

# Recent Progress on NLS type Equations

J. Colliander

UNC Lectures ; 1 Feb. 2009

Describe Three Results:

- C-Roy:  $NLS_3^+(R^2)$  GWP in  $H^s$ ,  $s > \frac{1}{3}$ .
- Geordie Richards: on blowup for elliptic-elliptic Davey-Stewartson.
- Imanziwari:  $NLS_3^-(R^3)$  blows up precisely on a circle.

①  $NLS_3^+(R^2)$  Cubic Defocusing NLS on  $R^2$ .

$$\begin{cases} i\partial_t u + \Delta u = -|u|^2 u \\ u(0) = u_0 \in H^s(R^2) \end{cases} \quad \text{GWP for which } s ?$$

- Conjecture  
GWP + Scattering expected  
for  $s \geq 0$ .
- $L^2 \cap \{\text{radial}\} \checkmark$  [KTV].

Incremental Progress toward Conjecture

<u>Author(s)</u>	<u>GWP Exponent</u>	<u>Tools/Ideas</u>
[GV]	$s \geq 1$	LWP + Energy
[B98]	$s > \frac{2}{3} \rightarrow \frac{3}{5}$	Duhommed Smoothing; high/low Freq. cut
[CKSTT02]	$s > \frac{4}{7}$	$H[\mathcal{I}u] ; N^{-\frac{3}{2}+}$
[CKSTT07]	$s > \frac{1}{2}$	Resonant Cut; $N^{-2+}$
[FG05]	$s \geq \frac{1}{2}$	$H[\mathcal{I}u] ; T^{1/2-} \text{-Morawetz}$
[CQTz07]	$s > \frac{2}{5}$	$H[\mathcal{I}u] ; T^{1/3-} \text{-I-Morawetz}$
<u>C-Roy 08</u>	$s > \frac{1}{3}$	Resonant Cut; $T^{1/3-} \text{-I-Morawetz}$ .
	$s > \frac{4}{13} ?$	
	$s > 0 ?$	
	$s = 0 ???$	

Theorem:  $NLS_3^+(R^2)$  GWP  
in  $H^s$ ,  $s > \frac{1}{3}$ .

Describe Proof Let  $u_0 \in H^s$ ,  $0 < s \leq \frac{2}{5}$ . (Eventually  $\frac{1}{3} < s$ )

Task: Construct  $u_0 \mapsto u(t)$   $\forall t \in [0, T]$  w.  $T$  large fixed.

$$u_\lambda(\tau, y) = \frac{1}{\lambda} u\left(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}\right) \quad \forall \lambda > 0 \text{ also solution.}$$

Equivalent Task: Construct  $u_\lambda(0) \mapsto u_\lambda(\tau) \quad \forall \tau \in [0, \lambda^2 T]$ .

I-operator

$N = N(T)$  selected later.  $I_N: H^s \rightarrow H^1$  via  $(I_N f)'(\tau) = M_N(\tau) f'(\tau)$

$$M_N(\tau) = M\left(\frac{\tau}{N}\right); \quad M(\tau) = \begin{cases} 1, & |\tau| \leq 1 \\ |\tau|^{5-1}, & |\tau| \geq 2 \end{cases} \text{ smooth, monotone}$$

Rough Solution  $H^s \ni u_0 \mapsto u$  induces finite energy reference evolution  $I_N u$ .

A priori controls on  $I_N u$  are used to globalize  $u$ .

Modified Energy

$$H[Iu(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla Iu(t, x)|^2 + \frac{1}{2} |Iu(t, x)|^4 dx$$

$$\|u(t)\|_{H^s}^2 \leq H[Iu(t)] + \|u_0\|_{L^2}^2$$

Scaling invariance + properties of  $I_N$ :

$$H[I_N u_\lambda(\cdot)] \leq C_0 \frac{N^{2(1-s)}}{\lambda^{2s}}; \quad C_0 = C(u_0).$$

$$\text{The choice of } \lambda = \boxed{\lambda(N) = C N^{\frac{1-s}{s}}} \rightarrow E[I_N u_\lambda(\cdot)] \leq \frac{1}{100}.$$

(Drop subscript  $\lambda$ ; keep in mind target  $[0, \lambda^2 T]$ .)

### Double Layer Decomposition

First, construct  $v(t) \forall t \in J_1 = [0, N^3] \subset [0, \lambda^2 T]$ .

Then, iterate that construction to cover  $[0, \lambda^2 T]$ .

Decompose  $J_1 = \bigcup_{i=1}^{\mathbb{I}} I_i$  where  $I_i$  disjoint LWP time intervals.

$$\|v\|_{L^4_{t \in I_i, x}}^4 \sim \varepsilon_0 \ll 1.$$

In principle the number  $\mathbb{I}$  of such intervals could be **HUGE**.

### Morawetz Input

However, the  $T^{1/3}$ -Morawetz estimate from [CGTZ07] gives

$$\|Iv\|_{L^4_{t \in J, x}}^4 \lesssim C_0 |J|^{1/3}, \quad (\forall J \text{ where solution exists})$$

Thus,  $\mathbb{I} \sim N$ .

### Resonant Decomposition I-Method Input

[CKSTT07]

delicate  
multilinear  
analysis

$\exists$  different modified energy  $\tilde{E}[v]$  which satisfies:

$\exists$  different modified energy  $\tilde{E}[Iv]$  at each time  $t$ :

- proximity to  $\#[Iv]$  at each time  $t$ :

$$|\#[Iv(t)] - \tilde{E}[v]| \lesssim N^{-1+} (\tilde{E}[Iv(t)])^2$$

- Almost Conservation Law:

$$\text{osc}_{t \in I_1} \tilde{E}[v] = \sup_{t \in I_1} \tilde{E}[v(t)] - \inf_{t \in I_1} \tilde{E}[v(t)] \leq C_0 N^{-2+}.$$

Bookkeeping

As  $t$  traverses  $I_1$ :

$H[I_{u(t)}]$  stays with  $O(N^{-1+})$  of  $\tilde{E}[v]$ .  
 $\tilde{E}[v]$  increments by at most  $CN^{-2+}$ .

Repeat process across  $I_2$ :

$$\underset{I_1 \cup I_2}{osc} \quad \tilde{E}[v] \leq 2CN^{-2+}.$$

By Monotone Control ( $I \in N$ ),

$$\underset{J_1 = \bigcup_{i=1}^I I_i}{osc} \quad \tilde{E}[v] \leq NCN^{-2+} = CN^{-1+}.$$

Let  $J_2 = [N^3, 2N^3] \subset [0, \lambda^2 T]$ . Decompose, iterate...

We can repeat process over  $J_3, J_4, \dots$  a total of  
 $N$  steps before  $\tilde{E}[v]$  doubles.

This process advances the solution over  $t \in [0, N^4]$ .

We need  $N^4 > \lambda^2 T \iff N^{-\frac{2}{3} + 6} \geq T$  so

$s > \frac{1}{3}$  suffices.

② Elliptic-Elliptic Davey-Stewartson Blowup

(Geordie Richards)

⑤

Davey-Stewartson System:

$$\begin{cases} i \partial_t v + \sigma u_{xx} + v_{yy} + \gamma |v|^2 v - \phi_x v = 0 \\ \alpha \phi_{xx} + \phi_{yy} + \gamma (|v|^2)_x = 0 \end{cases}$$

$\tau, \alpha; \gamma, \gamma \quad (\gamma = \pm 1, \gamma \geq 0)$

+ + ← elliptic-elliptic

$\phi$  equation:  $(\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2) \hat{\phi} = -i \gamma \tilde{\gamma}_1 (|v|^2)^\wedge$

$$(\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2) \hat{\phi}_x = -\gamma \tilde{\gamma}_1^2 (|v|^2)^\wedge$$

$$\hookrightarrow \hat{\phi}_x = -\gamma \frac{\tilde{\gamma}_1^2}{\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2} (|v|^2)^\wedge = -\gamma B (|v|^2)^\wedge$$

$$\hat{\phi}_x = -\gamma B (|v|^2)$$

General system collapses in elliptic-elliptic ( $\sigma=1, \alpha=1$ ) case:

$$DS_{\text{ell, ell}}(\mathbb{R}^2) \left\{ \begin{array}{l} i \partial_t v + \Delta v + \mathcal{L} (|v|^2) v = 0 \\ v(0, x) = v_0(x) \in H^s(\mathbb{R}^2) \end{array} \right.$$

$$\mathcal{L} = \gamma I + \gamma B \quad ; \quad \gamma = \pm 1, \gamma \geq 0$$

There is very similar to  $NLS_3(\mathbb{R}^2)$

① LWP Theory similar to  $NLS_3(\mathbb{R}^2)$ . (See Cazenave Rio lectures)  
 (e.g. maximality criterion; mass conservation; ...)

•  $L^2$ -critical;  $L^2$  LWP

• PC Invariance

• Subcritical  
Scaling behavior

H<sup>1</sup>-setting  
Conserved Energy (H<sup>1</sup> data)

$$E[u] = \frac{1}{2} \int |v u|^2 - \frac{1}{4} \int \mathcal{L}(|u|^2) |u|^2 dx$$

- [PSSW]:

$$\int \mathcal{L}(|u|^2) |u|^2 dx \leq C_{\text{opt}} + \|v u\|_{L^2}^2 \|u\|_{L^2}^2$$

$$C_{\text{opt}} = \frac{2}{\|R\|_2} \quad \text{for some } R \in H^1, \quad R(x) > 0, \quad u(t, x) = e^{itR(x)} \text{ solves } DS_{e,e}.$$

→  $\exists$  explicit blowups.

(These are not observed numerically; R unique?)

- [Ghidaglia-Saut]:

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If  $-\gamma \geq \max(\gamma, 0)$  all H<sup>1</sup>-solutions are global

$$(a) \text{ If } -\gamma \geq \max(\gamma, 0) \text{ then } E[v] < 0 \iff -\gamma < \max(\gamma, 0).$$

$$(b) \exists v \in \Sigma \text{ s.t. } E[v] < 0 \iff -\gamma < \max(\gamma, 0).$$

$$(c) \text{ If } u_0 \in \Sigma \text{ s.t. } E[u_0] < 0 \text{ then } u_0 \mapsto u \text{ blows up in finite time.}$$

### Theorem (G. Richards 08)

Let  $H^1 \ni u_0 \mapsto u \in D_{\mathbb{R}^n, e}$  which blows up as  $t \nearrow T^* < \infty$ .

Fix ~~and~~  $\lambda(t) > 0$  such that  $\lambda(t) \| \nabla u(t) \|_2 \rightarrow +\infty$  as  $t \nearrow T^*$ .

Then  $\exists x(t) \in \mathbb{R}^2$  s.t.

$$\liminf_{t \nearrow T^*} \int_{|x-x(t)| < \lambda(t)} |u(t, x)|^2 dx \geq \|R\|_{L^2}^2$$

- Analog of Merle-Tsutsumi; Nawa results

- Richards proof is based on profile decomposition due to Hamdi-Keraani: 05.

### Theorem (G. Richards 08)

Let  $L^2 \ni u_0 \mapsto u \in D_{\mathbb{R}^n, e}$  with  $T^* < \infty$ . Then

$$\limsup_{t \nearrow T^*} \sup_{\substack{\text{squares } Q \\ \text{side}(Q) < (T^*-t)^{1/2}}} \int_Q |u(t, x)|^2 dx \geq n(\|u_0\|_{L^2}).$$

- analog of [B98] mass concentration result.

- pretty much a direct consequence of  $T^* < \infty$  + refined linear estimates using the [B98] ideas.

③ Standing Ring Blowup Solutions of Cubic NLS (Ivan Zwiers) ⑧

$$\text{NLS}_3^- \quad \begin{cases} i\partial_t u + \Delta u = -|u|^2 u \\ u(0, \underline{x}) = u_0(\underline{x}) ; \quad \underline{x} \in \mathbb{R}^{2+K} \end{cases}$$

Cylindrical Coordinates

$$\underline{x} = (r, z, \theta) \in [0, +\infty) \times (-\infty, \infty) \times S^K$$

Theorem (Ivan Zwiers 2008)

$\exists$  cylindrically symmetric data  $u_0$  for which the evolution  $u(t)$  has maximum lifetime  $T^* < \infty$  and

- $\exists \lambda(t) > 0, r(t) > 0, z(t) \in \mathbb{R}, \gamma(t) \in \mathbb{R}$  s.t.

$$u(t, r, z, \theta) = \frac{1}{\lambda(t)} Q \left( \frac{(r, z) - (r(t), z(t))}{\lambda(t)} \right) e^{-i\gamma(t)} \xrightarrow{l^2} u^*(r, z, \theta)$$

as  $t \uparrow T^*$ .

Furthermore,  $(r(t), z(t)) \rightarrow (r(T^*), z(T^*)) \sim (1, 0)$

as  $t \uparrow T^*$ .

- log log blowup rate ; ...



- $\forall R > 0$

$$u^* \in H^{\frac{N+K-1}{2}}(|(r, z) - (r(T), z(T))| > R)$$

(persistence of regularity outside singularity)

## RKS:

- Exploits 2D  $L^2$ -central log log machinery of [me].
- Inspired by Raphaël's ring blowup for

$$NLS_5^- (R^{t+\epsilon})$$

- Builds from regularity persistence tools recently appearing in [Raphaël-Szeftel] for

$$NLS_5^- (R^{t+1})$$

- Provides another example of blowup also in energy supercritical rough.