

Conjecture $H^1 \ni v_0 \mapsto v$ (5) ell-ell DS on $[0, T^*)$, forward maximal, $T^* < \infty$.

$$\liminf_{t \uparrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0| \leq (T^*-t)^{\frac{1}{2}}} |v(t, x)|^2 dx \geq \|R\|_{L^2}^2$$

where R is the optimizer of the PSSW inequality

$$\int \int (|v|^2) |v|^2 \leq C_{\text{optimal}} \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2.$$

Recall: $\mathcal{L}(|v|^2) = r_2 |v|^2 + \beta(|v|^2)$; $r_2 = -1$.

$$\beta(f) = \left[\gamma (\alpha \partial_x^2 + \partial_t^2)^{-1} \partial_{xx} \right] (f)$$

DS: $\partial_t v + \Delta v = \mathcal{L}(|v|^2) v$; $\mathcal{L}(|v|^2) \leq 0$?

$$H[v] = \int_{\mathbb{R}^2} |\nabla v|^2 + c \int (|v|^2) |v|^2 dx$$

Hadi Keraani (Compensated Compactness)

$v_n : \mathbb{R}^2 \rightarrow \mathbb{C}$: "snapshots at (rescaled) NLS dynamics"

$$\left\{ \begin{array}{l} \limsup_{n \rightarrow \infty} \|\nabla v_n\|_{L^2} \leq M \quad \text{kinetic energy upper bound} \\ \liminf_{n \rightarrow \infty} \|v_n\|_{L^4} \geq m \quad \text{potential energy lower bound.} \end{array} \right.$$

$\Rightarrow \exists \{x_n\} \subset \mathbb{R}^2$ s.t. (v_n to subsequence)

$$v_n(\cdot + x_n) \xrightarrow[H']{} V$$

with $\|V\|_{L^2(\mathbb{R}^2)} \geq \left(\frac{1}{2}\right)^{\frac{1}{2}} \frac{m^2}{M} \|Q\|_{L^2}$. (optimal).

Application: Mass concentration for L^2 -critical NLS blowup

Suppose $H^1 \ni u \mapsto u(t)$ solves $NLS_3(\mathbb{R}^2)$ blows up at time $T^* < \infty$.

A scaling argument implies a lower bound on the blowup rate:

$$\|\nabla u(t)\|_{L^2} \geq \frac{c}{\sqrt{T-t}}.$$

Why? For time t near T^* , rescale the solution

$$u_\lambda(t, y) = \frac{1}{\lambda} u\left(\frac{t-T}{\lambda^2}, \frac{y}{\lambda}\right)$$

$$\|\nabla u_\lambda(t)\|_{L^2} = \frac{1}{\lambda} \|\nabla u\left(\frac{T-t}{\lambda^2}\right)\|_{L^2}.$$

choose $\lambda = \lambda(t) = \|\nabla u(t)\|_{L^2}$.

Then $\|\nabla u_\lambda(t)\|_{L^2} = 1$ so u_λ lives for $t \in I$, $|I| \sim 1$.

But $t + \frac{T-t}{\lambda^2} < T^*$ so $\frac{1}{\lambda^2} < T^* - t$.

$$\text{Define } \rho(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t, \cdot)\|_{L^2}}; \quad v(t, x) = \rho(t, p(x)) \cdot v(t), \quad \rho(t) \downarrow 0 \text{ as } t \rightarrow T^+$$

Fix any $t_n \nearrow T^*$. $\rho_n = \rho(t_n)$, $v_n = v(t_n)$.

We have

$$\|v_n\|_{L^2} = \|v_0\| \quad (\text{Mass conservation; } L^2\text{-invariant setting})$$

$$\|\nabla v_n\| = \|\nabla Q\|_{L^2} \quad (\text{by choice of } \rho(t)).$$

$$E[v_n] = \rho_n^2 E[u_0] \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{[Yellow speech bubble]}$$

Thus

$$\|v_n\|_{L^4}^4 \longrightarrow 2 \|\nabla Q\|_{L^2}^2 \quad \text{as } n \rightarrow \infty.$$

This shows the family satisfies HK compensated compactness w..

$$M^4 = 2 \|\nabla Q\|_{L^2}^2; \quad M = \|\nabla Q\|_{L^2}$$

$$\Rightarrow \exists \{x_n\} \subset \mathbb{R}^n \text{ and } \exists v \in H^1 \text{ s.t. } \|v\|_{L^2} \geq \|\nabla Q\|_{L^2}$$

s.t. (u_l to subsequence)

$$\rho_n^{\frac{d}{2}} v(t_n, \rho_n(\cdot) + x_n) \rightharpoonup v \quad \text{weakly}$$

$$\Rightarrow \liminf_{n \rightarrow \infty} \int_{|x| \leq A} \rho_n^2 |v(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \geq A} |v|^2 dx$$

$$\int_{|x| < A} \rho_n^{-2} |u(t_n, x_n + \rho_n x)|^2 dx = \int_{|y - x_n| < \rho_n A} |u(t_n, y)|^2 dy.$$

$$y = x_n + \rho_n x \quad |x| < A \iff |y - x_n| < \rho_n A.$$

$$dy = \rho_n^{-2} dx$$

Therefore

$$\liminf_{n \rightarrow \infty} \sup_{\substack{y \in \mathbb{R}^2 \\ |y - x_n| < \rho_n A}} |u(t_n, y)|^2 dy \geq \int_{|x| \geq A} |V|^2 dx.$$

Proposition 3.1. Let $v = \{v_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then, there exist a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted $\{v_n\}_{n=1}^\infty$), a family $\{x^j\}_{j=1}^\infty$ of sequences in \mathbb{R}^d , and a sequence $\{V^j\}_{j=1}^\infty$ of H^1 functions, such that

- (i) for every $k \neq j$, $|x_n^k - x_n^j| \xrightarrow{n \rightarrow \infty} +\infty$;
- (ii) for every $\ell \geq 1$ and every $x \in \mathbb{R}^d$,

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x_n^j) + v_n^{\ell}(x), \quad (3.1)$$

with

$$\limsup_{n \rightarrow \infty} \|v_n^{\ell}\|_{L^p(\mathbb{R}^d)} \xrightarrow{\ell \rightarrow \infty} 0, \quad (3.2)$$

for every $p \in]2, 2^*[$.

Moreover, as $n \rightarrow +\infty$,

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^{\ell}\|_{L^2}^2 + o(1), \quad (3.3)$$

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^{\ell}\|_{L^2}^2 + o(1). \quad (3.4)$$

□