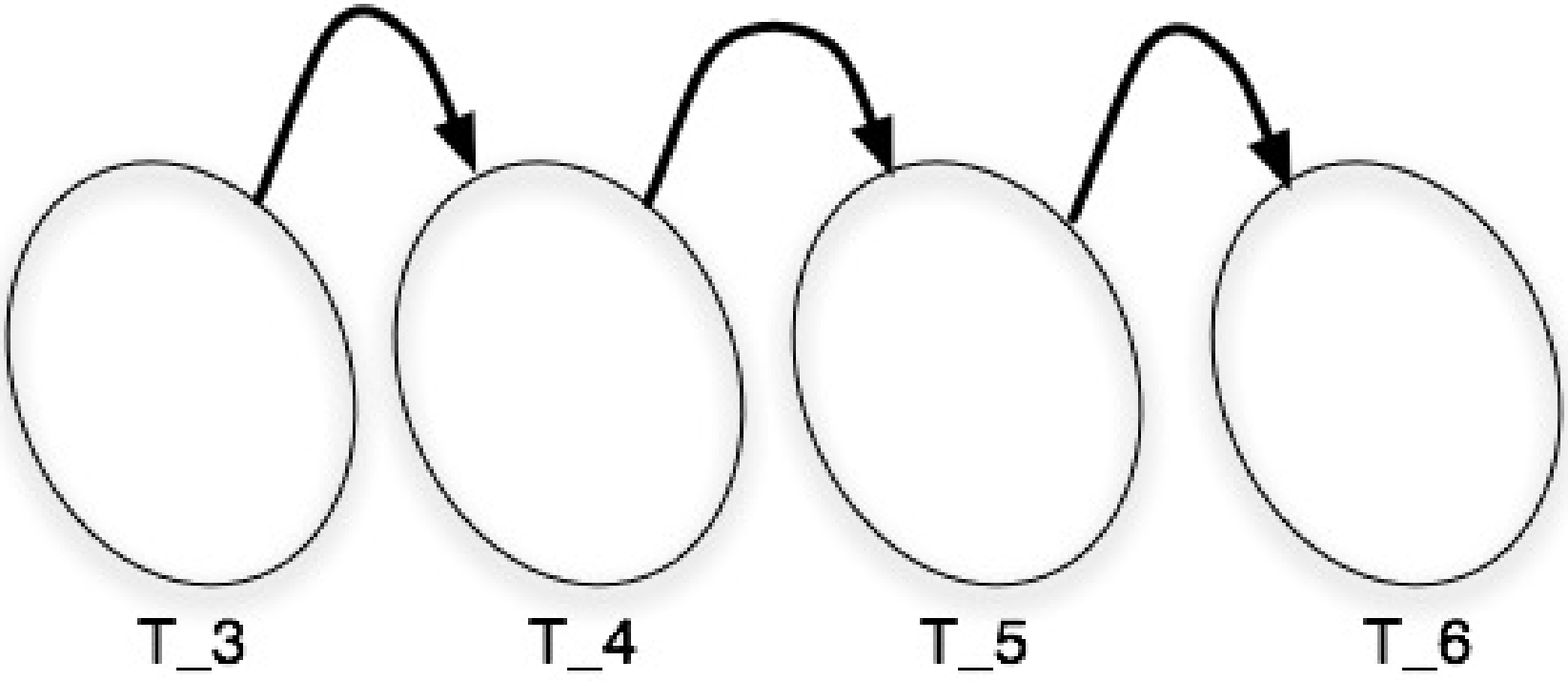
Explicit Slider Solution



Two Explicit Solution Families



Concatenated Sliders: Idea of Proof



Arnold Diffusion for Toy Model Statement

Theorem

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time t such that

$$S(t)x_3 = x_{M-2}.$$

Remark

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time t_0 and then arbitrarily concentrated at mode $j = M - 2$ at later time t.

Construction of Resonant Set Λ

The task is to construct a finite set $\Lambda \subset \mathbb{Z}^2$ satisfying the properties that led to the Toy Model ODE. We do this in two steps:

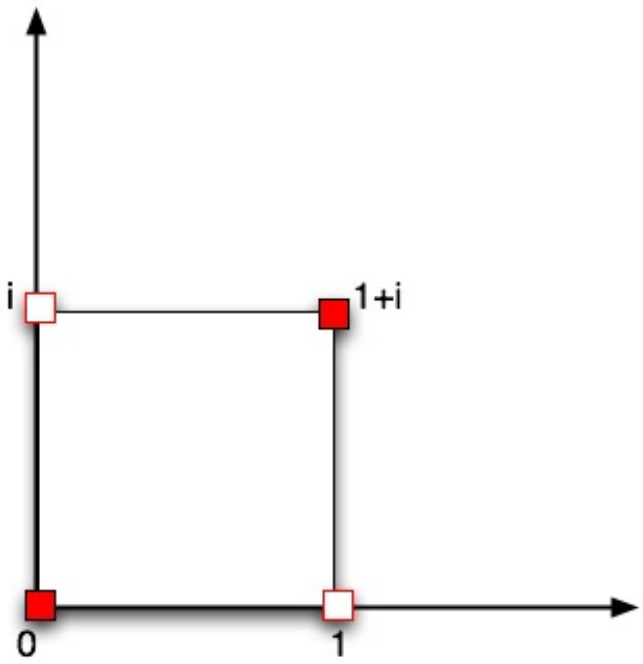
1. Build combinatorial model of Λ called $\Sigma \subset \mathbb{C}^{M-1}$.
2. Build a map $f : \mathbb{C}^{M-1} \to \mathbb{R}^2$ which gives

$$f(\Sigma) = \Lambda \subset \mathbb{Z}^2$$

satisfying the properties.

Construction of Combinatorial Model Σ

- Standard Unit Square: $S = \{0, 1, 1+i, i\} \subset C, S = S_1 \cup S_2$ where $S_1 = \{1, i\}$ and $S_2 = \{0, 1+i\}$



- $\mathbb{Z}^2 \equiv \mathbb{Z}[i]; (n_1, n_2) \equiv n_1 + in_2$

Construction of Combinatorial Model Σ

- We define

$$\Sigma_j = \{(z_1, z_2, \ldots, z_{M-1}) : z_1, \ldots, z_{j-1} \in S_2, z_j, \ldots, z_{M-1} \in S_1\}$$

  with the properties
  - $\Sigma_j = S_2^{j-1} \times S_1^{M-j} \subset \mathbb{C}^{M-1}$
  - $|\Sigma_j| = 2^{M-1}$
- Next, we define

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M.$$

  - $|\Sigma| = M2^{M-1}$.
  - Σ_j is called a generation.

Combinatorial Nuclear Family

- Consider the set $F = \{F_0, F_1, F_{1+i}, F_i\} \subset \Sigma$ defined by

$$F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n)$$

  with $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_n \in S_2$ and $w \in S$.
  - The elements $F_0, F_{1+i} \in \Sigma_{j+1}$ are called *children*.
  - The elements F_1, F_i are called *parents*.
  - The four element set F is called a combinatorial nuclear family connecting the generations Σ_j and Σ_{j+1}.
- $\forall\ j\ \exists\ 2^{M-2}$ combinatorial nuclear families connecting generations Σ_j and Σ_{j+1}.
- The set Σ satisfies
  - Existence and uniqueness of spouse and children (of sibling and parents).
  - Sibling is never also a spouse.

Construction of the Placement Function

We need to map $\Sigma \subset \mathbb{C}^{M-1}$ into the frequency lattice $\mathbb{Z}^2$.

- We first define $f_1 : \Sigma_1 \to \mathbb{C}$.
- $\forall\ 1 \leq j \leq M$ and each combinatorial nuclear family F connecting generations Σ_j and Σ_{j+1}, we associate an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.
- Given f_1 and the angles of all the families, we define placement functions $f_j : \Sigma_j \to \mathbb{C}$ recursively by the rule: Suppose $f_j : \Sigma_j \to \mathbb{C}$ has been defined. We define $f_{j+1} : \Sigma_{j+1} \to \mathbb{C}$:

$$\begin{aligned} f_{j+1}(F_{1+i}) &= \frac{1+e^{i\theta(F)}}{2} f_j(F_1) + \frac{1-e^{i\theta(F)}}{2} f_j(F_i) \\ f_{j+1}(F_0) &= \frac{1+e^{i\theta(F)}}{2} f_j(F_1) - \frac{1-e^{i\theta(F)}}{2} f_j(F_i) \end{aligned}$$

for all combinatorial nuclear families connecting Σ_j to Σ_{j+1}.

Theorem: Good Placement Function

Let $M \geq 2$, $s > 1$, and let N be a sufficiently large integer (depending on M). $\exists$ an initial placement function $f_1 : \Sigma_1 \to \mathbb{C}$ and choices of angles $\theta(F)$ for each nuclear family F (and thus an associated complete placement function $f : \Sigma \to \mathbb{C}$) with the following properties:

- (Non-degeneracy) The function f is injective.
- (Integrality) We have $f(\Sigma) \subset \mathbb{Z}[i]$.
- (Magnitude) We have $C(M)^{-1}N \leq |f(x)| \leq C(M)N$ for all $x \in \Sigma$.
- (Closure/Faithfulness) If x_1, x_2, x_3 are distinct elements of Σ are such that $f(x_1), f(x_2), f(x_3)$ form a right-angled triangle, then x_1, x_2, x_3 belong to a combinatorial nuclear family.
- (Wide Diaspora/Norm Explosion) We have

$$\sum_{n \in f(\Sigma_M)} |n|^{2s} > \frac{1}{2} 2^{(s-1)(M-1)} \sum_{n \in f(\Sigma_1)} |n|^{2s}.$$