Weak Turbulence for a 2D periodic Schrödinger equation

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1. Introduction

The NLS Initial Value Problem

[Joint work with **Keel, Staffilani, Takaoka and Tao**]
We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \text{where} \ x \in \mathbb{T}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x,t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$

$$\text{Energy} = E(u) = \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4 \, dx = E(0)$$

We want to understand the shape of $|\hat{u}(t,\xi)|$. The conservation laws impose L^2-moment constraints on this object.

Notion of Weak Turbulence

Definition

Weak turbulence is the phenomenon of global-in-time defocusing solutions shifting their mass toward increasingly high frequencies.

This shift is also called a forward cascade.

- A way to measure weak turbulence is to study

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,\xi)|^2 |\xi|^{2s} d\xi$$

  and prove that it grows for large times t.
- Turbulence is incompatible with scattering and integrability.
- Finite time blowup behavior is not weak turbulence.

Incompatible with Scattering & Integrability

- Scattering: $\forall$ global solution $u(t,x) \in H^s$ $\exists$ $u_0^+ \in H^s$ such that,
$$\lim_{t \to +\infty} \|u(t,x) - e^{it\Delta} u_0^+(x)\|_{H^s} = 0.$$
Note: $\|e^{it\Delta} u_0^+\|_{H^s} = \|u_0^+\|_{H^s} \implies \|u(t)\|_{H^s}$ is bounded.
Proofs rely on Morawetz-type (global dispersive) estimate.

- Complete Integrability: The 1d equation
$$(i\partial_t + \partial_x^2)u = |u|^2 u$$
has infinitely many conservation laws. Combining them in the right way one gets that $\|u(t)\|_{H^s} \le C_s$ for all times.

Distinctions from finite time blowup setting

- Glassey's virial identity shows corresponding focusing problem

$$\begin{cases} (-i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \ \text{where} \ \ x \in \mathbb{R}^2. \end{cases} \qquad (NLS^-(\mathbb{R}^2))$$

  has many finite time blowup solutions.

- The associated energy has a changed sign:

$$E(u) = \int \frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{4}|u(x,t)|^4 \, dx.$$

- Blowup solutions explode in H^1 in finite time.

Bourgain's Glassey versus Morawetz Example

Bourgain's Glassey versus Morawetz Example

Bourgain's Glassey versus Morawetz Example

Bourgain constructed a coupled NLS system with Energy

$$\begin{aligned} E(u,v) &= \int_{\mathbb{R}} |\partial_x u|^2 - \frac{1}{3}|u|^6 dx + \int_{\mathbb{R}^3} |\nabla v|^2 dy \\ &+ \left(\int (1+|u|^2)^3 |u|^2 dx \right) \left(\int |v|^4 e^{-|y|^2} dy \right). \end{aligned}$$

- If $(u(0), v(0)) \in H^s \times H^s,\ s \geq 1$ and $v(0) \neq 0$ there is a global solution $(u(t), v(t)) \in H^s \times H^s$.
- However, if $\int |u(0)|^2 x^2 dx < \infty$ and $E(u(0), v(0)) < \infty$ then

$$\limsup_{t \to \infty} (\|u(t)\|_{H^1} + \|v(t)\|_{H^1}) = \infty.$$

Past Results (defocusing case)

- Bourgain: (late 90's)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove
  $$\|u(t)\|_{H^s}^2 \leq C_s|t|^{4s}.$$
  The idea is to improve the local estimate for $t \in [-1, 1]$
  $$\|u(t)\|_{H^s} \leq C_s\|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1$$
  ($\implies \|u(t)\|_{H^s} \lesssim C^{|t|}$ upper bounds) to obtain
  $$\|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s\|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,$$
  for some $\delta > 0$. This iterates to give
  $$\|u(t)\|_{H^s} \leq C_s|t|^{1/\delta}.$$
- Improvements: Staffilani, Colliander-Delort-Kenig-Staffilani.

Past Results

- Bourgain: (late 90's)
  Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x,t)$ to the modified wave equation
  $$(\partial_{tt} - \tilde{\Delta})u = u^p$$
  such that $\|u(t)\|_{H^s} \sim |t|^m$.
- Physics: Weak turbulence theory: Hasselmann & Zakharov.
  Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ DO NOT exhibit weak turbulence. $\exists$ solutions to dispersive equations on $\mathbb{T}^d$ that exhibit weak turbulence. In particular for NLS($\mathbb{T}^2$) there exists $u(x,t)$ s. t.

$$\|u(t)\|^2_{H^s} \to \infty \text{ as } t \to \infty.$$

Main Result

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

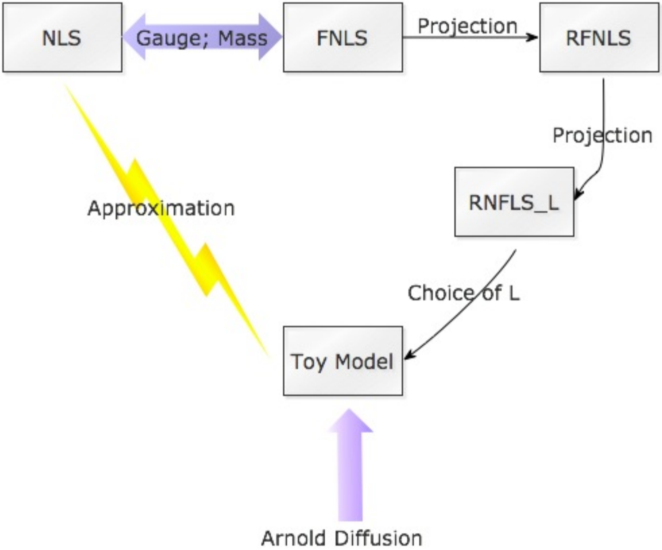
Let $s > 1$, $k \gg 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x,t)$ and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$\|u(t)\|_{H^s}^2 \geq K.$$

2. Overview of Proof



Preliminary reductions

- **Gauge Freedom:**
  If u solves NLS then $v(t,x) = e^{-i2Gt}u(t,x)$ solves

$$\begin{cases} i\partial_t v + \Delta v = (2G + |v|^2)v \\ \qquad v(0,x) = v_0(x), \qquad\qquad x \in \mathbb{T}^2. \end{cases} \qquad (NLS_G)$$

- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)}.$$

$$\begin{cases} i\partial_t a_n = 2Ga_n + \displaystyle\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\ \qquad a_n(0) = \widehat{u_0}(n), \qquad\qquad n \in \mathbb{Z}^2. \end{cases} \qquad (\mathcal{F}NLS_G)$$

Preliminary reductions

- **Diagonal decomposition of sum:**

$$\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} = \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n\\ n\neq n_1,n_3}} + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n\\ n=n_1}} + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n\\ n=n_3}} - \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n\\ n=n_1=n_3}}$$

- **Choice of G:**

$$G = -\|u_0\|_{L^2}^2.$$

Resonant truncation

- *NLS* dynamic is recast as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a}_{n_2}a_{n_3}e^{i\omega_4 t}. \quad (\mathcal{F}NLS)$$

  where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.$$

-

$$\begin{aligned}\Gamma_{res}(n) &= \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}.\\ &= \{\text{ Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n_4) \text{ is a rectangle } \}\end{aligned}$$

- The *resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma_{res}(n)} b_{n_1}\overline{b}_{n_2}b_{n_3}. \quad (R\mathcal{F}NLS)$$

Finite dimensional resonant truncation

- A set $\Lambda \subset \mathbb{Z}^2$ is *closed under resonant interactions* if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$

- A *finite dimensional resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma_{res}(n)\cap\Lambda^3} b_{n_1}\overline{b}_{n_2} b_{n_3}. \quad (R\mathcal{F}NLS_\Lambda)$$

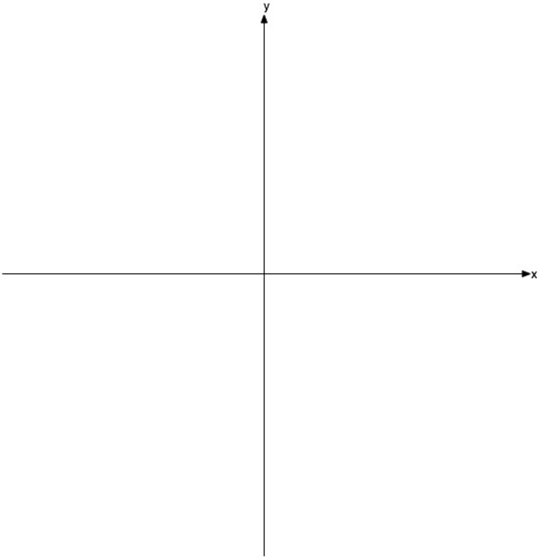
- $\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $R\mathcal{F}NLS_\Lambda$ is an ODE.
- If $\operatorname{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}NLS$-evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $R\mathcal{F}NLS_\Lambda$-ODE $a_n(0) \longmapsto b_n(t)$.
- Given ϵ, s, K, build Λ so that $R\mathcal{F}NLS_\Lambda$ has weak turbulence.

Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...

Imagine a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ with properties. Define a nuclear family to be a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the 'parents') live in generation Λ_j and n_2, n_4 ('children') live in generation Λ_{j+1}.

- $\forall\ 1 \leq j < M$ and $\forall\ n_1 \in \Lambda_j\ \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- $\forall\ 1 \leq j < M$ and $\forall\ n_2 \in \Lambda_{j+1}\ \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- The sibling of a frequency is never its spouse.
- Besides nuclear families, Λ contains no other rectangles.
- The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.

Cartoon Construction of Λ



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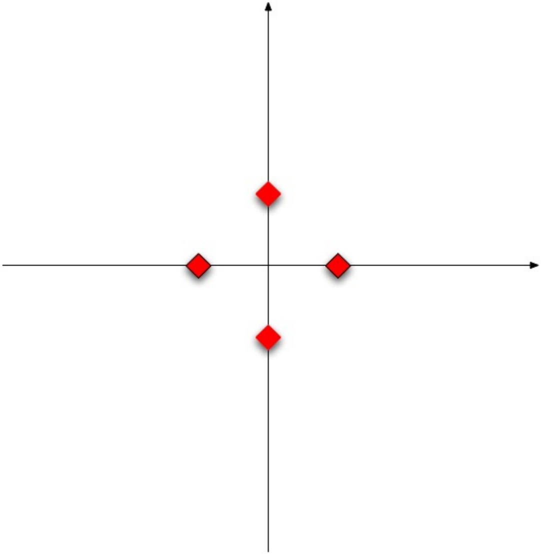

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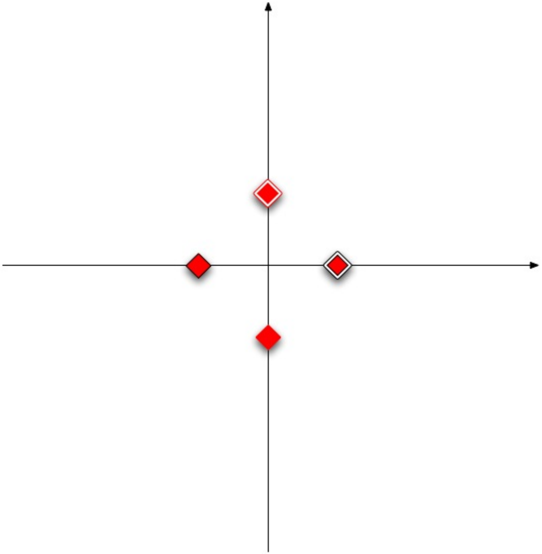

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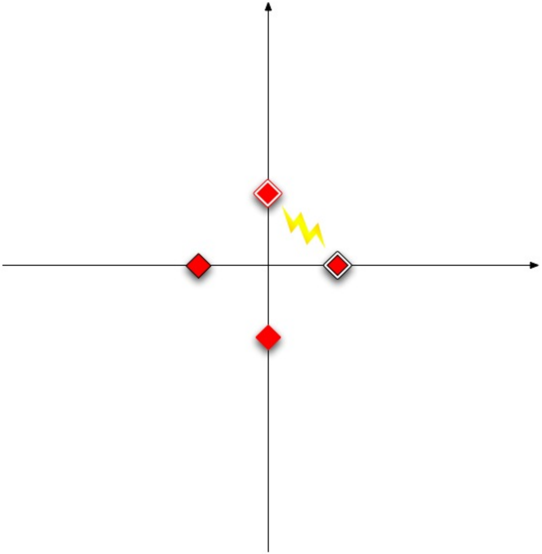

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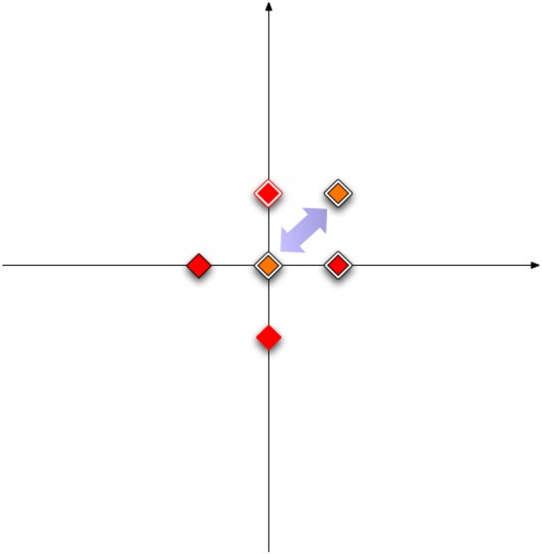

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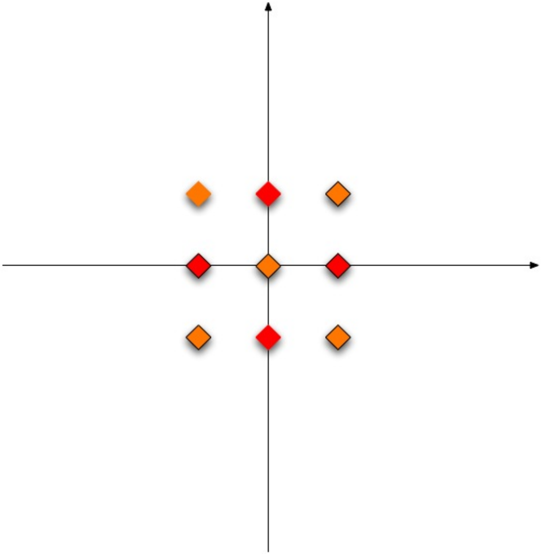

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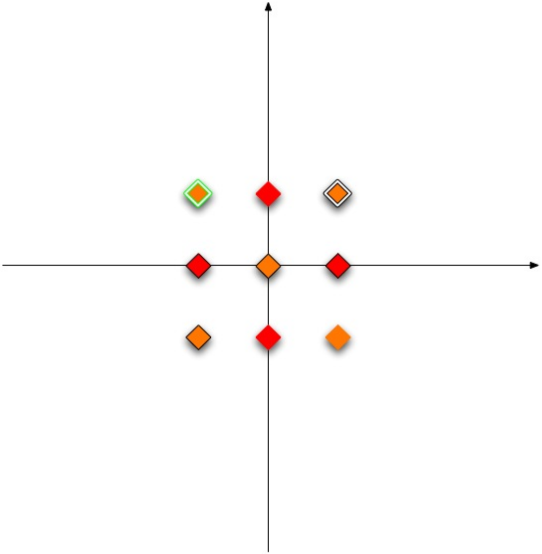

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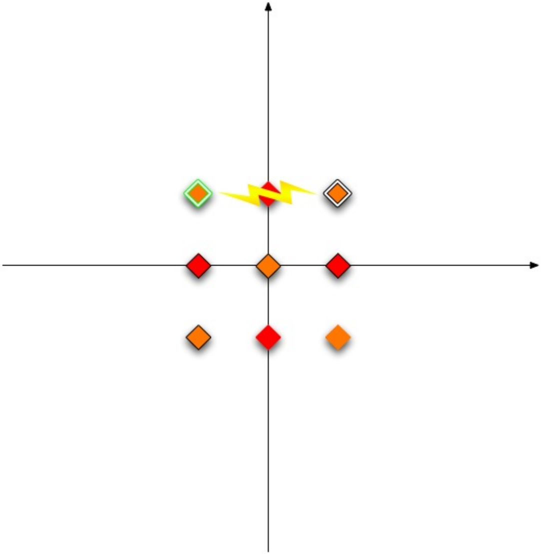

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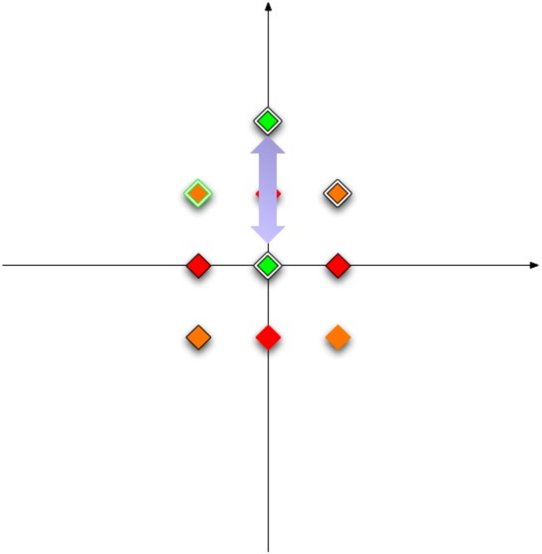

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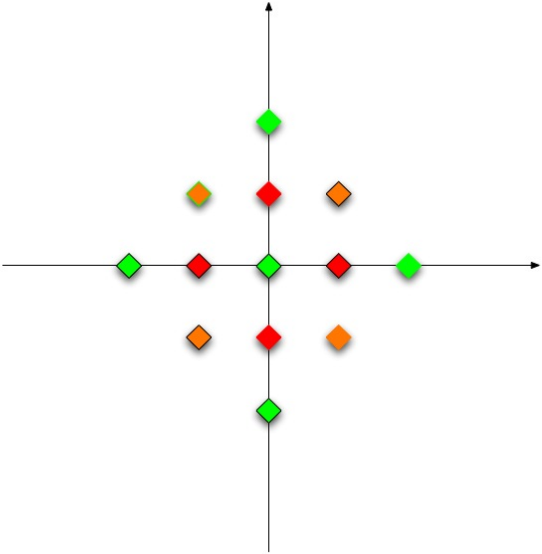

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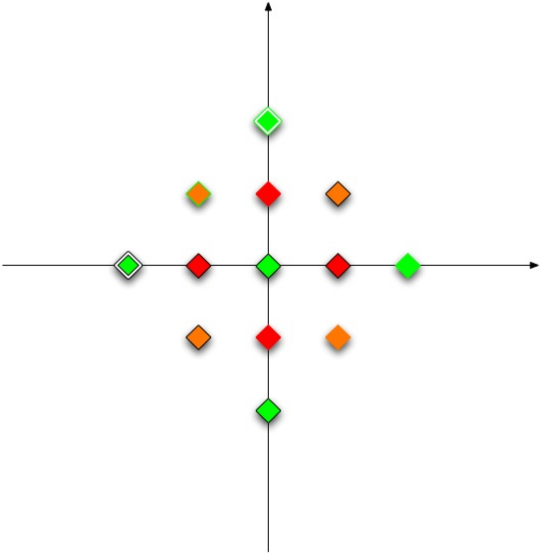

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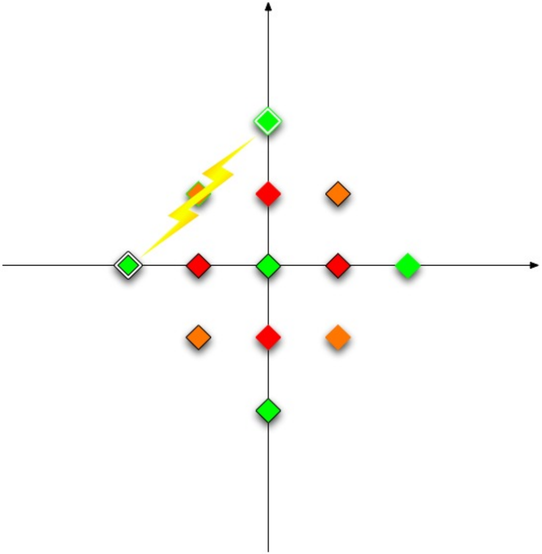

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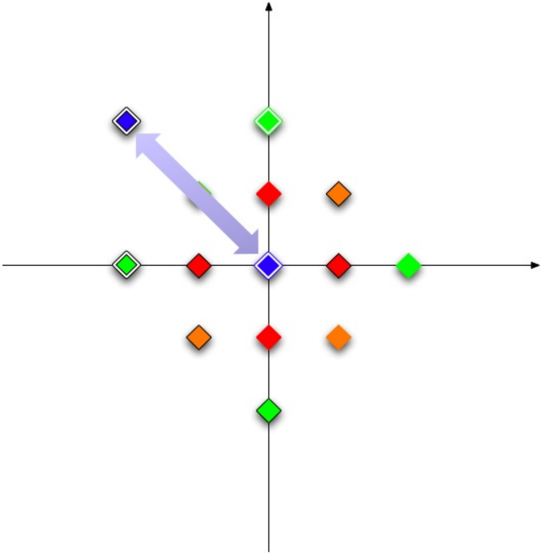

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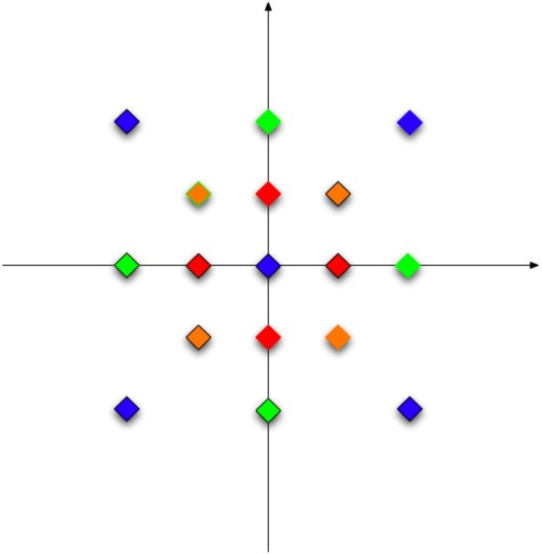

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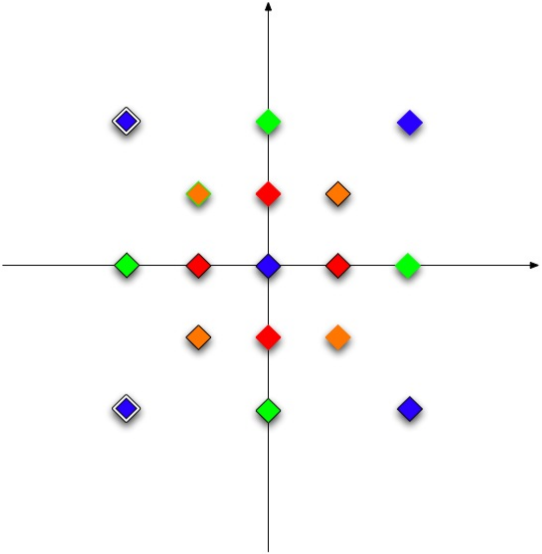

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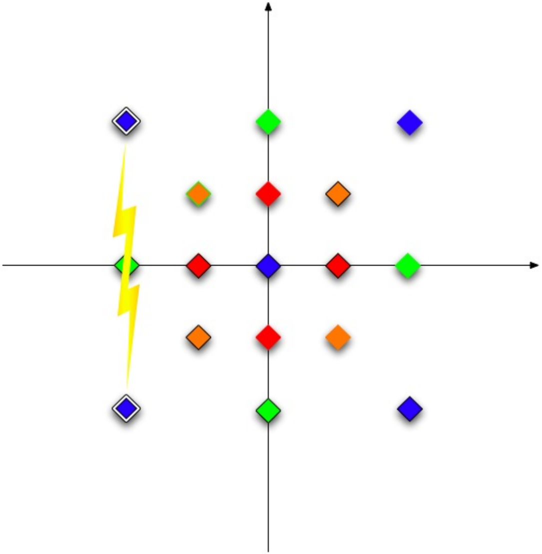


The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $R\mathcal{F}NLS_\Lambda$ simplifies to the toy model ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\bar{b}_j(t)[b_j(t)^2 - b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 (\sum_{n \in \Lambda_j} |n|^{2s}).$$

We also want Λ to satisfy Wide Diaspora Property

$$\sum_{n \in \Lambda_M} |n|^{2s} \gg \sum_{n \in \Lambda_1} |n|^{2s}.$$

Conservation laws for the *ODE* system

$$Mass = \sum_j |b_j(t)|^2 = C_0$$

$$Momentum = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} n = C_1,$$

$$Energy = K + P = C_2,$$

where

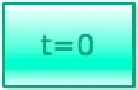
$$K = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} |n|^2,$$

$$P = \frac{1}{2}\sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2.$$

Conservation laws for ODE do not involve Fourier moments!

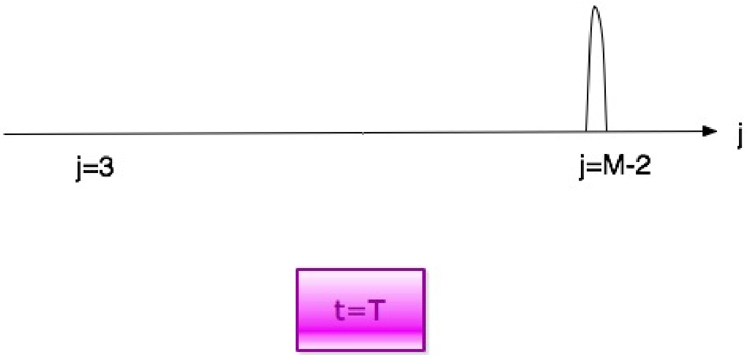
3. Arnold Diffusion for the Toy Model ODE

Using dynamical systems methods, we construct a Toy Model ODE evolution such that:



2. Arnold Diffusion for the Toy Model ODE

Using dynamical systems methods, we construct a Toy Model ODE evolution such that:



A travelling wave through the generations.

Properties of the Toy Model *ODE*

- Solution of the Toy Model is a vector flow $t \to b(t) \in \mathbb{C}^M$

$$b(t) = (b_1(t), \ldots, b_M(t)) \in \mathbb{C}^M; b_j = 0 \ \forall\ j \leq 0, j \geq M+1.$$

- Local Well-Posedness; Let $S(t)$ denote associated flowmap.
- Mass Conservation: $|b(t)|^2 = |b(0)|^2 \implies$
    - Toy Model ODE is Globally Well-Posed.
    - Invariance of the sphere: $\Sigma = \{x \in \mathbb{C}^M : |x|^2 = 1\}$

$$S(t)\Sigma = \Sigma.$$

Properties of the Toy Model *ODE*

- Support Conservation:

$$\begin{aligned} \partial_t |b_j|^2 &= 2Re(\overline{b_j}\partial_t b_j) \\ &= 4Re(i\overline{b_j}^2[b_{j-1}^2 - b_{j+1}^2]) \\ &\leq C|b_j|^2. \end{aligned}$$

  Thus, if $b_j(0) = 0$ then $b_j(t) = 0$ for all t.

- Invariance of coordinate tori:

$$\mathbb{T}_j = \{(b_1, \ldots, b_M \in \Sigma) : |b_j| = 1, b_k = 0 \ \forall \ k \neq j\}$$

  Mass Conservation $\implies S(T)\mathbb{T}_j = \mathbb{T}_j$.
  Dynamics on the invariant tori is easy:

$$b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \ \forall \ k \neq j.$$

Explicit Slider Solution

Consider $M = 2$. Then *ODE* is of the form

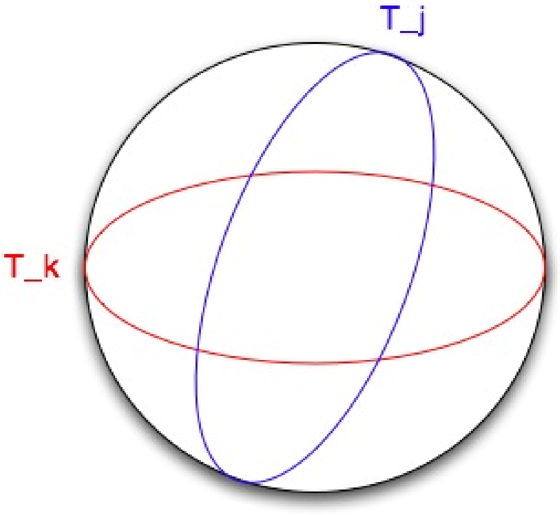
$$\begin{aligned} \partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1} b_2^2 \\ \partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2} b_1^2. \end{aligned}$$

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

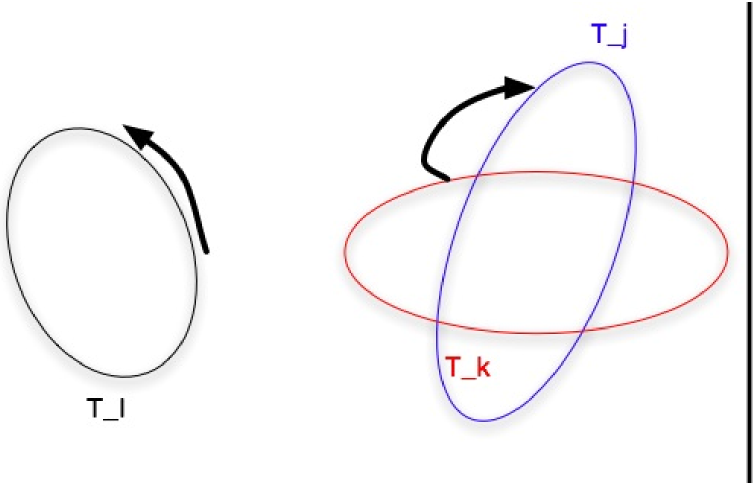
$$b_1(t) = \frac{e^{-it}}{\sqrt{1+e^{2\sqrt{3}t}}}\omega \ \ , b_2(t) = \frac{e^{-it}}{\sqrt{1+e^{-2\sqrt{3}t}}}\omega^2.$$

- As $t \to -\infty, (b_1(t), b_2(t)) \to (e^{-it}\omega, 0) \in \mathbb{T}_1$.
- As $t \to +\infty, (b_1(t), b_2(t)) \to (0, e^{-it}\omega^2) \in \mathbb{T}_2$.

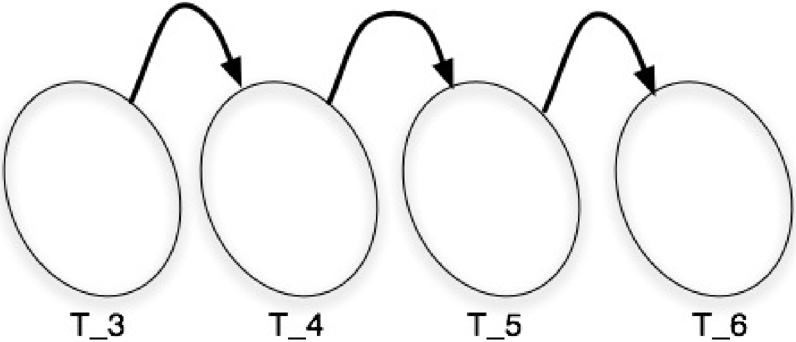
Explicit Slider Solution



Two Explicit Solution Families



Concatenated Sliders: Idea of Proof



Arnold Diffusion for Toy Model Statement

Theorem

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time t such that

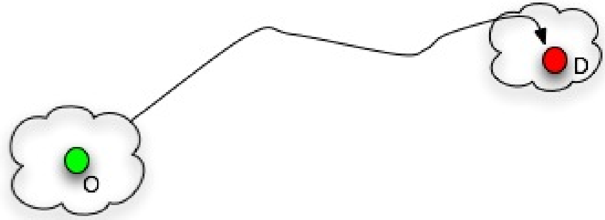
$$S(t)x_3 = x_{M-2}.$$

Remark

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time t_0 and then arbitrarily concentrated at mode $j = M - 2$ at later time t.

Targets and Covering

Let O, D denote points in our phase space Σ. Can we flow along $S(t)$ from *nearby* the origin point 0 to *nearby* the destination point D? More generally, suppose O and D are subsets of Σ.



The notion of a **target** quantifies this question.

Targets

- Let $\mathcal{M}$ denote a subset of Σ. Let d be a (pseudo)metric on Σ. Let $R > 0$ be a radius.
- The Target $(\mathcal{M}, d, R) := \{x \in \Sigma : d(x, M) < R\}$.
- Given $x, y \in \Sigma$.
  We say x hits y if $y = S(t)x$ for some $t \geq 0$.

Covering

Given an initial target (M_1, d_1, R_1) and a final target (M_2, d_2, R_2). We say (M_1, d_1, R_1) can cover (M_2, d_2, R_2) and write

$$(M_1, d_1, R_1) \implies (M_2, d_2, R_2)$$

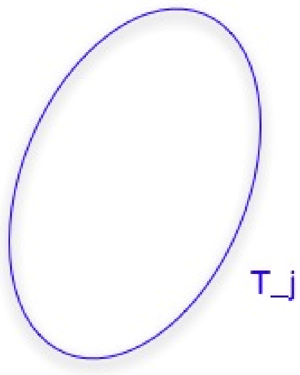
if:
$\forall\ x_2 \ \in M_2\ \exists\ x_1 \in M_1$ such that $\ \forall\ y_1 \in \Sigma$ with $d_1(x_1, y_1) < R_1\ \exists\ y_2 \in \Sigma$ with $d(x_2, y_2) < R_2$ such that y_1 hits y_2.

- The flowout of (M_1, d_1, R_1) is **surjective** onto (M_2, d_2, R_2).
- Covering also includes a notion of stability.

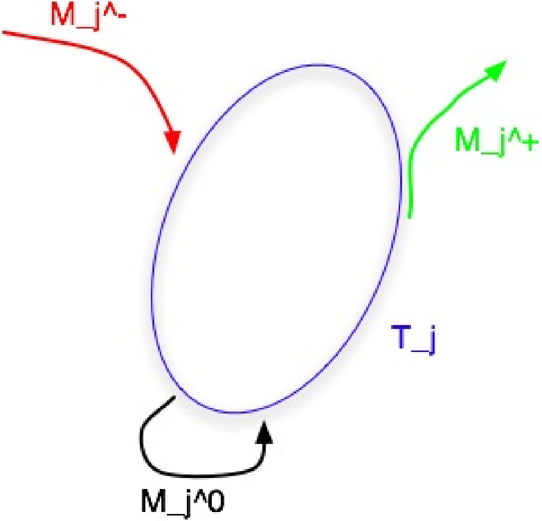
Strategy of Proof

- **Transitivity of Covering:** If $(M_1, d_1, r_1) \implies (M_2, d_2, r_2)$ and $(M_2, d_2, r_2) \implies (M_3, d_3, r_3)$
  then $(M_1, d_1, r_1) \implies (M_3, d_3, r_3)$.
- $\forall\ j \in 3, \ldots, M-2$ we define 3 targets close to $\mathbb{T}_j$:
  - Incoming Target (M_j^-, d_j^-, R_j^-)
  - Ricochet Target (M_j^0, d_j^0, R_j^0)
  - Outgoing Target (M_j^+, d_j^+, R_j^+)
- $\forall\ j = 3, \ldots, M-2$ with appropriate $d_j^{-,0,+}, R_j^{-,0,+}$, prove:
  - $(M_j^-, d_j^-, R_j^-) \implies (M_j^0, d_j^0, R_j^0)$
  - $(M_j^0, d_j^0, R_j^0) \implies (M_j^+, d_j^+, R_j^+)$
  - $(M_j^+, d_j^+, R_j^+) \implies (M_{j+1}^+, d_{j+1}^+, R_{j+1}^+)$

Targets Around T_j



Targets Around T_j



4. Construction of Resonant Set Λ

The task is to construct a finite set $\Lambda \subset \mathbb{Z}^2$ satisfying the properties that led to the Toy Model ODE. We do this in two steps:

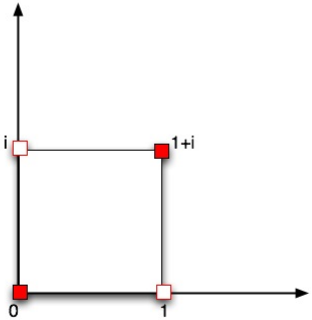
1. Build combinatorial model of Λ called $\Sigma \subset \mathbb{C}^{M-1}$.
2. Build a map $f : \mathbb{C}^{M-1} \to \mathbb{R}^2$ which gives

$$f(\Sigma) = \Lambda \subset \mathbb{Z}^2$$

   satisfying the properties.

Construction of Combinatorial Model Σ

- Standard Unit Square: $S = \{0, 1, 1+i, i\} \subset C, S = S_1 \cup S_2$ where $S_1 = \{1, i\}$ and $S_2 = \{0, 1+i\}$



- $\mathbb{Z}^2 \equiv \mathbb{Z}[i]; (n_1, n_2) \equiv n_1 + in_2$

Construction of Combinatorial Model Σ

- We define

  $$\Sigma_j = \{(z_1, z_2, \dots, z_{M-1}) : z_1, \dots, z_{j-1} \in S_2, z_j, \dots, z_{M-1} \in S_1\}$$

  with the properties
  - $\Sigma_j = S_2^{j-1} \times S_1^{M-j} \subset \mathbb{C}^{M-1}$
  - $|\Sigma_j| = 2^{M-1}$
- Next, we define

  $$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M.$$

  - $|\Sigma| = M2^{M-1}$.
  - Σ_j is called a generation.

Combinatorial Nuclear Family

- Consider the set $F = \{F_0, F_1, F_{1+i}, F_i\} \subset \Sigma$ defined by

$$F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n)$$

  with $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_n \in S_2$ and $w \in S$.
  - The elements $F_0, F_{1+i} \in \Sigma_{j+1}$ are called *children*.
  - The elements F_1, F_i are called *parents*.
  - The four element set F is called a combinatorial nuclear family connecting the generations Σ_j and Σ_{j+1}.
- $\forall\ j\ \exists\ 2^{M-2}$ combinatorial nuclear families connecting generations Σ_j and Σ_{j+1}.
- The set Σ satisfies
  - Existence and uniqueness of spouse and children (of sibling and parents).
  - Sibling is never also a spouse.

Construction of the Placement Function

We need to map $\Sigma \subset \mathbb{C}^{M-1}$ into the frequency lattice $\mathbb{Z}^2$.

- We first define $f_1 : \Sigma_1 \to \mathbb{C}$.
- $\forall\ 1 \leq j \leq M$ and each combinatorial nuclear family F connecting generations Σ_j and Σ_{j+1}, we associate an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.
- Given f_1 and the angles of all the families, we define placement functions $f_j : \Sigma_j \to \mathbb{C}$ recursively by the rule: Suppose $f_j : \Sigma_j \to \mathbb{C}$ has been defined. We define $f_{j+1} : \Sigma_{j+1} \to \mathbb{C}$:

$$\begin{aligned} f_{j+1}(F_{1+i}) &= \frac{1+e^{i\theta(F)}}{2} f_j(F_1) + \frac{1-e^{i\theta(F)}}{2} f_j(F_i) \\ f_{j+1}(F_0) &= \frac{1+e^{i\theta(F)}}{2} f_j(F_1) - \frac{1-e^{i\theta(F)}}{2} f_j(F_i) \end{aligned}$$

for all combinatorial nuclear families connecting Σ_j to Σ_{j+1}.

Theorem: Good Placement Function

Let $M \geq 2$, $s > 1$, and let N be a sufficiently large integer (depending on M). $\exists$ an initial placement function $f_1 : \Sigma_1 \to \mathbb{C}$ and choices of angles $\theta(F)$ for each nuclear family F (and thus an associated complete placement function $f : \Sigma \to \mathbb{C}$) with the following properties:

- (Non-degeneracy) The function f is injective.
- (Integrality) We have $f(\Sigma) \subset \mathbb{Z}[i]$.
- (Magnitude) We have $C(M)^{-1}N \leq |f(x)| \leq C(M)N$ for all $x \in \Sigma$.
- (Closure/Faithfulness) If x_1, x_2, x_3 are distinct elements of Σ are such that $f(x_1), f(x_2), f(x_3)$ form a right-angled triangle, then x_1, x_2, x_3 belong to a combinatorial nuclear family.
- (Wide Diaspora/Norm Explosion) We have

$$\sum_{n \in f(\Sigma_M)} |n|^{2s} > \frac{1}{2} 2^{(s-1)(M-1)} \sum_{n \in f(\Sigma_1)} |n|^{2s}.$$