Rough Blowup Solutions of Cubic Focusing NLS on $\mathbb{R}^2$

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1. The initial value problem $NLS_3^-(\mathbb{R}^2)$

Cauchy problem for (physical; canonical) focusing *NLS* :

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^-(\mathbb{R}^2))$$

Dilation Symmetry

- Suppose $u : [0, T]_t \times \mathbb{R}^2_x \to \mathbb{C}$ solves *NLS*.
- $\forall\ \lambda > 0$, define $u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y)$.
- Then $u^\lambda : [0, \lambda^2 T]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$ is also a solution of NLS.
- $NLS_3^-(\mathbb{R}^2)$ is L^2-critical.

Other Symmetries

- Phase, space, time translation solution symmetries $\implies$ mass, momentum, energy conservation laws.
- One solution spawns solution family by symmetry orbit.

Conservation Structure

- Global Conserved Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = E[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx - \frac{1}{2} |u(t)|^4 dx.$$

- Local conservation relations [e.g. $\partial_t |u|^2 + \nabla \Im(\overline{u} \nabla u) = 0$]
  - Mass & Momentum localizations $\implies$ virial; Morawetz.
  - Energy (frequency) localization $\implies$ AC laws; I-method.

2. LWP & Maximal-in-time Implications

Strichartz Estimates, Duhamel, Contraction

- Optimal (Sobolev H^s) regularity: $s \geq s_c = 0$ [CW], [KPV].
- Maximality/Blowup Criteria: If $T^* < \infty$
  - Strichartz Divergence, e.g.

$$\|u\|_{L^4([0,t)\times\mathbb{R}^2)} \text{ diverges as } t \nearrow T^*.$$

  - Subcritical Scaling Lower Bound,

$$\|u(t)\|_{H^s(\mathbb{R}^2)} \gtrsim \frac{1}{(T^*-t)^{s/2}},\ 0 < s.$$

  - What blowup speeds are realized by NLS evolutions?
- Small Data Scattering Theory: $\exists\ \gamma_0 > 0$ such that

$$\|u_0\|_{L^2} < \gamma_0 \implies u(t) \text{ global, asymptotically linear.}$$

Strichartz Refinements

Advances around Fourier Restriction Phenomena led to...

- LWP for spaces of initial data larger than L^2 [MVV], [B]. . . .
- "Small" data scattering valid for certain large L^2 data.
- Further Implications of $T^* < \infty$:
  - Critical Norm (Mass) Concentration (along time sequence) [B].
  - Asymptotic Compactness Modulo Symmetries [MV].
- Links between rates of blowup quantities [B], [C-Roudenko].

Qualitative Aspects of Small Data Theory

- Robust, open set in L^2.
- Asymptotically linear behavior.
- Smallness brutally controls solution via fixed point argument.
- What is the boundary of small data scattering portion of phase space L^2?

3. Known Maximal-in-Time Solution Scenarios

1. Soliton solutions exist: $u(t,x) = e^{it}R(x)$
   - $Q(x)$ ground state; also excited states.
   - non-scattering; Strichartz S^0 norms diverge global-in-time.
   - a priori H^1 control for H^1 data s.t. $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [W]
2. $\{\text{radial}\} \cap L^2 \ni u_0 \longmapsto u$ scatters if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. [KTV]?
3. $\mathcal{PC}$ transformation + solitons $\implies$ explicit (fast) $\frac{1}{t}$-blowups.
   - $\mathcal{PC}$ is a Stricharz S^0 isometry.
   - There exists an enlarged class of $\frac{1}{t}$-blowups [BW]. (Stability?)
4. Virial Blowup Solutions
   - Obstructive argument
   - Qualitative properties?

Ground State

- H^1-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$ [W]:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u,\, T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- Q is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

Pseudoconformal Symmetry

- Pseudoconformal transformation:
$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$
- $\mathcal{PC}$ is L^2-critical *NLS* solution symmetry:
Suppose $0 < t_1 < t_2 < \infty$. If
$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$
then
$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$
solves
$$i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v.$$
- $\mathcal{PC}$ is an L^2-Strichartz isometry:
If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then
$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}},$$

  is an explicit blowup solution.

- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in S is conically concentrated into a point.

- Minimal mass H^1 blowup solution characterization:
  $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [M]

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

  $$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

  and

  $$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

  Conservation identities lead to the generalized virial identity

  $$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

  $$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

NLS Blowup Dynamic?

Question: What are the dynamical properties of $NLS_3^-(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation

4. log log BLOWUP REGIME

- Numerical/Persuasive arguments [LPSS] led to:
  - Prediction of blowups with log log speed:

$$\|u(t)\|_{H^1} \sim \sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}.$$

  - Prediction that such blowups are generic/stable/observed.
  - Identification of certain mechanisms forecasting log log.
- $NLS_5^-(\mathbb{R}^1)$ has log log blowup solutions. [P]
- Detailed Description of log log regime in series by [MR].

Qualitative Aspects of $\log\log$ regime

- Robust, open set in H^1.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomona in critical space (L^2 instability?).
- Conjectured quantization properties?
- Boundary of $\log\log$ regime in phase space?

Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- Small Excess Mass: $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.
- Negative Total Energy: $H[u_0] < 0$.

The associated solution $u_0 \longmapsto u$ explodes with $T^* < \infty$ and

- $\exists\ (\lambda(t), x(t), \gamma(t) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t.

$$u(t) - \frac{1}{\lambda(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2.$$

- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.
- Sharp log log speed law holds:

$$\lambda(t)\sqrt{\frac{\log|\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$

- $u^* \notin H^s$ for $s > 0$; $u^* \notin L^p$ for $p > 2$. (Rough residual)

Theorem (Raphaël): H^1 Stability of $\log\log$

- Fact: $\mathcal{PC} + \log\log$ for $E < 0 \implies \exists\ \log\log$ with $E > 0$.
- H^1-Stability Theorem: The set of data with $u_0 \in H^1$ with small excess mass blowing up in $\log\log$ regime is open in H^1.
- Develops **bootstrap** approach to *constructing* $\log\log$.
- Further Bootstrap/stability applications [PR:Ω], [R:Ring].

Theorem (C-RAPHAËL): H^s STABILITY OF $\log\log$

- Let $u_0 \in H^1$ evolve into the $\log\log$ regime.
- $\forall\ s > 0\ \exists\ \epsilon = \epsilon(s, u_0) > 0$ such that $\forall\ v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^-(\mathbb{R}^2)$ solution $v_0 \longmapsto v$ blows up in $\log\log$ regime.

Thus, the H^1 $\log\log$ blowup solutions constructed by [MR] are contained in an open superset of $\log\log$ blowups in H^s, $\forall\ s > 0$.

Remarks about the H^s stability of $\log\log$

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of H^s $\log\log$ blowups.
- The condition $s > 0$ is expected to be optimal.
  Small L^2 (but huge H^s) perturbation destroys rough residual mass ($u^* \notin H^s, \ \forall \ s > 0$) leading to fast $\frac{1}{t}$-blowup? (Zwiers)
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via I-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled I-method?

Aspects of the [MR] Analysis

- Geometrical description of $\log\log$ blowup solutions.
  - Various profiles $Q, Q_b, \widetilde{Q}_b, \widetilde{Q}_{b(t)} + \zeta_{b(t)}$. (Obscure Notation)
  - Modulation parameters related to solution symmetries.
  - Three zones: blowup core, radiation, distant/decoupled.
- Virial/Coercivity constraints; Orthogonality conditions.
- Signals of $\log\log$ blowup speed.
- A key role played by Energy conservation.

Geometrical Description

- Near T^*, $\log\log$ blowups satisfy **geometrical ansatz**

$$u(t,x) = \frac{1}{\lambda(t)}(Q_{b(t)} + \epsilon)\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}.$$

- Parameters $(\lambda(t), x(t), \gamma(t), b(t))$ solve ODEs forced by $F(\epsilon)$.
- ODEs emerge from geometrical ansatz, taking inner products with equation, imposing orthogonality conditions.
  (These choices change across the [MR] works.)

Signals of the $\log\log$ blowup

- $T^* < \infty$, $\frac{ds}{dt} = \frac{1}{\lambda^2}$. ($s$ is a rescaled time)
- Core size λ linked with deformation b:

$$\lambda(s) \sim e^{-e^{C\frac{1}{b(s)}}}, \ -\frac{\lambda_s}{\lambda} \sim b.$$

  Formal manipulations (eliminating b) lead to $\log\log$ law for λ.
- Control of ϵ:

$$\int |\nabla\epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

Energy Conservation in [MR] analysis

- Control of ϵ:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- Energy conservation and $\lambda \searrow 0 \implies$

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)

5. Dynamical Rescaled l-method bootstrap