Rough Blowup Solutions of Cubic Focusing NLS on $\mathbb{R}^2$

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1. The initial value problem \( \textit{NLS}_3^-(\mathbb{R}^2) \)

Cauchy problem for (physical; canonical) focusing \( \textit{NLS} \):

\[
\begin{align*}
(i\partial_t + \Delta)u &= -|u|^2 u \\
u(0, x) &= u_0(x).
\end{align*}
\]

\( \textit{NLS}_3^-(\mathbb{R}^2) \)

\textbf{Dilation Symmetry}

- Suppose \( u : [0, T]_t \times \mathbb{R}^2_x \to \mathbb{C} \) solves \( \textit{NLS} \).
- \( \forall \lambda > 0 \), define \( u^\lambda(\tau, y) = \lambda^{-1}u(\lambda^{-2}\tau, \lambda^{-1}y) \).
- Then \( u^\lambda : [0, \lambda^2 T]_\tau \times \mathbb{R}^2_y \to \mathbb{C} \) is also a solution of \( \textit{NLS} \).
- \( \textit{NLS}_3^-(\mathbb{R}^2) \) is \( L^2 \)-critical.

\textbf{Other Symmetries}

- Phase, space, time translation solution symmetries \( \Longrightarrow \) mass, momentum, energy conservation laws.
- One solution spawns solution family by symmetry orbit.
Global Conserved Quantities

\[ \text{Mass} = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx. \]

\[ \text{Momentum} = 2\mathbb{S} \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) \, dx. \]

\[ \text{Energy} = E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 \, dx - \frac{1}{2} |u(t)|^4 \, dx. \]

Local conservation relations [e.g. \( \partial_t |u|^2 + \nabla \mathbb{S}(\overline{u} \nabla u) = 0 \)]

- Mass & Momentum localizations \( \implies \) virial; Morawetz.
- Energy (frequency) localization \( \implies \) AC laws; I-method.
2. LWP & Maximal-in-time Implications

Strichartz Estimates, Duhamel, Contraction

- Optimal (Sobolev $H^s$) regularity: $s \geq s_c = 0$ [CW], [KPV].
- Maximality/Blowup Criteria: If $T^* < \infty$
  - Strichartz Divergence, e.g.
    \[
    \|u\|_{L^4([0,t] \times \mathbb{R}^2)} \text{ diverges as } t \to T^*.
    \]
  - Subcritical Scaling Lower Bound,
    \[
    \|u(t)\|_{H^s(\mathbb{R}^2)} \gtrsim \frac{1}{(T^* - t)^{s/2}}, \quad 0 < s.
    \]
- What blowup speeds are realized by NLS evolutions?
- Small Data Scattering Theory: \(\exists \gamma_0 > 0\) such that
  \[
  \|u_0\|_{L^2} < \gamma_0 \implies u(t) \text{ global, asymptotically linear.}
  \]
Advances around **Fourier Restriction Phenomena** led to...

- LWP for spaces of initial data larger than $L^2$ [MVV], [B]. . . .
- “Small” data scattering valid for certain large $L^2$ data.
- Further Implications of $T^* < \infty$:
  - Critical Norm (Mass) Concentration (along time sequence) [B].
  - Asymptotic Compactness Modulo Symmetries [MV].
- Links between rates of blowup quantities [B], [C-Roudenko].
Qualitative Aspects of Small Data Theory

- Robust, open set in $L^2$.
- Asymptotically linear behavior.
- Smallness brutally controls solution via fixed point argument.
- What is the boundary of small data scattering portion of phase space $L^2$?
3. **Known Maximal-in-Time Solution Scenarios**

1. **Soliton solutions exist:** \( u(t, x) = e^{it}R(x) \)
   - \( Q(x) \) ground state; also excited states.
   - non-scattering; Strichartz \( S^0 \) norms diverge global-in-time.
   - a priori \( H^1 \) control for \( H^1 \) data s.t. \( \|u_0\|_{L^2} < \|Q\|_{L^2} \). [W]

2. \( \{\text{radial}\} \cap L^2 \ni u_0 \mapsto u \) scatters if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \). [KTV]?

3. \( \mathcal{PC} \) transformation + solitons \( \implies \) explicit (fast) \( \frac{1}{t} \)-blowups.
   - \( \mathcal{PC} \) is a Strichartz \( S^0 \) isometry.
   - There exists an enlarged class of \( \frac{1}{t} \)-blowups [BW]. (Stability?)

4. **Virial Blowup Solutions**
   - Obstructive argument
   - Qualitative properties?
**Ground State**

- \( H^1 \)-GWP mass threshold for \( NLS_{3}^- (\mathbb{R}^2) \) [W]:

\[
\| u_0 \|_{L^2} < \| Q \|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty,
\]

based on optimal Gagliardo-Nirenberg inequality on \( \mathbb{R}^2 \)

\[
\| u \|_{L^4}^4 \leq \left[ \frac{2}{\| Q \|_{L^2}^2} \right] \| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2.
\]

- \( Q \) is the ground state solution to \( -Q + \Delta Q = -Q^3 \).
- The ground state soliton solution to \( NLS_{3}^- (\mathbb{R}^2) \) is

\[
u(t, x) = e^{it} Q(x).
\]
Pseudoconformal Symmetry

- Pseudoconformal transformation:
  \[ \mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u \left( -\frac{1}{\tau}, \frac{y}{\tau} \right), \]

- \( \mathcal{PC} \) is \( L^2 \)-critical NLS solution symmetry:
  Suppose \( 0 < t_1 < t_2 < \infty \). If
  \[ u : [t_1, t_2] \times \mathbb{R}^2_x \rightarrow \mathbb{C} \text{ solves } \text{NLS}^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d) \]
  then
  \[ \mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}] \tau \times \mathbb{R}^2_y \rightarrow \mathbb{C} \]
  solves
  \[ i\partial_\tau v + \Delta_y v = \pm |v|^{4/d} v. \]

- \( \mathcal{PC} \) is an \( L^2 \)-Strichartz isometry:
  If \( \frac{2}{q} + \frac{d}{r} = \frac{d}{2} \) then
  \[ \| \mathcal{PC}[u] \|_{L^q_t L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \| u \|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}. \]
Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it} Q(x)$,

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t}} + i,$$

is an explicit blowup solution.

- $S$ has minimal mass:

$$\| S(-1) \|_{L^2_x} = \| Q \|_{L^2}.$$

All mass in $S$ is *conically* concentrated into a point.

- Minimal mass $H^1$ blowup solution characterization:

$u_0 \in H^1, \| u_0 \|_{L^2} = \| Q \|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [M]
Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \rightarrow \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t, x) \, dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\mathbb{R}(\overline{\phi}\nabla \phi) \, dx.$$ 

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta \Delta a)|\phi|^2 + 4a_{jk} \mathbb{R}(\overline{\phi_j \phi_k}) - a_{jj} |u|^4 \, dx.$$ 

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 \, dx = 16H[u_0].$$

- $H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 \, dx < \infty$ blows up.
- How do these solutions blow up?
**NLS Blowup Dynamic?**

**Question:** What are the dynamical properties of $\text{NLS}_3^-(\mathbb{R}^2)$ blowup solutions?

maximality criteria; critical norm behavior
asymptotic compactness; profile decompositions
conservation structure; virial ideas; parameter modulation
4. **log log BLOWUP REGIME**

- Numerical/Persuasive arguments [LPSS] led to:
  - Prediction of blowups with \( \log \log \) speed:
    \[
    \| u(t) \|_{\mathcal{H}^1} \sim \sqrt{\frac{\log | \log (T^* - t)|}{T^* - t}} \gg \frac{1}{\sqrt{T^* - t}}.
    \]
  - Prediction that such blowups are generic/stable/observed.
  - Identification of certain mechanisms forecasting \( \log \log \).
- \( NLS_5^- (\mathbb{R}^1) \) has \( \log \log \) blowup solutions. [P]
- **Detailed Description** of \( \log \log \) regime in series by [MR].
Qualitative Aspects of $\log \log$ regime

- Robust, open set in $H^1$.
- Asymptotically nonlinear with subtle interaction.
- Delicate phenomena in critical space ($L^2$ instability?).
- Conjectured quantization properties?
- Boundary of log log regime in phase space?
Theorem (Merle-Raphaël): log log Regime

Consider any initial data $u_0 \in H^1$ such that

- **Small Excess Mass:** $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$.  
- **Negative Total Energy:** $H[u_0] < 0$.

The associated solution $u_0 \mapsto u$ explodes with $T^* < \infty$ and

- There exist $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}^*_+ \times \mathbb{R}^2 \times \mathbb{R})$ and $u^* \in L^2$ s.t. 
$$u(t) - \frac{1}{\lambda(t)} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i \gamma(t)} \to u^* \text{ in } L^2.$$  
- $x(t) \to x(T^*)$ in $\mathbb{R}^2$ as $t \nearrow T^*$.  
- Sharp log log speed law holds:
$$\lambda(t) \sqrt{\frac{\log |\log(T^* - t)|}{T^* - t}} \to \sqrt{2\pi} \text{ as } t \nearrow T^*.$$  
- $u^* \notin H^s$ for $s > 0$; $u^* \notin L^p$ for $p > 2$. (Rough residual)
Theorem (Raphaël): \( H^1 \) Stability of log log

- **Fact:** \( \mathcal{P}C + \log \log \) for \( E < 0 \) \( \implies \exists \) log log with \( E > 0 \).
- **\( H^1 \)-Stability Theorem:** The set of data with \( u_0 \in H^1 \) with small excess mass blowing up in log log regime is open in \( H^1 \).
- Develops **bootstrap** approach to *constructing* log log.
- Further Bootstrap/stability applications [PR:Ω], [R:Ring].
Theorem (C-Raphaël): $H^s$ Stability of log log

- Let $u_0 \in H^1$ evolve into the log log regime.
- $\forall \ s > 0 \ \exists \ \epsilon = \epsilon(s, u_0) > 0$ such that $\forall \ v_0 \in H^s(\mathbb{R}^2)$

$$\|u_0 - v_0\|_{H^s} < \epsilon,$$

$NLS_3^{-}(\mathbb{R}^2)$ solution $v_0 \mapsto v$ blows up in log log regime.

Thus, the $H^1$ log log blowup solutions constructed by [MR] are contained in an open superset of log log blowups in $H^s$, $\forall \ s > 0$. 
Remarks about the $H^s$ stability of log log

- The theorem implies existence of rough blowup solutions.
- Proof does not apply to perturbations of $H^s$ log log blowups.
- The condition $s > 0$ is expected to be optimal.
  Small $L^2$ (but huge $H^s$) perturbation destroys rough residual mass ($u^* \not\in H^s$, $\forall s > 0$) leading to fast $\frac{1}{t}$-blowup? (Zwiers)
- Strategy of proof
  - Isolate roles of energy conservation in [MR] analysis.
  - Relax to almost conserved modified energy via $I$-method.
  - Big Bootstrap.
- Other Applications of Dynamical Rescaled $I$-method?
Aspects of the [MR] Analysis

- Geometrical description of log log blowup solutions.
  - Various profiles $Q, Q_b, \tilde{Q}_b, \tilde{Q}_{b(t)} + \zeta_{b(t)}$. (Obscure Notation)
  - Modulation parameters related to solution symmetries.
  - Three zones: blowup core, radiation, distant/decoupled.

- Virial/Coercivity constraints; Orthogonality conditions.

- Signals of log log blowup speed.

- A key role played by Energy conservation.
Near $T^*$, log log blowups satisfy geometrical ansatz

$$u(t, x) = \frac{1}{\lambda(t)} (Q_b(t) + \epsilon) \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)}.$$

Parameters $(\lambda(t), x(t), \gamma(t), b(t))$ solve ODEs forced by $F(\epsilon)$.

ODEs emerge from geometrical ansatz, taking inner products with equation, imposing orthogonality conditions. (These choices change across the [MR] works.)
**Signals of the log log blowup**

- $T^* < \infty, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}$. (s is a rescaled time)
- Core size $\lambda$ linked with deformation $b$:
  \[
  \lambda(s) \sim e^{-e^{\frac{1}{b(s)}}}, \quad -\frac{\lambda s}{\lambda} \sim b.
  \]

  Formal manipulations (eliminating $b$) lead to log log law for $\lambda$.
- Control of $\epsilon$:
  \[
  \int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{c}{b}} + \lambda^2 |E(u)|.
  \]
Energy Conservation in [MR] analysis

- Control of $\epsilon$:

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$  

- Energy conservation and $\lambda \downarrow 0 \implies$  

$$\int |\nabla \epsilon|^2 dx \lesssim e^{-\frac{C}{b}} + \lambda^2 |E(u)|.$$  

- We can maintain same conclusion if $|E(u)| \ll \frac{1}{\lambda^2}$.  
  (Observation in [CRSW]; Led to [C-Raphaël] collaboration)
5. Dynamical Rescaled I-method bootstrap