Nonlinear Hamiltonian PDE

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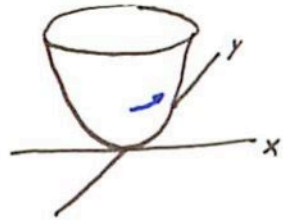
$H : \mathbb{R}^2_{x,y} \longmapsto \mathbb{R}$ e.g. $H(x,y) = x^2 + y^2$.



$\mathbb{R} \ni t \longmapsto (x(t), y(t))$ is a flow on $\mathbb{R}^2$ s.t.

$$\text{(H)} \quad \begin{cases} \dot{x} = H_y \\ \dot{y} = -H_x \end{cases} \qquad \text{Hamilton's Equations} \qquad \dot{} = \frac{d}{dt}.$$

This is a special case of the general 1st order ODE in 2 variables:

$$\begin{cases} \dot{x} = F(x,y) \\ \dot{y} = G(x,y). \end{cases}$$

Along the Hamilton flow $t \longmapsto (x(t), y(t))$,

$$\frac{d}{dt} H(x(t), y(t)) = H_x \dot{x} + H_y \dot{y} = H_x H_y + H_y(-H_x) = 0.$$

The flow moves along level sets of H.

Reexpressions of (H).

$$\frac{d}{dt} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\underline{w}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{:= \mathbb{J}} \begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

$$\boxed{\dot{\underline{w}} = \mathbb{J} \nabla H.}$$

Vector representation of (H)

$\underline{x} \in \mathbb{R}^2$.

- $\mathbb{J} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

  So $\mathbb{J}$ rotates vectors by $-\frac{\pi}{2}$ radians.

- Motion is $\perp \nabla H$ so it stays in level set.

- $\mathbb{J}^2 = -\mathbb{I}$

  so some connection to $\mathbb{C}$?

$z = x + iy$ chain rule $\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial x}\frac{\partial}{\partial \bar z}$

$\bar z = x - iy.$ $\frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \bar z}{\partial y}\frac{\partial}{\partial \bar z}.$

algebra:

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

$$\dot z = \dot x + i\dot y = H_y - iH_x = -i\{H_x + iH_y\} = -i2H_{\bar z}.$$

$$\dot z = -i2\, H_{\bar z}.$$

$$\boxed{\dot z = -i\, H_{\bar z}}$$

Complex Representation of (H)

<u>Example</u> $H(x,y) = x^2 + y^2$ or $H(z,\bar z) = |z|^2 = z\bar z.$

$\dot x = 2y$

$\dot y = -2x$

standard

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

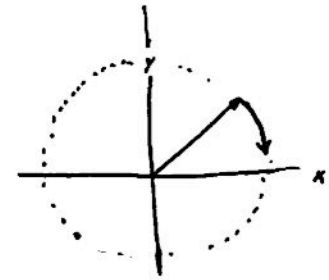
vector

$\dot z = -iz.$

complex

solution: $z_0 \longmapsto z(t)$

$z(t) = e^{-it} z_0.$



Think of these dynamics sitting "above" a point. ⟶

Jacked up Example

$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$

$j = 1, \dots, n$

$\mathbb{R}^{2n} \updownarrow \mathbb{C}^n$

$\{(x_1, y_1), \dots, (x_n, y_n)\}$

$(\underline{x}, \underline{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.

etc. $\quad \underline{w} = (\underline{x}, \underline{y})$.

$$\begin{cases} \dot{x}_j = H_{y_j} \\ \dot{y}_j = -H_{y_j} \end{cases}$$

$$\dot{\underline{w}} = \mathbb{J} \nabla H$$

$$\mathbb{J} = \begin{bmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{bmatrix}$$

$$\dot{z}_j = -i H_{\bar{z}_j}$$

standard $\quad$ vector $\quad$ complex

$$H(\underline{x}, \underline{y}) = \sum_{j=1}^{n} j^2 (x_j^2 + y_j^2) = \sum_{j=1}^{n} j^2 z_j \bar{z}_j$$

$$\dot{x}_j = (2j^2)\, y_j$$

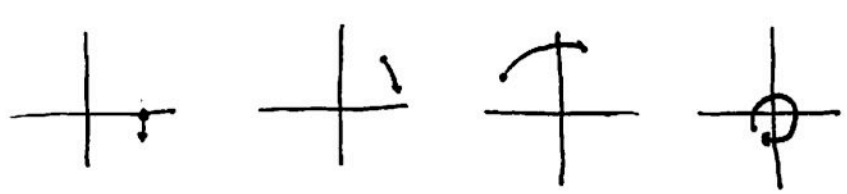
$$\dot{y}_j = -(2j^2)\, x_j$$

standard

$$\dot{z}_j = -i j^2 z_j$$

solution: $\underline{z}_0 \longmapsto \underline{z}(t)$

$$z_j(t) = e^{-i j^2 t} z_{0j}$$



$j=1$ $\quad$ $j=2$ $\quad$ $j=3$ $\quad$ $j=4$ $\quad$ $j=5$

Now consider

$$H : \{\text{smooth functions on } S^1\} \times \overline{\{\text{smooth functions on } S^1\}} \longmapsto \mathbb{R}.$$

e.g. $x \in S^1$, $u : S^1 \longrightarrow \mathbb{C}$ smooth.

$$H[u, \bar{u}] = \int_{S^1} (\partial_x u)(\partial_x \bar{u})\, dx$$

Hamilton Flow: $t \longmapsto u(t) \in \{\text{smooth functions on } S^1\}$.

$$\dot{u} = -i H_{\bar{u}}.$$

What is $H_{\bar{u}}$?

$$\langle H_{\bar{u}}, w \rangle := \lim_{\varepsilon \to 0} \frac{H[u, \bar{u} + \varepsilon \bar{w}] - H[u, \bar{u}]}{\varepsilon}$$

directional derivative

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{S^1} \left[\partial_x u \, \partial_x(\bar{u} + \varepsilon \bar{w}) - \partial_x u \, \partial_x \bar{u} \right] dx$$

$$= \int_{S^1} \partial_x u \, \partial_x \bar{w} \, dx = \int_{S^1} (-\partial_x^2 u)\, \bar{w} \, dx$$

IBP

$$= \langle \underbrace{(-\partial_x^2 u)}, \bar{w} \rangle$$

$\Longrightarrow$ $\dot{u} = -i H_{\bar{u}}$ becomes $\dot{u} = -i(-\partial_x^2 u)$

$$\boxed{i u_t + \partial_x^2 u = 0}$$

Schrödinger's Eq.

Fourier Series

$$u : \mathbb{R}_t \times \underset{\substack{\uparrow \\ \text{periodic}}}{S^1_x} \longrightarrow \mathbb{C}$$

$$u(t, x) = \sum_{n \in \mathbb{Z}} a_n(t)\, e^{inx} .$$

Schrödinger Flow — reduces on $t \longmapsto a_n(t)$.

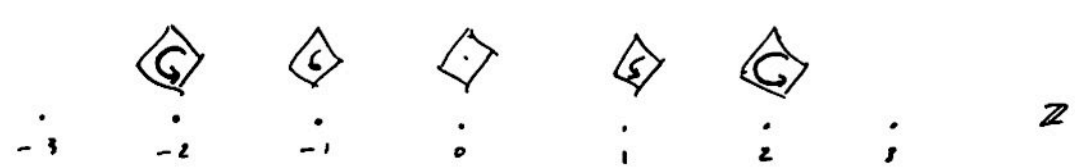
$$H[u, \bar{u}] = \int_{S^1} \partial_x \Big(\sum_{n \in \mathbb{Z}} a_n(t)\, e^{inx} \Big) \, \partial_x \overline{\Big(\sum_{k \in \mathbb{Z}} a_k(t)\, e^{ikx} \Big)} \, dx$$

$$= \int_{S^1} \Big(\sum_{n \in \mathbb{Z}} in\, a_n(t)\, e^{inx} \Big) \Big(\sum_{k \in \mathbb{Z}} \overline{a_k(t)}\, (-ik)\, e^{-ikx} \Big) \, dx$$

$$= \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (in)(-ik)\, a_n(t)\, \overline{a_k(t)} \underbrace{\int_{S^1} e^{i(n-k)x} \, dx}_{\substack{= 0 \text{ unless } n = k \\ = 1 \text{ otherwise.}}}$$

$$= \sum_{n \in \mathbb{Z}} n^2\, a_n(t)\, \overline{a_n(t)}$$

Thus, the linear Schrödinger flow is just the jacked up example

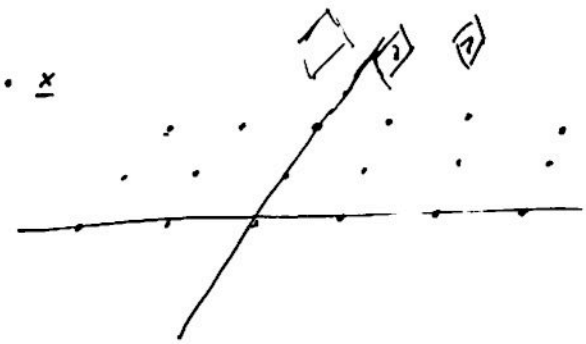


We have infinitely many decoupled Harmonic Oscillators.

We can generalize these constructions:

$$u : \mathbb{R}_t \times \underbrace{\left[S^1_{x_1} \times S^1_{x_2} \times \cdots \times S^1_{x_d} \right]}_{\mathbb{T}^d_{\underline{x}}} \longmapsto \mathbb{C}$$

$$u(t, \underline{x}) = \sum_{\underline{n} \in \mathbb{Z}^d} a_{\underline{n}}(t)\, e^{i \underline{n} \cdot \underline{x}}$$



$$u : \mathbb{R}_t \times \mathbb{R}^d_x \longmapsto \mathbb{C}.$$

$$u(t, \underline{x}) = \int \widehat{u(t)}(\underline{\xi})\, e^{i \underline{\xi} \cdot \underline{x}}\, d\underline{\xi}.$$

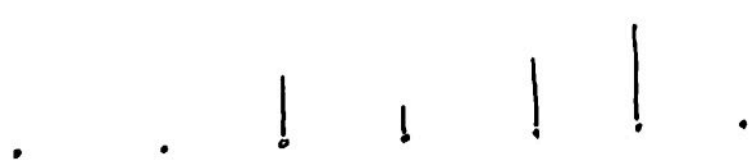
For the linear Schrödinger flow, we have

$$|a_{\underline{n}}(t)| = |a_{\underline{n}}(0)| \quad \text{for all time } t.$$

$$|\widehat{u(t)}(\xi)| = |\widehat{u(0)}(\xi)| \quad \text{"}$$

This means that the spatial oscillation properties of $u(t,x)$ forever resembles those of $u(0,x)$.

$|a_n(t)|$ frozen forever in time.



Nonlinear Coupling

Many physical systems have wave phenomena described by the cubic nonlinear Schrödinger equation:

$$NLS_3^{\pm} \quad \begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u(0,x) = u_0(x). \end{cases} \iff \begin{cases} \dot{u} = -i H_{\bar{u}}. \\ u(0) = u_0 \end{cases}$$

This is also Hamiltonian:

$$H[u, \bar{u}] = \int |\nabla u(t)|^2 \pm \frac{1}{2} |u(t)|^4 \, dx$$

$$H_{\bar{u}} = -\Delta u \pm |u|^2 u. \qquad \|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

The $|u|^2 u$ nonlinearity couples the $a_n(t)$ dynamics. What happens? This question and other closely related questions has been the focus of much of my research to date.

Two infinite Dimensional phenomena.

Weak Turbulence $NLS_3^+(\mathbb{T}^2)$.

Singularity Formation $NLS_3^-(\mathbb{R}^2)$.

The NLS Initial Value Problem

[Joint work with **Keel, Staffilani, Takaoka and Tao**]
We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x,t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$
$$\text{Energy} = E(u) = \int (\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4)\, dx = E(0)$$

We want to understand the shape of $|\hat{u}(t,\xi)|$. The conservation laws impose L^2-moment constraints on this object.

Notion of Weak Turbulence

Definition

Weak turbulence is the phenomenon of global-in-time defocusing solutions shifting their mass toward increasingly high frequencies.

This shift is also called a forward cascade.

- A way to measure weak turbulence is to study

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,\xi)|^2 |\xi|^{2s} d\xi$$

  and prove that it grows for large times t.
- Turbulence is incompatible with scattering and integrability.

Past Results

- Bourgain: (late 90's)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove

$$\|u(t)\|_{H^s}^2 \leq C_s |t|^{4s}.$$

  The idea is to improve the local estimate for $t \in [-1, 1]$

$$\|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1$$

  ($\Longrightarrow$ $\|u(t)\|_{H^s} \lesssim C^{|t|}$ upper bounds) to obtain

$$\|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,$$

  for some $\delta > 0$. This iterates to give

$$\|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}.$$

- Improvements: Staffilani, Colliander-Delort-Kenig-Staffilani.

Past Results

- Bourgain: (late 90's)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x,t)$ to the modified wave equation
  $$(\partial_{tt} - \tilde{\Delta})u = u^p$$
  such that $\|u(t)\|_{H^s} \sim |t|^m$.
- Physics: Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ DO NOT exhibit weak turbulence. $\exists$ solutions to dispersive equations on $\mathbb{T}^d$ that exhibit weak turbulence. In particular for $NLS(\mathbb{T}^2)$ there exists $u(x,t)$ s. t.

$$\|u(t)\|_{H^s}^2 \to \infty \text{ as } t \to \infty.$$

Main Result

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x,t)$ and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$\|u(T)\|_{H^s}^2 \geq K.$$

Preliminary reductions

- **Gauge Freedom:**
  If u solves NLS then $v(t,x) = e^{-i2Gt}u(t,x)$ solves

$$\begin{cases} i\partial_t v + \Delta v = (2G + |v|^2)v \\ v(0,x) = v_0(x), \qquad x \in \mathbb{T}^2. \end{cases} \tag{NLS_G}$$

- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)}.$$

$$\begin{cases} i\partial_t a_n = 2Ga_n + \displaystyle\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\ a_n(0) = \widehat{u_0}(n), \qquad n \in \mathbb{Z}^2. \end{cases} \tag{$\mathcal{F}NLS_G$}$$

Resonant truncation

- *NLS* dynamic is recast as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a}_{n_2}a_{n_3}e^{i\omega_4 t}. \quad (\mathcal{F}NLS)$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.$$

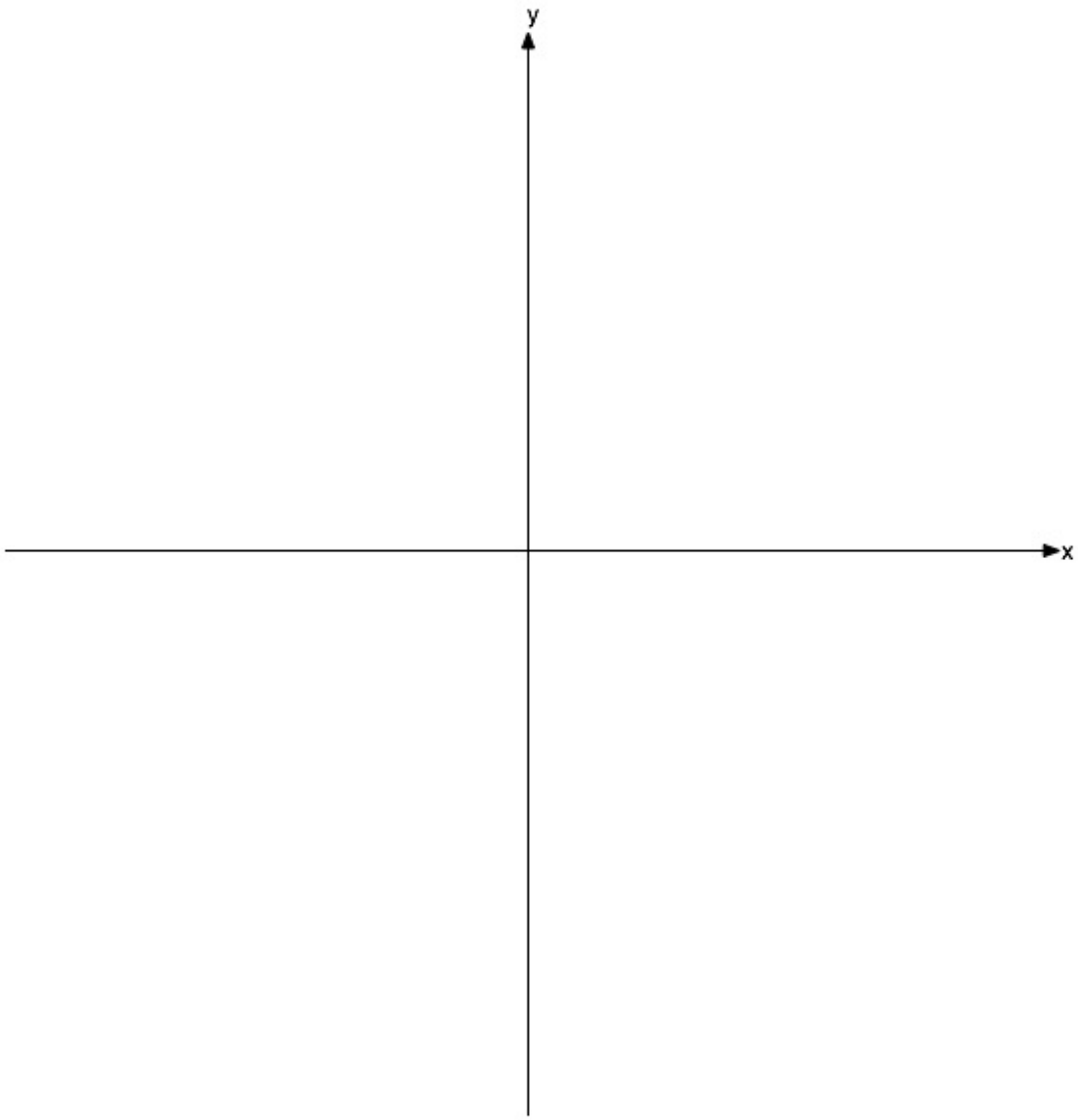
-

$$\begin{aligned}\Gamma_{res}(n) &= \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}. \\ &= \{ \text{ Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n_4) \text{ is a rectangle } \}\end{aligned}$$

- The *resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma_{res}(n)} b_{n_1}\overline{b}_{n_2}b_{n_3}. \quad (R\mathcal{F}NLS)$$

Cartoon Construction of Λ



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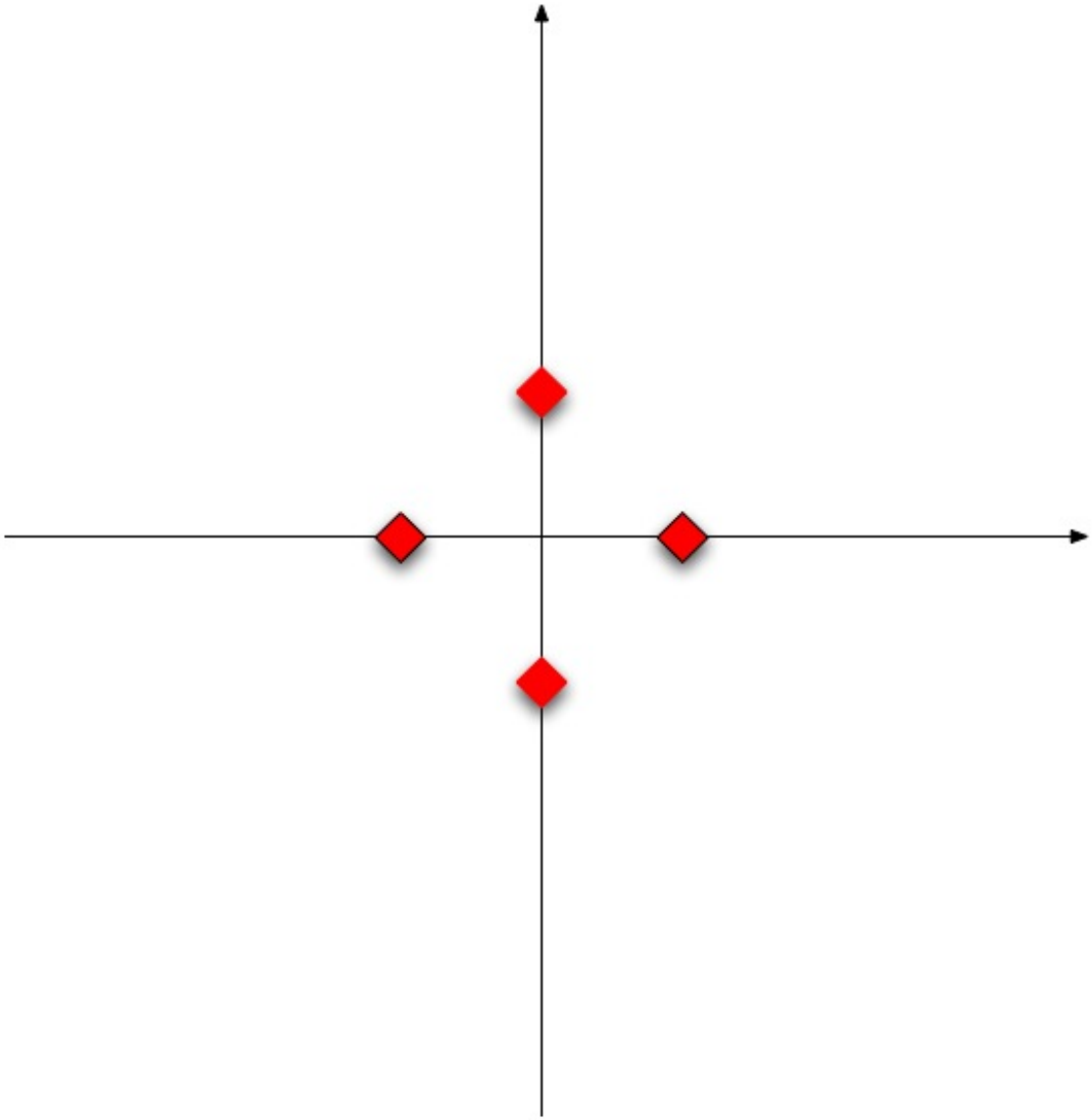

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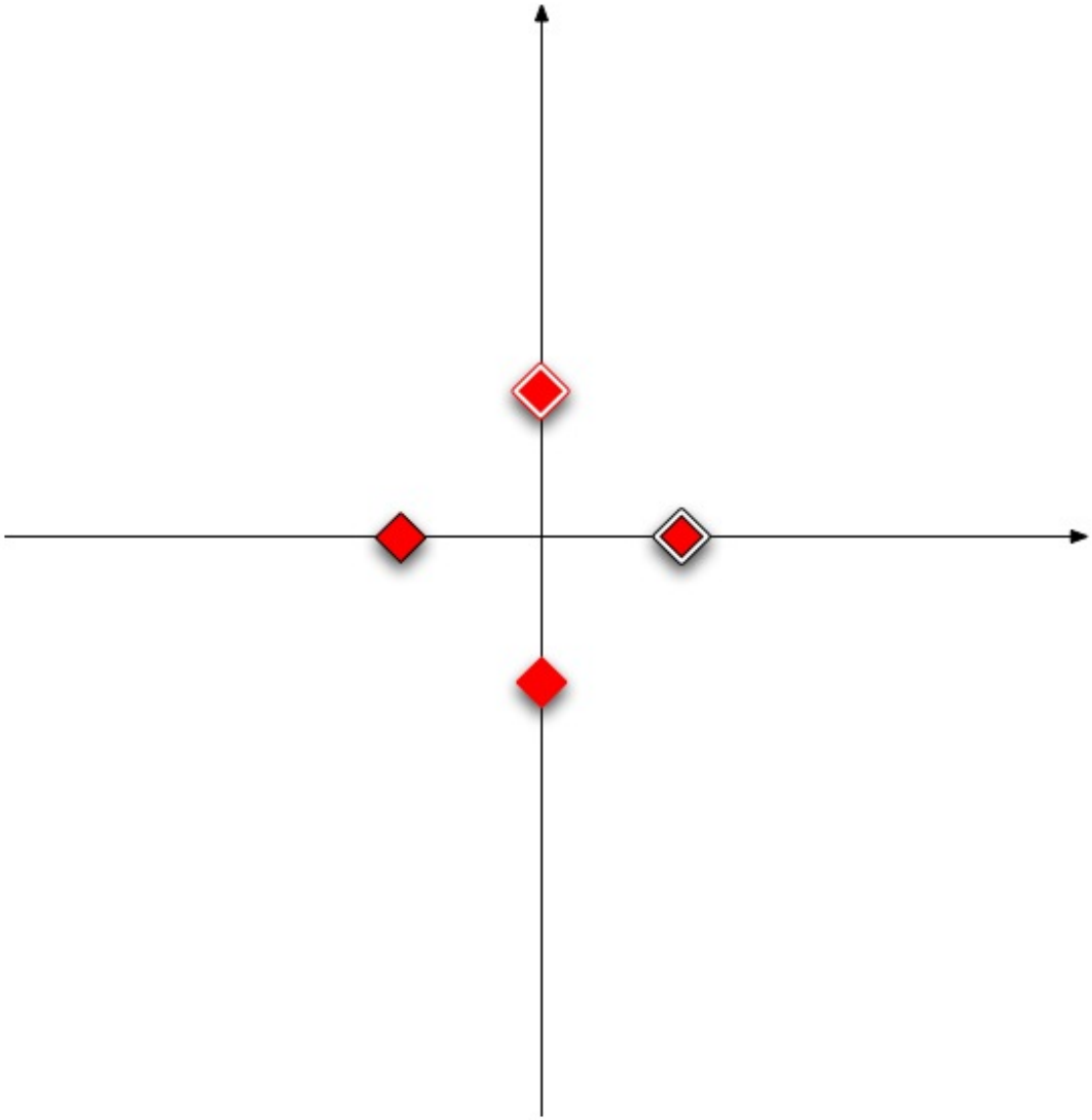

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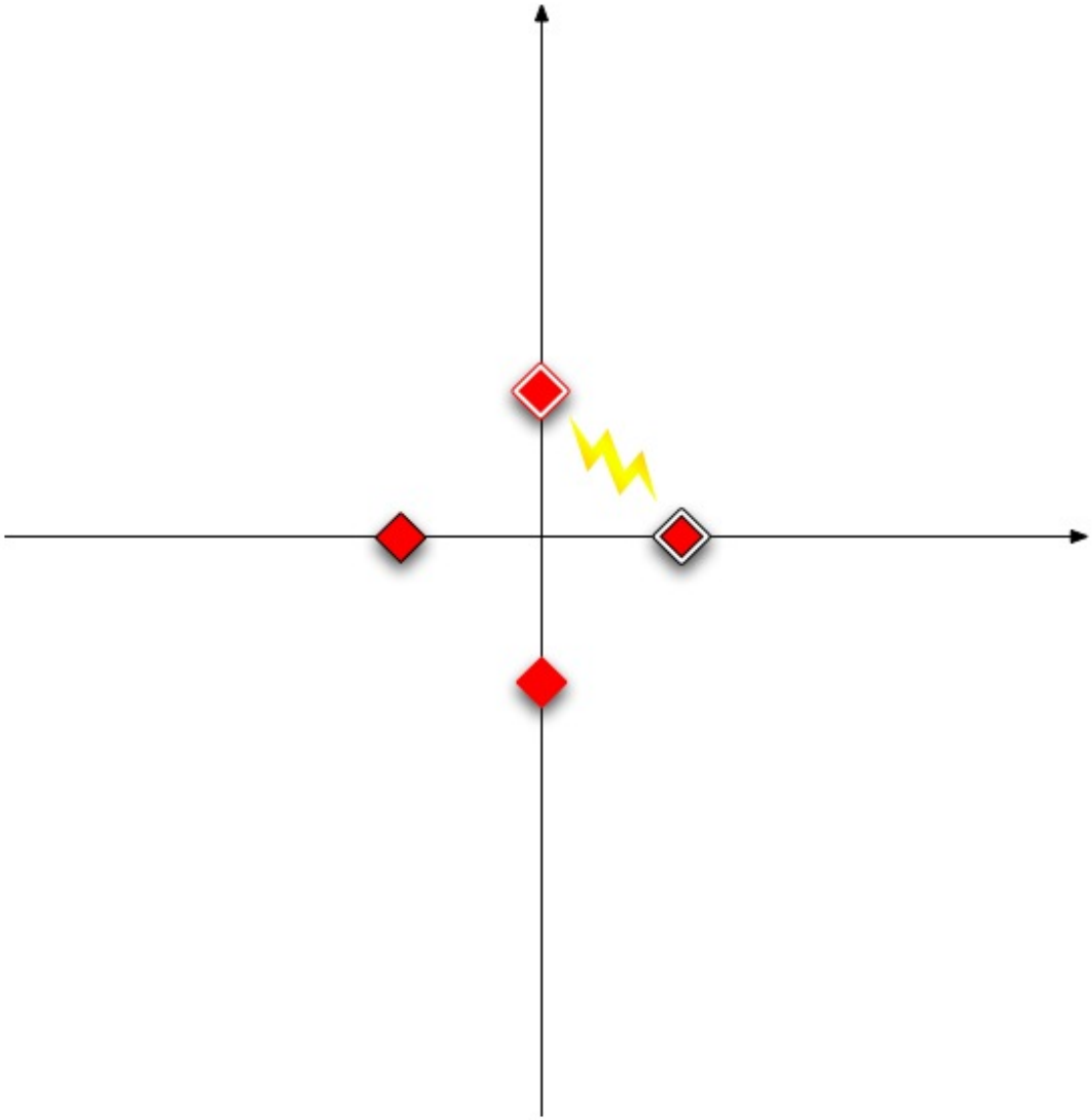

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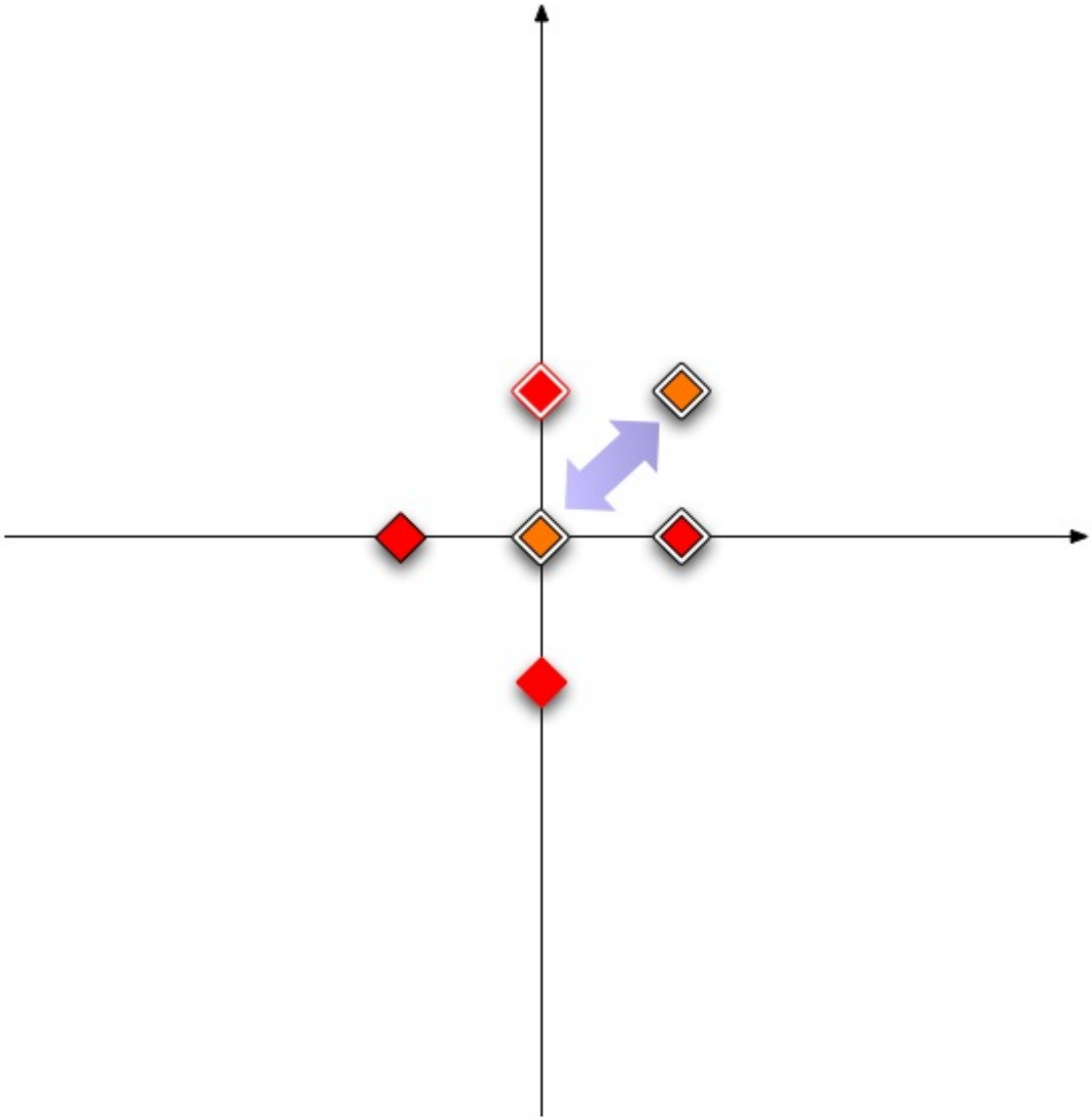

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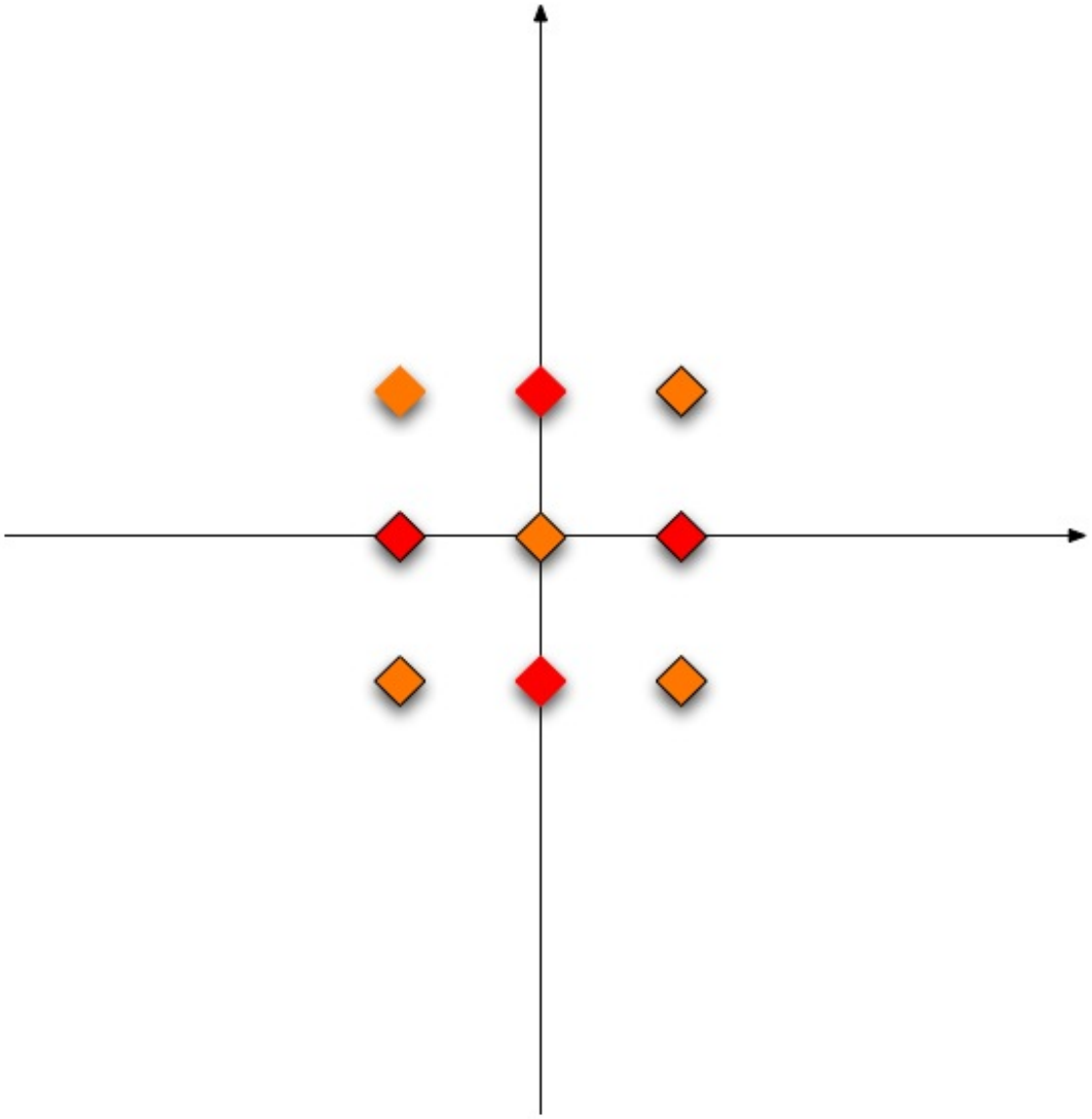

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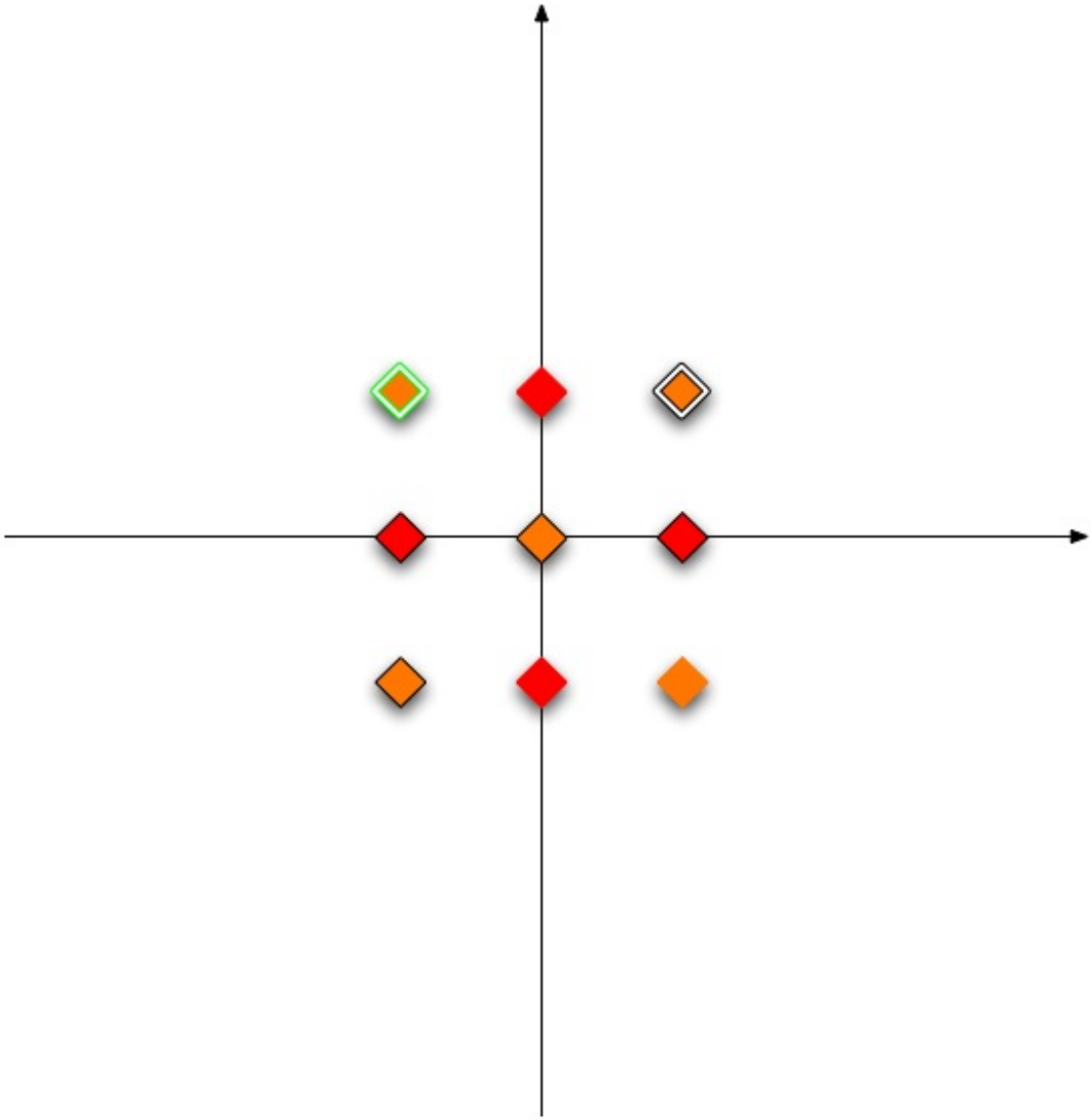

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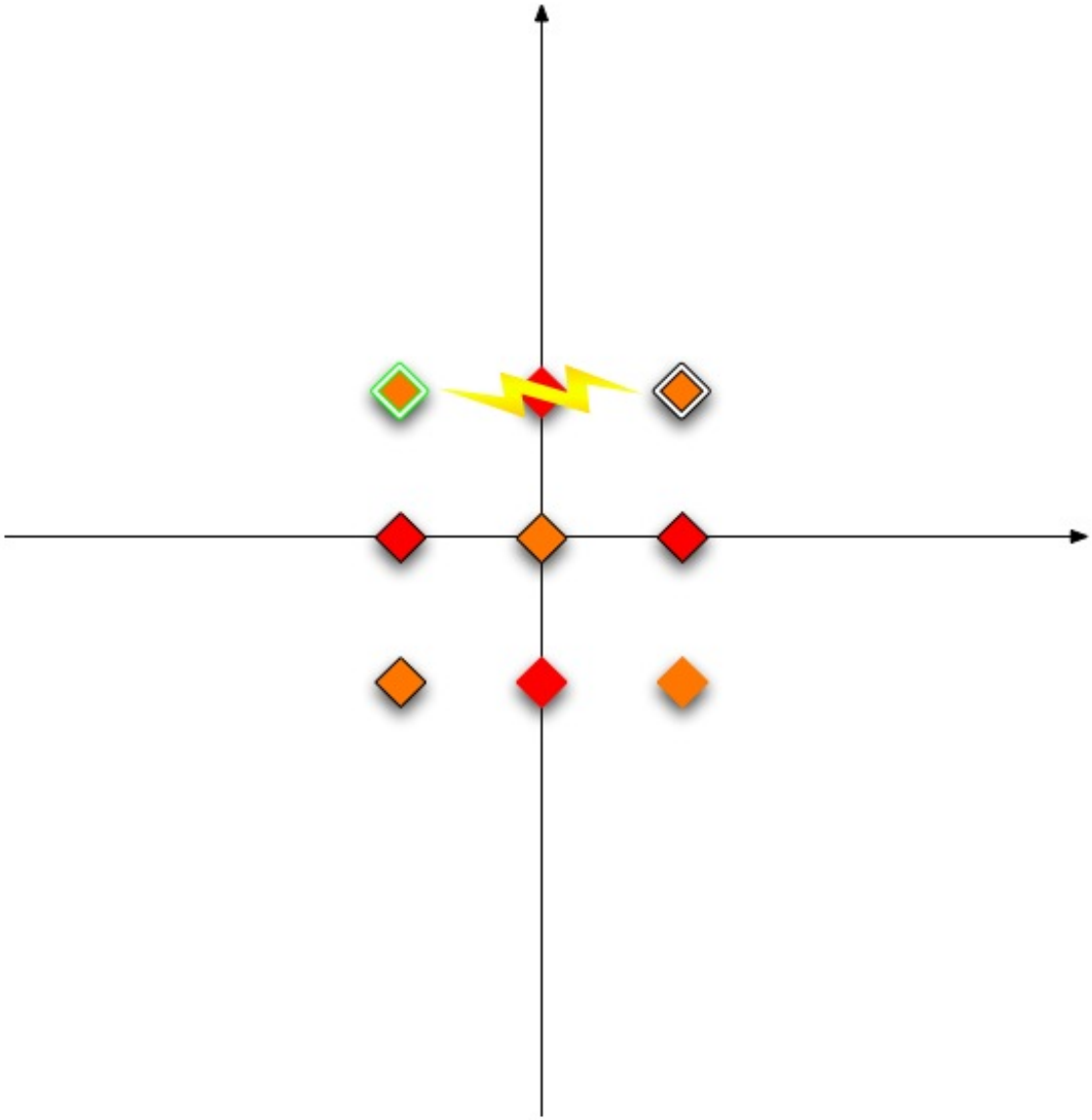

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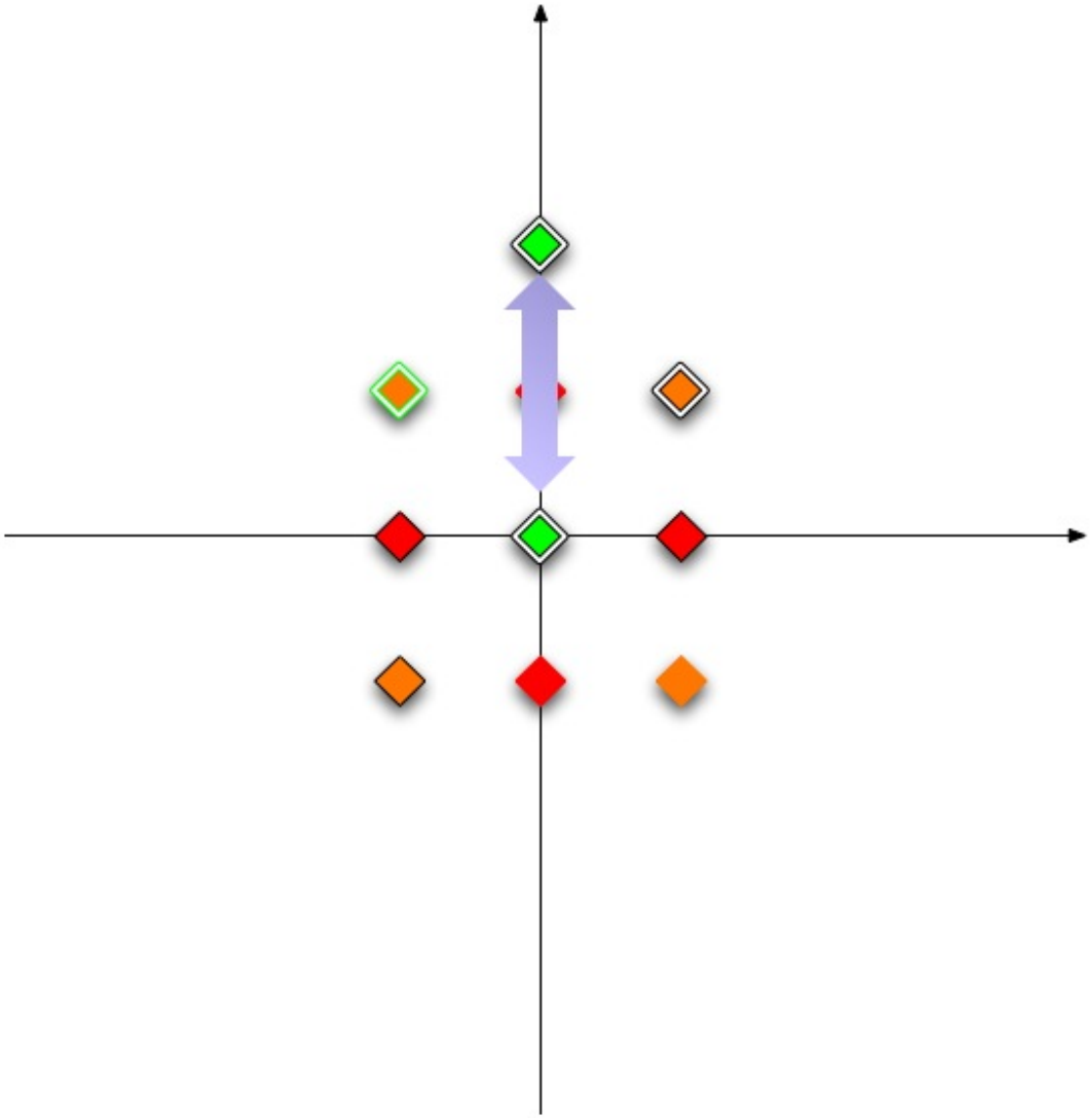

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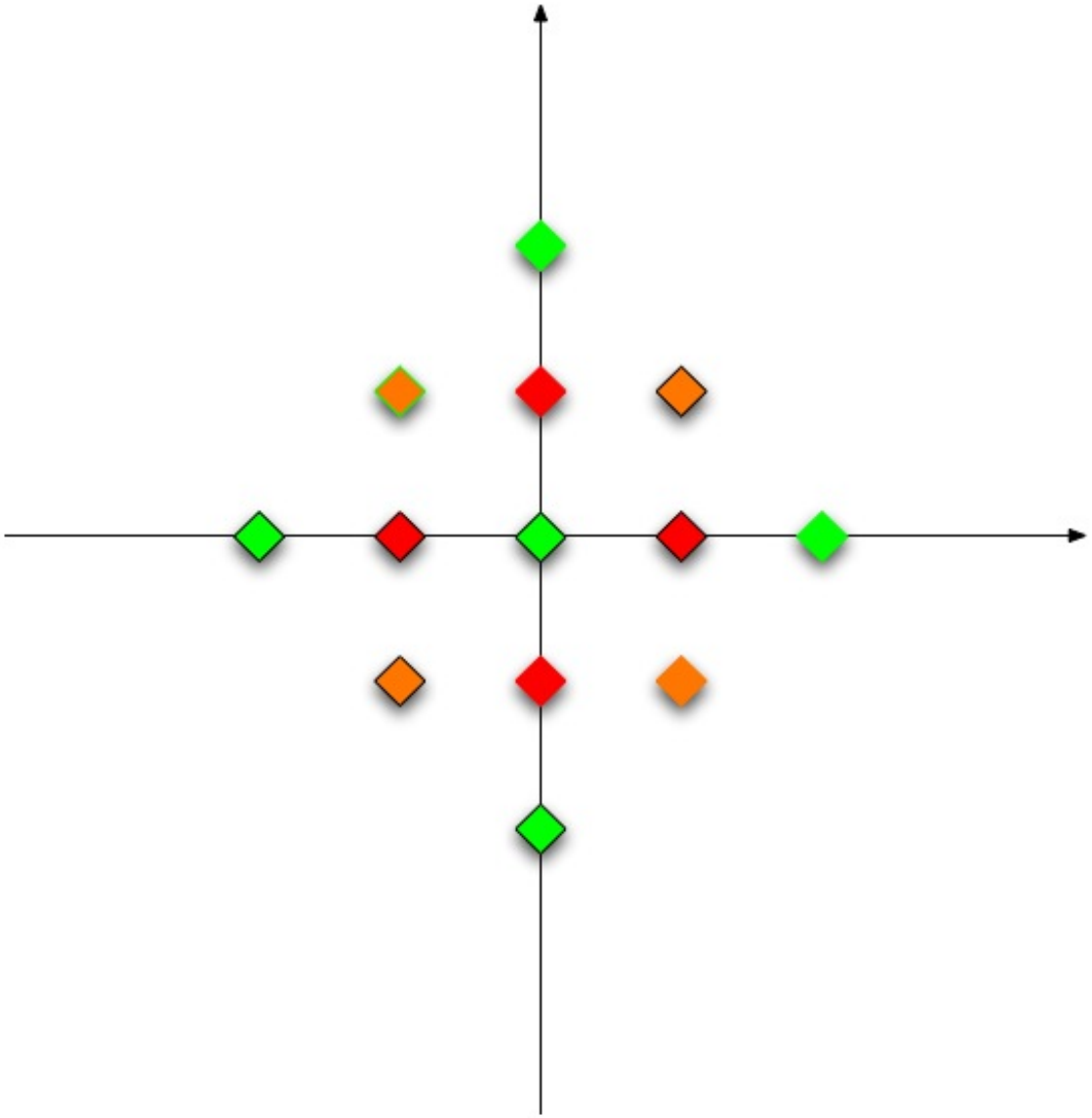

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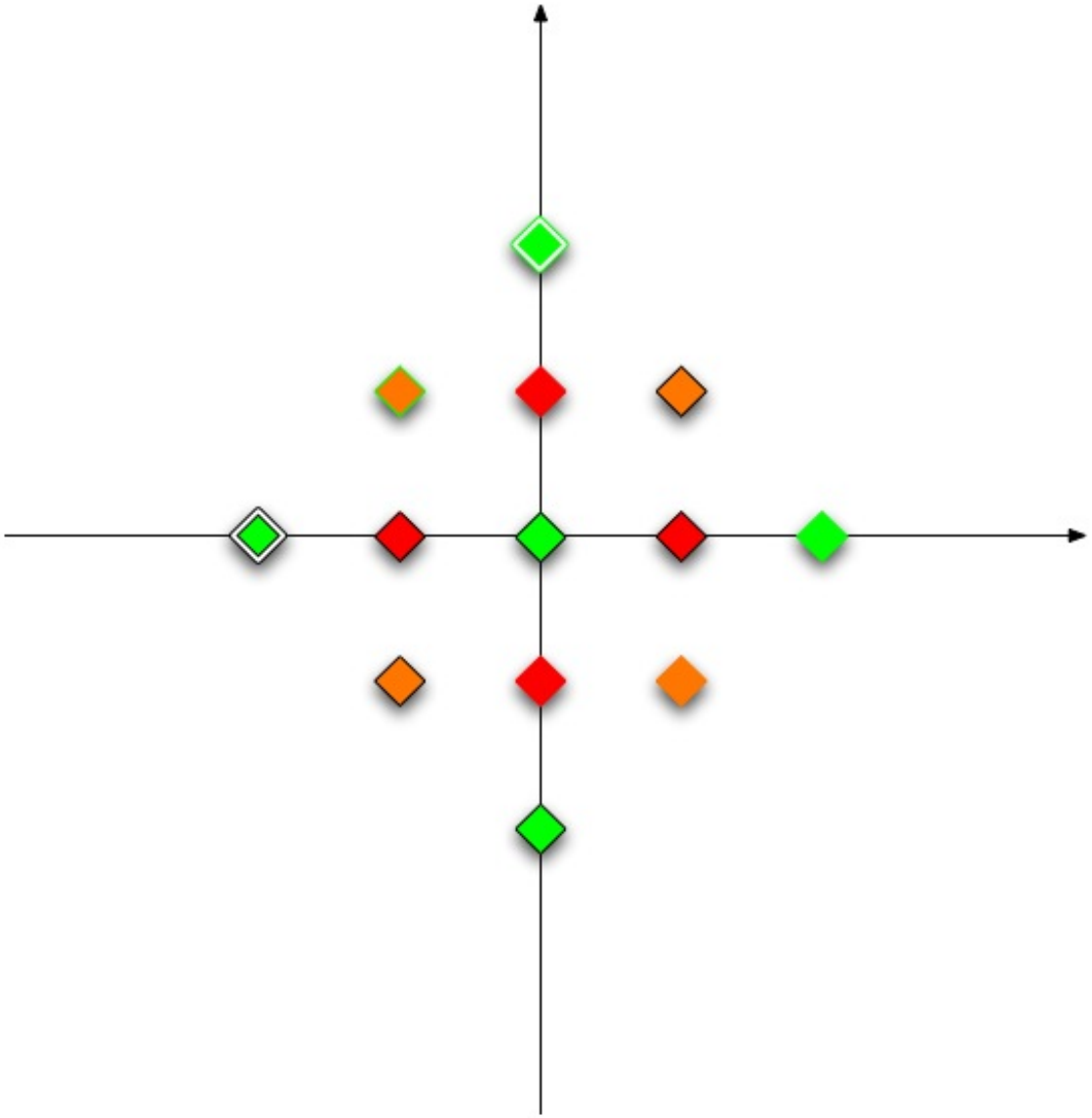

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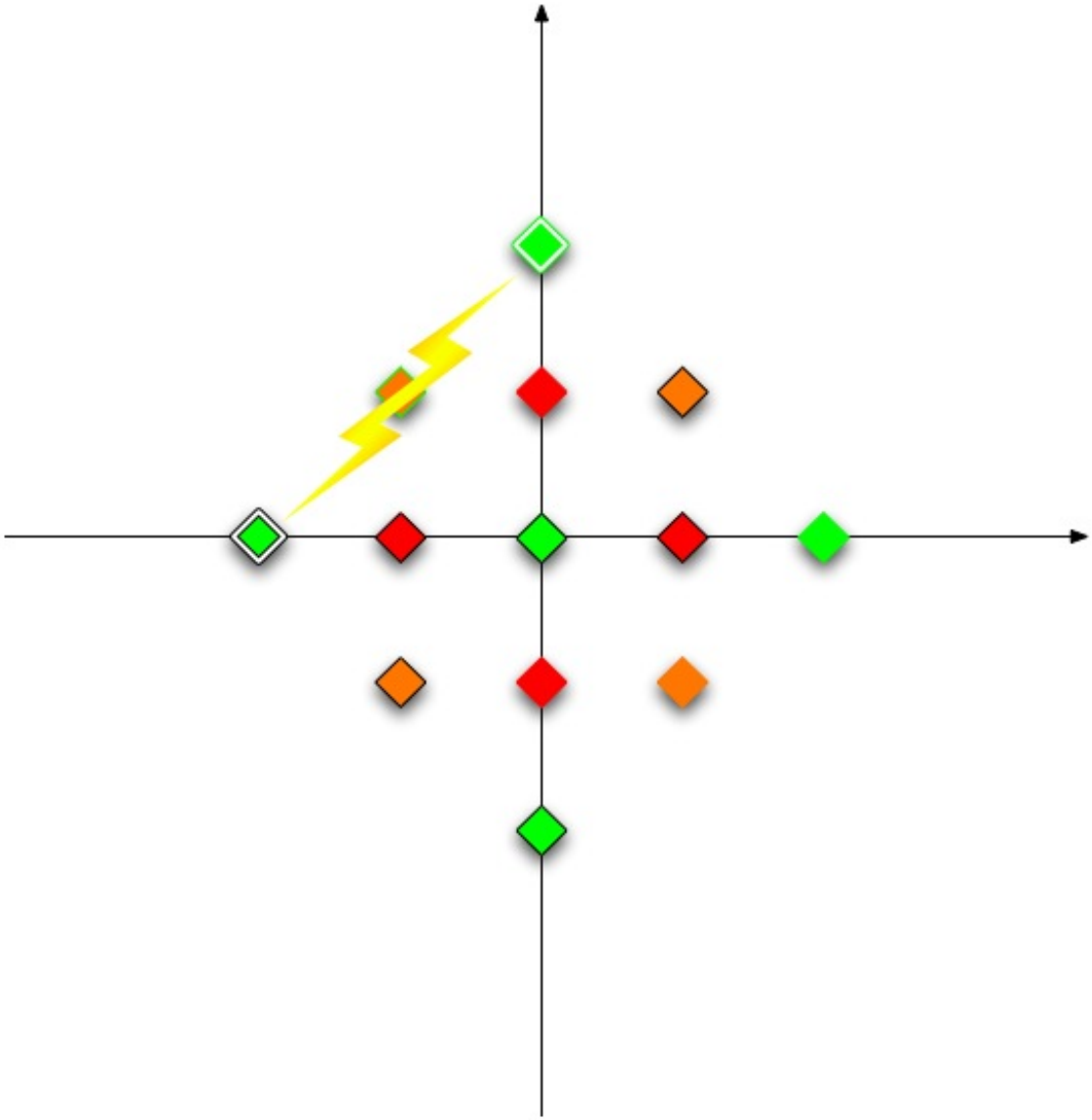

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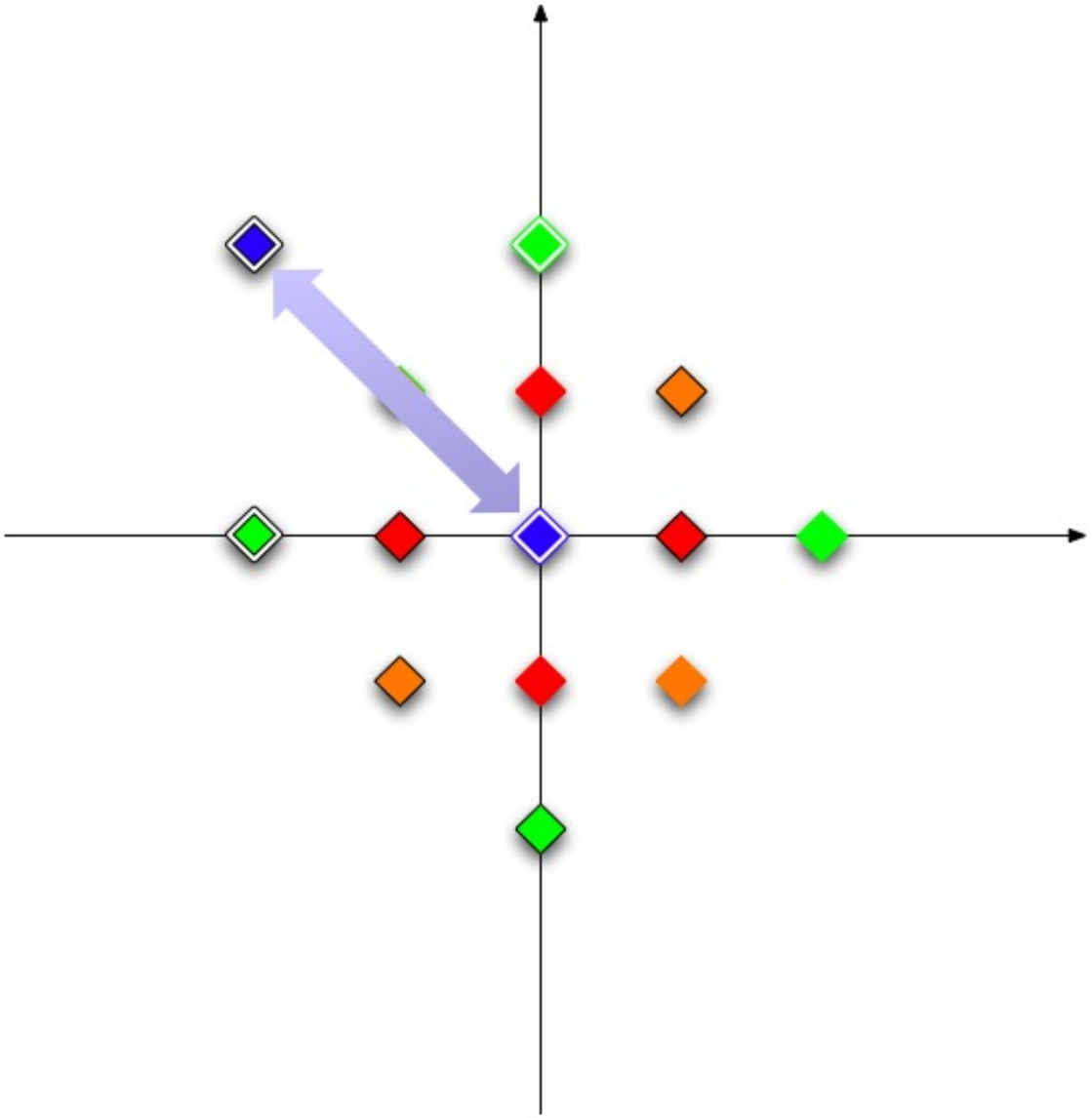

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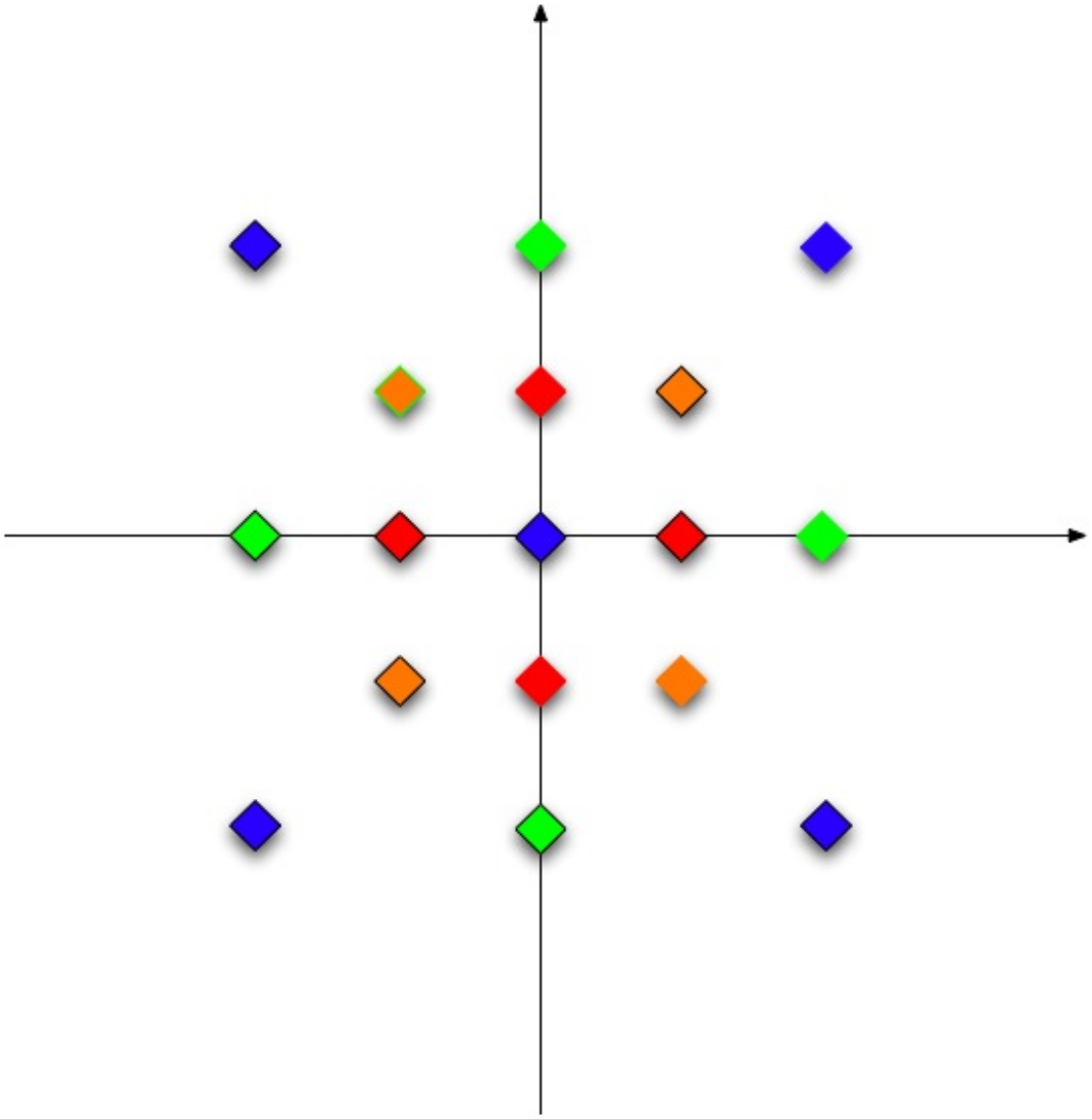

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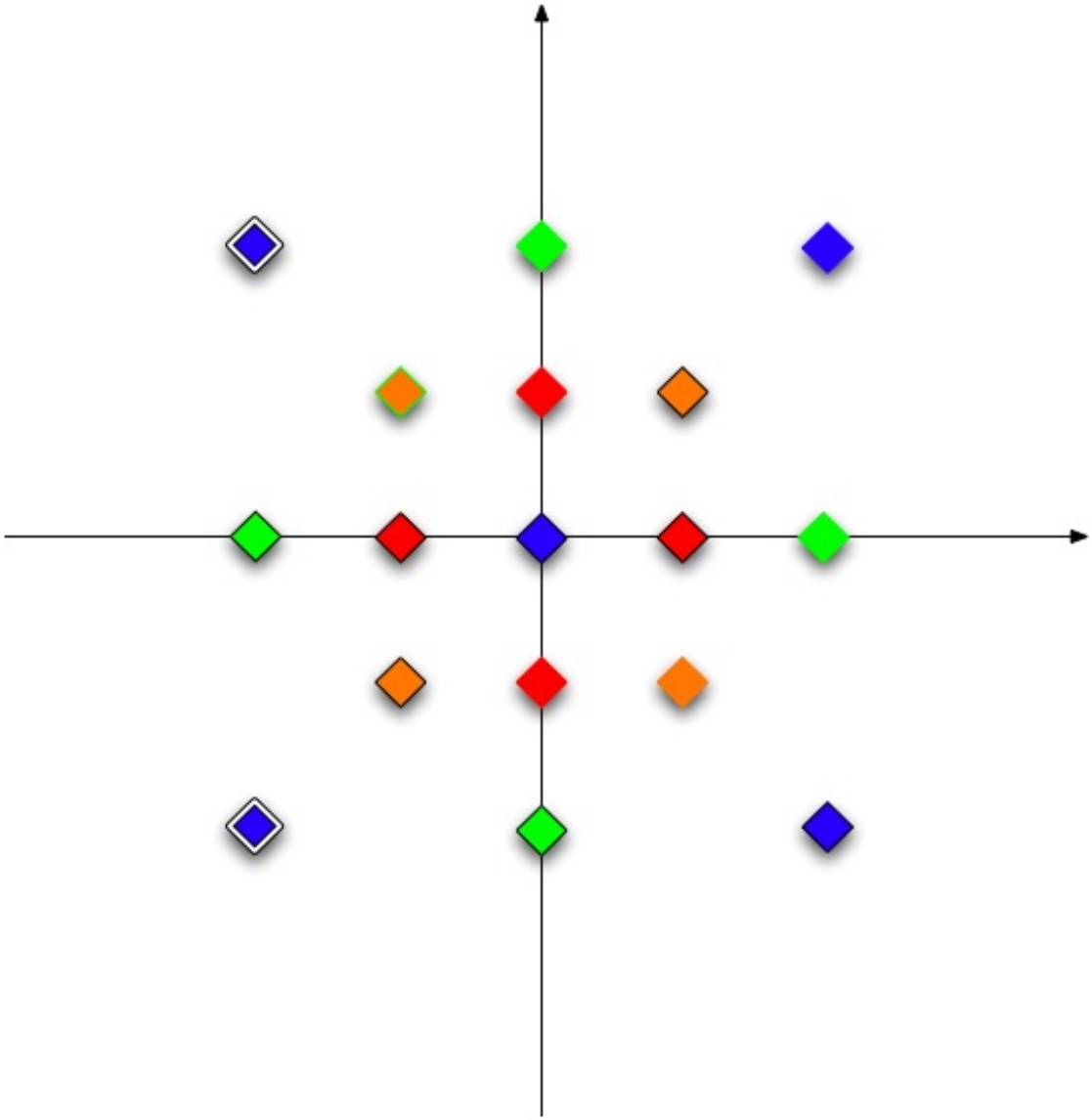

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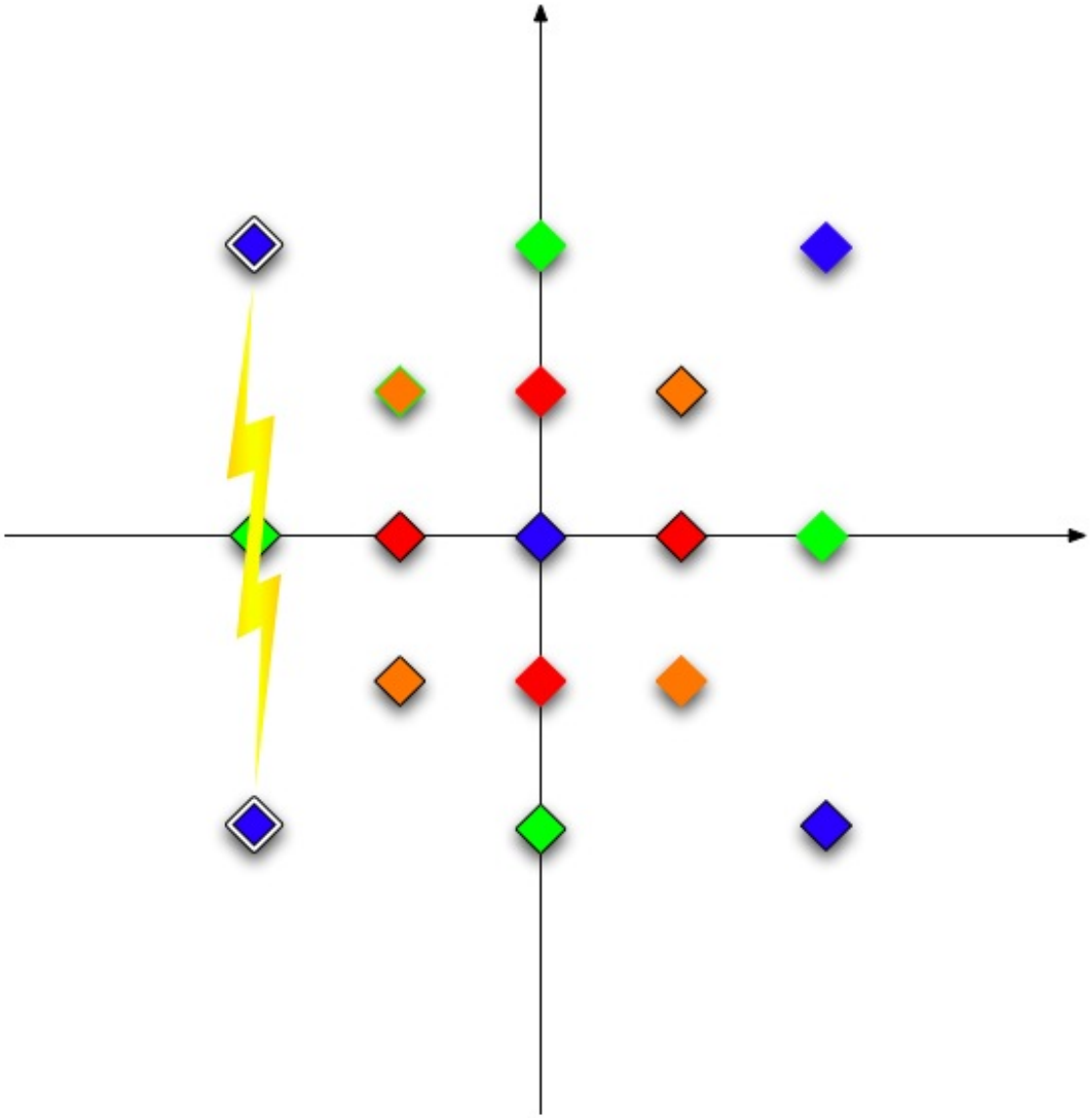


The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $R\mathcal{F}NLS_\Lambda$ simplifies to the toy model ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\overline{b}_j(t)[b_j(t)^2 - b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 (\sum_{n\in\Lambda_j} |n|^{2s}).$$

We also want Λ to satisfy Wide Diaspora Property

$$\sum_{n\in\Lambda_M} |n|^{2s} \gg \sum_{n\in\Lambda_1} |n|^{2s}.$$

Properties of the Toy Model *ODE*

- Solution of the Toy Model is a vector flow $t \to b(t) \in \mathbb{C}^M$

$$b(t) = (b_1(t), \ldots, b_M(t)) \in \mathbb{C}^M; b_j = 0 \ \forall \ j \leq 0, j \geq M+1.$$

- Local Well-Posedness; Let $S(t)$ denote associated flowmap.
- Mass Conservation: $|b(t)|^2 = |b(0)|^2 \implies$
  - Toy Model ODE is Globally Well-Posed.
  - Invariance of the sphere: $\Sigma = \{x \in \mathbb{C}^M : |x|^2 = 1\}$

$$S(t)\Sigma = \Sigma.$$

Properties of the Toy Model *ODE*

- Support Conservation:

$$\begin{aligned}\partial_t |b_j|^2 &= 2Re(\overline{b_j}\partial_t b_j) \\ &= 4Re(i\overline{b_j}^2[b_{j-1}^2 - b_{j+1}^2]) \\ &\leq C|b_j|^2.\end{aligned}$$

  Thus, if $b_j(0) = 0$ then $b_j(t) = 0$ for all t.

- Invariance of coordinate tori:

$$\mathbb{T}_j = \{(b_1, \ldots, b_M \in \Sigma) : |b_j| = 1, b_k = 0 \ \forall \ k \neq j\}$$

  Mass Conservation $\implies S(T)\mathbb{T}_j = \mathbb{T}_j$.
  Dynamics on the invariant tori is easy:

$$b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \ \forall \ k \neq j.$$

Explicit Slider Solution

Consider $M = 2$. Then *ODE* is of the form

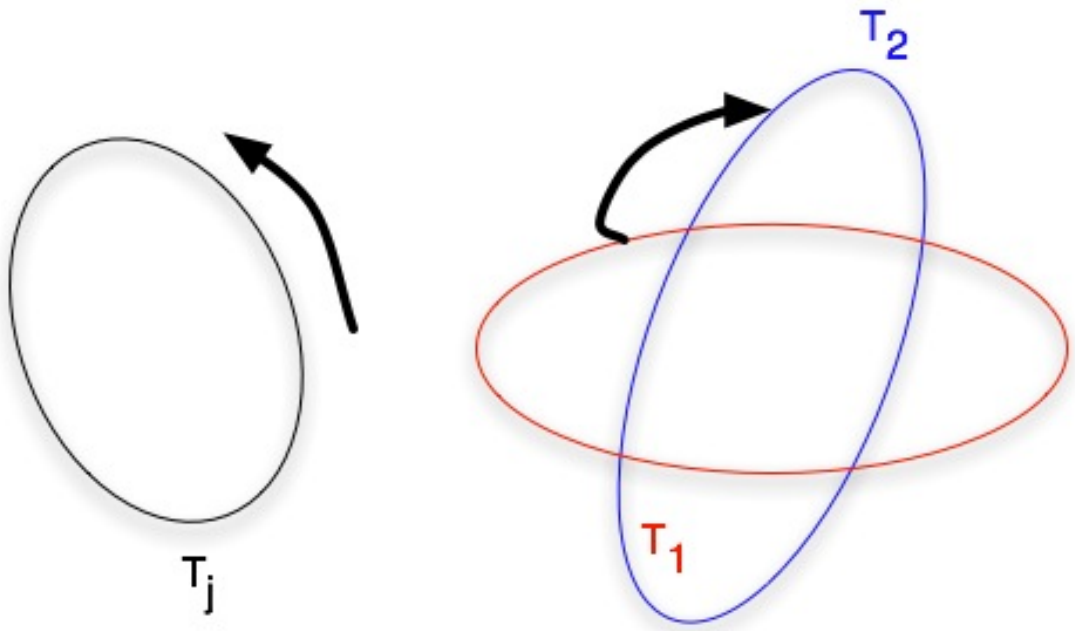
$$\begin{aligned} \partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1} b_2^2 \\ \partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2} b_1^2. \end{aligned}$$

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

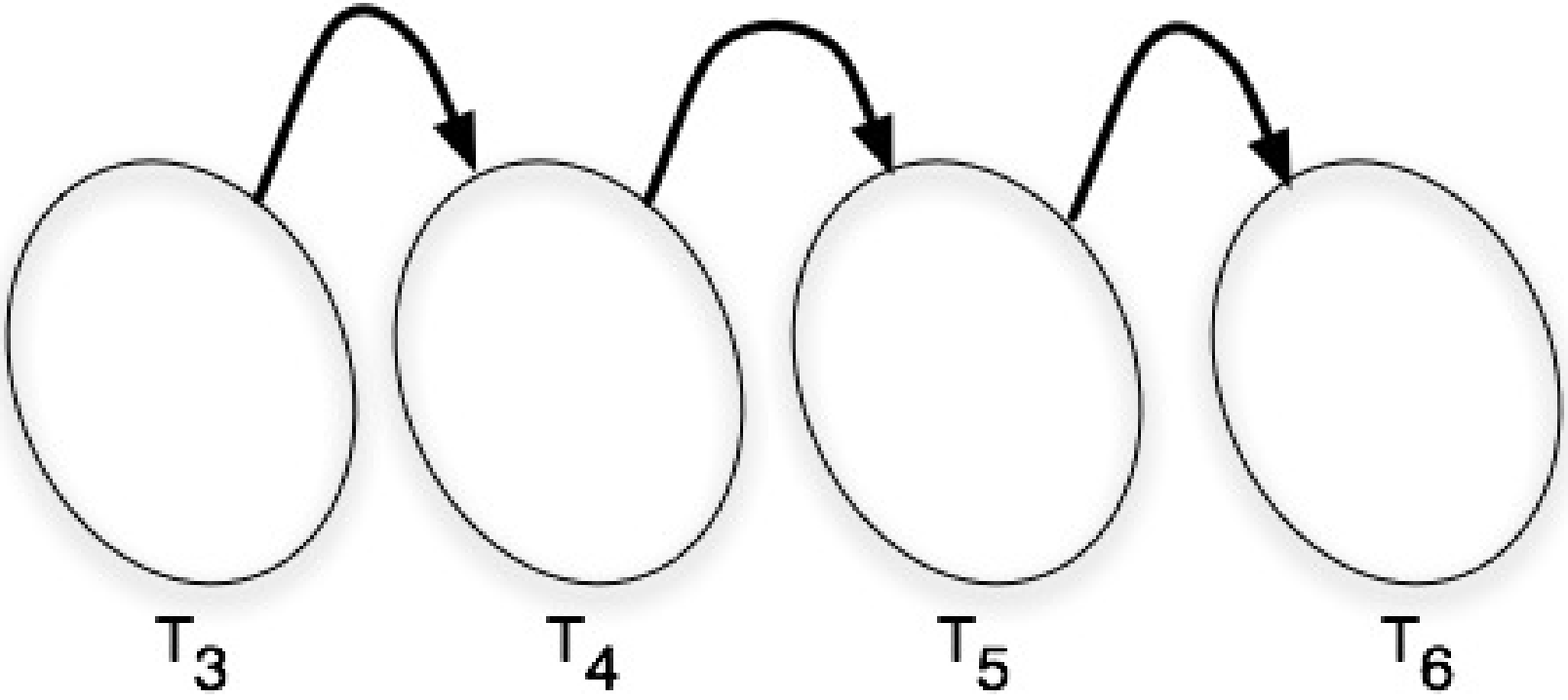
$$b_1(t) = \frac{e^{-it}}{\sqrt{1+e^{2\sqrt{3}t}}}\omega \ \ , b_2(t) = \frac{e^{-it}}{\sqrt{1+e^{-2\sqrt{3}t}}}\omega^2.$$

- As $t \to -\infty, (b_1(t), b_2(t)) \to (e^{-it}\omega, 0) \in \mathbb{T}_1$.
- As $t \to +\infty, (b_1(t), b_2(t)) \to (0, e^{-it}\omega^2) \in \mathbb{T}_2$.

Two Explicit Solution Families



Concatenated Sliders: Idea of Proof



Arnold Diffusion for Toy Model Statement

Theorem

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time t such that

$$S(t)x_3 = x_{M-2}.$$

Remark

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time t_0 and then arbitrarily concentrated at mode $j = M - 2$ at later time t.