

WEAK TURBULENCE FOR A 2D PERIODIC SCHRÖDINGER EQUATION

J. Colliander (University of Toronto)

23 February 2007



1 INTRODUCTION

2 OVERVIEW OF PROOF

3 ARNOLD DIFFUSION FOR TOY MODEL

4 CONSTRUCTION OF RESONANT SET Λ

1. INTRODUCTION

THE NLS INITIAL VALUE PROBLEM

[Joint work with **Keel, Staffilani, Takaoka and Tao**]

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \quad (\text{NLS}(\mathbb{T}^2))$$

Smooth solution $u(x, t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$

$$\text{Energy} = E(u) = \int \left(\frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(x, t)|^4 \right) dx = E(0)$$

We want to understand the shape of $|\hat{u}(t, \xi)|$. The conservation laws impose L^2 -moment constraints on this object.

NOTION OF WEAK TURBULENCE

DEFINITION

Weak turbulence is the phenomenon of global-in-time solutions shifting their mass toward increasingly high frequencies.

This shift is also called a **forward cascade**.

- A way to measure weak turbulence is to study

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t, \xi)|^2 |\xi|^{2s} d\xi$$

and prove that it grows for large times t .

- Turbulence is incompatible with **scattering** and **integrability**.

INCOMPATIBLE WITH SCATTERING & INTEGRABILITY

- **Scattering:** \forall global solution $u(t, x) \in H^s \exists u_0^+ \in H^s$ such that,

$$\lim_{t \rightarrow +\infty} \|u(t, x) - e^{it\Delta} u_0^+(x)\|_{H^s} = 0.$$

Note: $\|e^{it\Delta} u_0^+\|_{H^s} = \|u_0^+\|_{H^s} \implies \|u(t)\|_{H^s}$ is bounded.

- **Complete Integrability:** The 1d equation

$$(i\partial_t + \Delta)u = -|u|^2 u$$

has infinitely many conservation laws. Combining them in the right way one gets that $\|u(t)\|_{H^s} \leq C_s$ for all times.

PAST RESULTS

- **Bourgain:** (late 90's)

For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove

$$\|u(t)\|_{H^s}^2 \leq C_s |t|^{4s}.$$

The idea is to improve the local estimate for $t \in [-1, 1]$

$$\|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s \gg 1$$

($\implies \|u(t)\|_{H^s} \lesssim C^{|t|}$ upper bounds) to obtain

$$\|u(t)\|_{H^s} \leq 1 \|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s \gg 1,$$

for some $\delta > 0$. This iterates to give

$$\|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}.$$

- Improvements: **Staffilani, Colliander-Delort-Kenig-Staffilani.**

PAST RESULTS

- Bourgain: (late 90's)

Given $m, s \gg 1$ there exist $\tilde{\Delta}$ and a global solution $u(x, t)$ to the modified wave equation

$$(\partial_{tt} - \tilde{\Delta})u = u^p$$

such that $\|u(t)\|_{H^s} \sim |t|^m$.

- Physics: Weak turbulence theory: Hasselmann & Zakharov.
Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.

CONJECTURE

Solutions to dispersive equations on \mathbb{R}^d DO NOT exhibit weak turbulence. \exists solutions to dispersive equations on \mathbb{T}^d that exhibit weak turbulence. In particular for $NLS(\mathbb{T}^2)$ there exists $u(x, t)$ s. t.

$$\|u(t)\|_{H^s}^2 \rightarrow \infty \text{ as } t \rightarrow \infty.$$

MAIN RESULT

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2. \end{cases} \quad (NLS(\mathbb{T}^2))$$

THEOREM (COLLIANDER-KEEL-STAFFILANI-TAKAOKA-TAO)

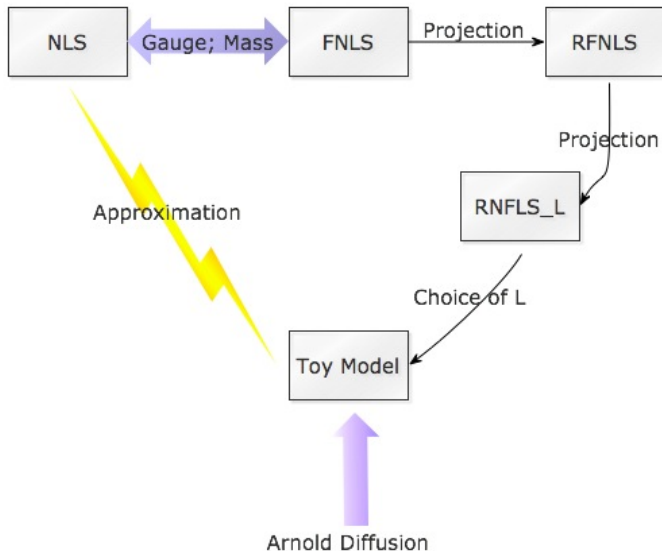
Let $s > 1$, $k \gg 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x, t)$ and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$\|u(t)\|_{H^s}^2 \geq K.$$

2. OVERVIEW OF PROOF



PRELIMINARY REDUCTIONS

- **Gauge Freedom:**

If u solves NLS then $v(t, x) = e^{-i2Gt}u(t, x)$ solves

$$\begin{cases} i\partial_t v + \Delta v = (2G + |v|^2)v \\ v(0, x) = v_0(x), \end{cases} \quad x \in \mathbb{T}^2. \quad (NLS_G)$$

- **Fourier Ansatz:** Recast the dynamics in Fourier coefficients,

$$v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)}.$$

$$\begin{cases} i\partial_t a_n = 2Ga_n + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \bar{a}_{n_2} a_{n_3} e^{i\omega_4 t} \\ a_n(0) = \hat{u}_0(n), \end{cases} \quad n \in \mathbb{Z}^2. \quad (\mathcal{FNLS}_G)$$

PRELIMINARY REDUCTIONS

■ Diagonal decomposition of sum:

$$\begin{aligned} \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} &= \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n \neq n_1, n_3}} + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_1}} \\ &+ \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_3}} - \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n = n_1 = n_3}} \end{aligned}$$

■ Choice of G :

$$G = -\|u_0\|_{L^2}^2.$$

RESONANT TRUNCATION

- *NLS* dynamic is recast as

$$-i\partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \bar{a}_{n_2} a_{n_3} e^{i\omega_4 t}. \quad (\mathcal{F}NLS)$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.$$

■

$$\begin{aligned} \Gamma_{res}(n) &= \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}. \\ &= \{ \text{Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n_4) \text{ is a rectangle} \} \end{aligned}$$

- The *resonant truncation* of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \bar{b}_{n_2} b_{n_3}. \quad (R\mathcal{F}NLS)$$

FINITE DIMENSIONAL RESONANT TRUNCATION

- A set $\Lambda \subset \mathbb{Z}^2$ is *closed under resonant interactions* if

$$n_1, n_2, n_3 \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$

- A *finite dimensional resonant truncation* of \mathcal{FNLS} is

$$-i\partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} \bar{b}_{n_2} b_{n_3}. \quad (R\mathcal{FNLS}_\Lambda)$$

- \forall resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $R\mathcal{FNLS}_\Lambda$ is an ODE.
- If $\text{spt}(a_n(0)) \subset \Lambda$ then \mathcal{FNLS} -evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $R\mathcal{FNLS}_\Lambda$ -ODE $a_n(0) \longmapsto b_n(t)$.
- Given ϵ, s, K , build Λ so that $R\mathcal{FNLS}_\Lambda$ has weak turbulence.

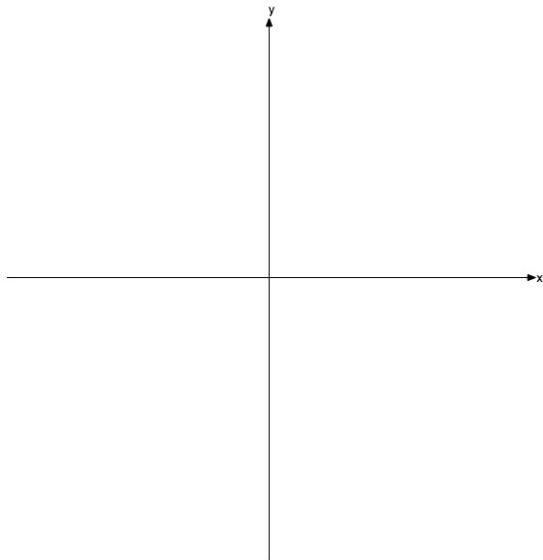
IMAGINE WE BUILD A RESONANT $\Lambda \subset \mathbb{Z}^2$ SUCH THAT...

Imagine a resonant-closed $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_M$ with **properties**.

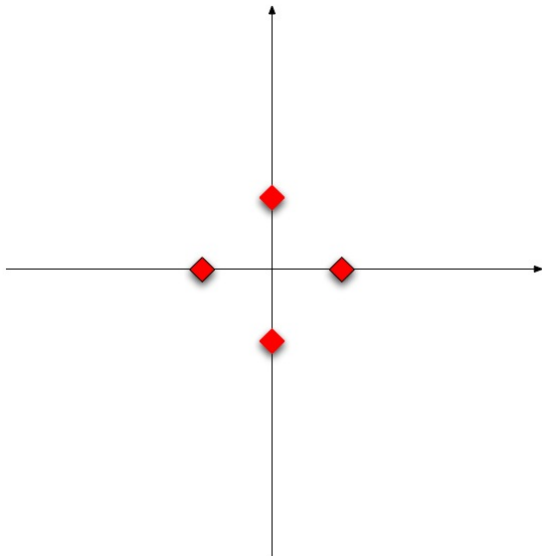
Define a **nuclear family** to be a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the 'parents') live in **generation** Λ_j and n_2, n_4 ('children') live in generation Λ_{j+1} .

- $\forall 1 \leq j < M$ and $\forall n_1 \in \Lambda_j \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.
- $\forall 1 \leq j < M$ and $\forall n_2 \in \Lambda_{j+1} \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.
- The sibling of a frequency is never its spouse.
- Besides nuclear families, Λ contains no other rectangles.
- The function $n \mapsto a_n(0)$ is constant on each generation Λ_j .

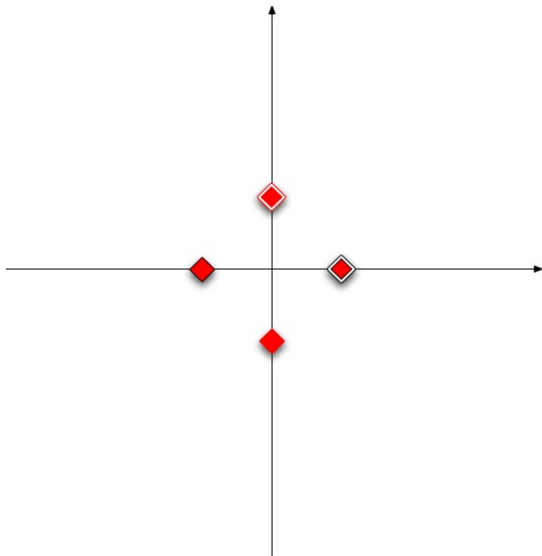
CARTOON CONSTRUCTION OF Λ



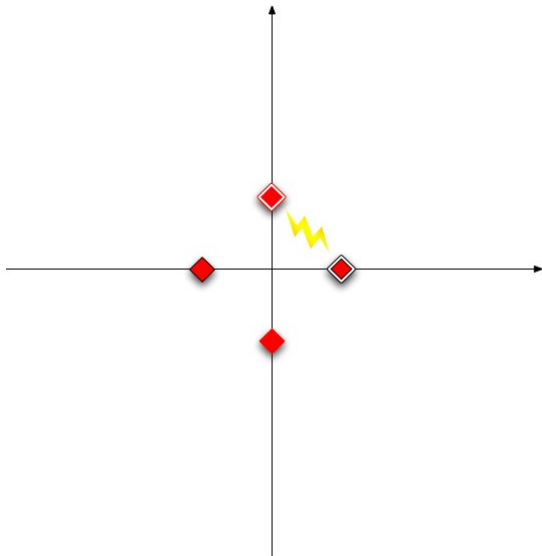
CARTOON CONSTRUCTION OF Λ



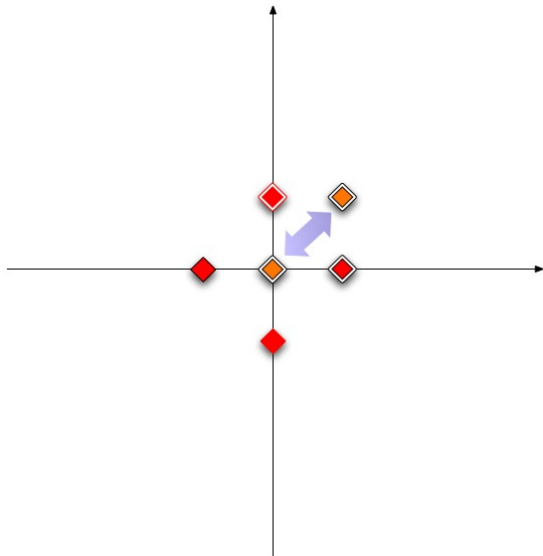
CARTOON CONSTRUCTION OF Λ



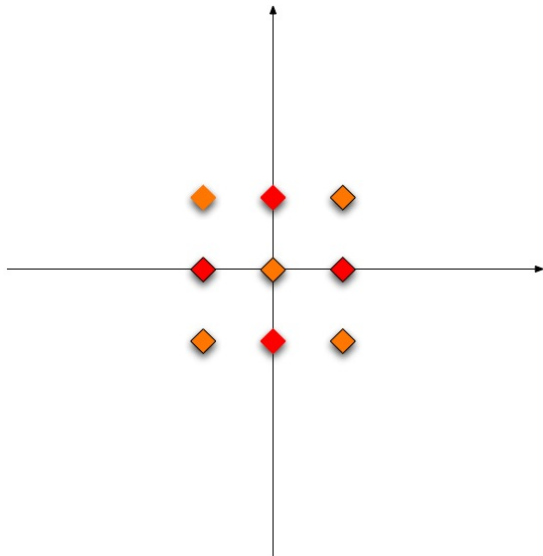
CARTOON CONSTRUCTION OF Λ



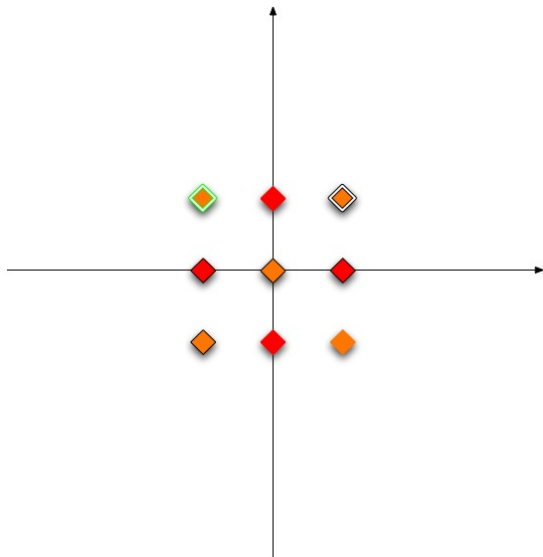
CARTOON CONSTRUCTION OF Λ



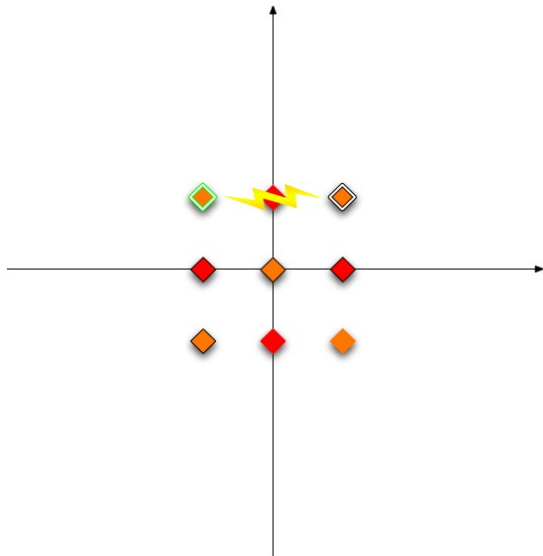
CARTOON CONSTRUCTION OF Λ



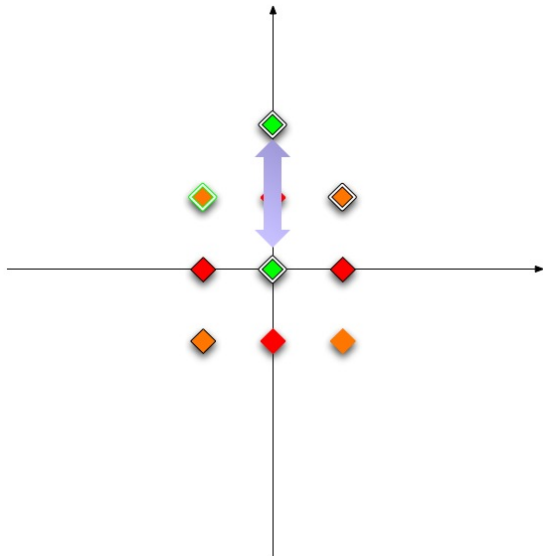
CARTOON CONSTRUCTION OF Λ



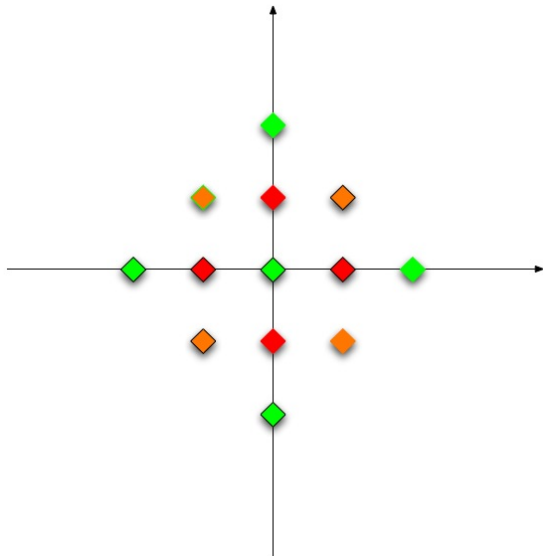
CARTOON CONSTRUCTION OF Λ



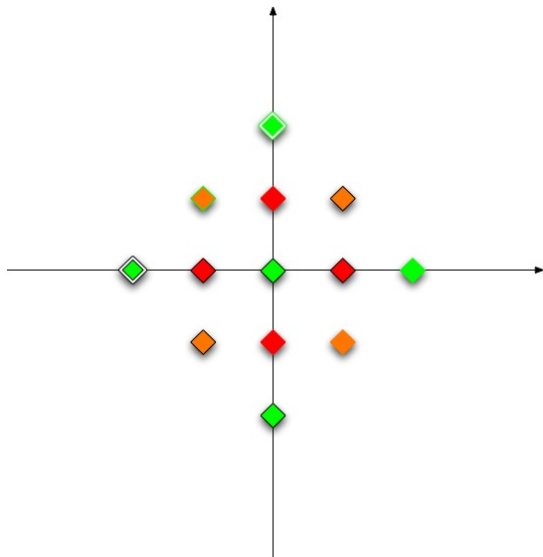
CARTOON CONSTRUCTION OF Λ



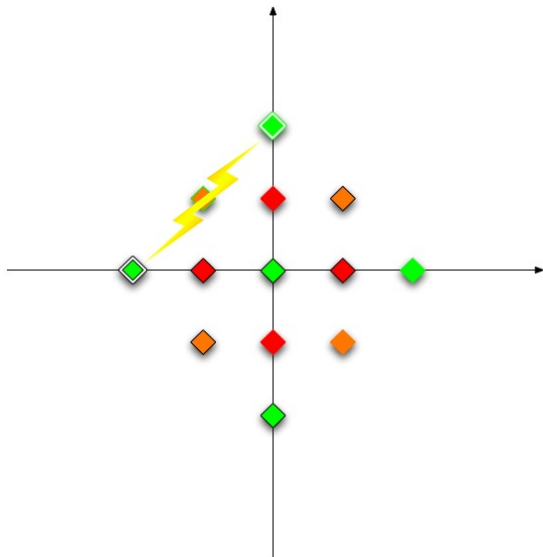
CARTOON CONSTRUCTION OF Λ



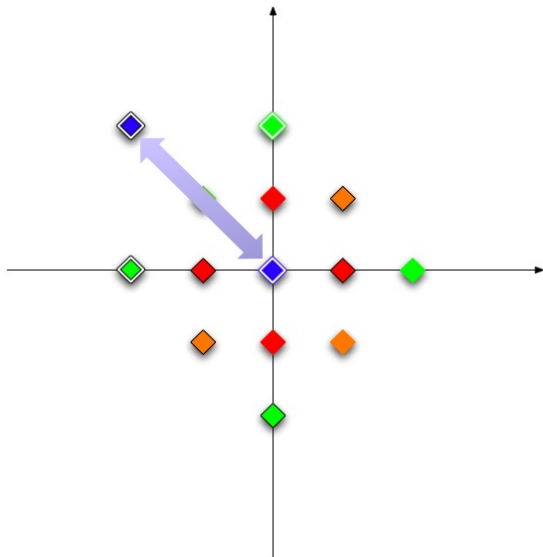
CARTOON CONSTRUCTION OF Λ



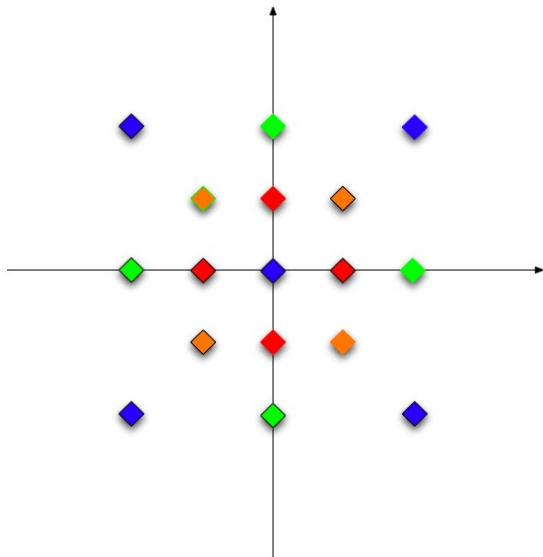
CARTOON CONSTRUCTION OF Λ



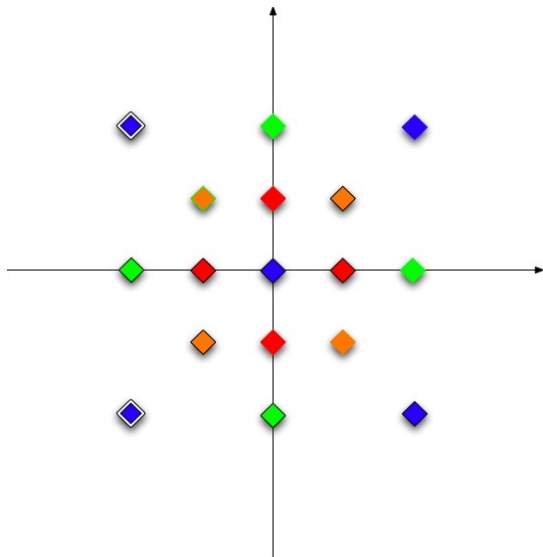
CARTOON CONSTRUCTION OF Λ



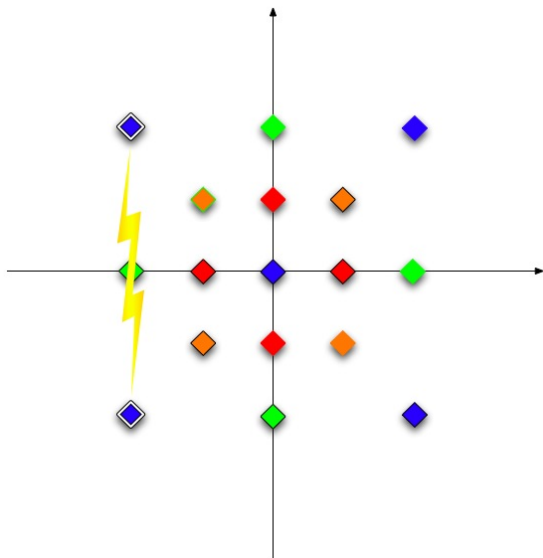
CARTOON CONSTRUCTION OF Λ



CARTOON CONSTRUCTION OF Λ



CARTOON CONSTRUCTION OF Λ



THE TOY MODEL ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_M$. The properties imply $R\mathcal{F}NLS_\Lambda$ simplifies to the **toy model** ODE

$$\partial_t b_j(t) = -i|b_j(t)|^2 b_j(t) + 2i\bar{b}_j(t)[b_j(t)^2 - b_{j+1}(t)^2].$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 \left(\sum_{n \in \Lambda_j} |n|^{2s} \right).$$

We also want Λ to satisfy **Wide Diaspora Property**

$$\sum_{n \in \Lambda_M} |n|^{2s} \gg \sum_{n \in \Lambda_1} |n|^{2s}.$$

CONSERVATION LAWS FOR THE *ODE* SYSTEM

$$\textit{Mass} = \sum_j |b_j(t)|^2 = C_0$$

$$\textit{Momentum} = \sum_j |b_j(t)|^2 \sum_{n \in \Lambda_j} n = C_1,$$

$$\textit{Energy} = K + P = C_2,$$

where

$$K = \sum_j |b_j(t)|^2 \sum_{n \in \Lambda_j} |n|^2,$$

$$P = \frac{1}{2} \sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2.$$

Conservation laws for ODE do not involve Fourier moments!

3. ARNOLD DIFFUSION FOR THE TOY MODEL ODE

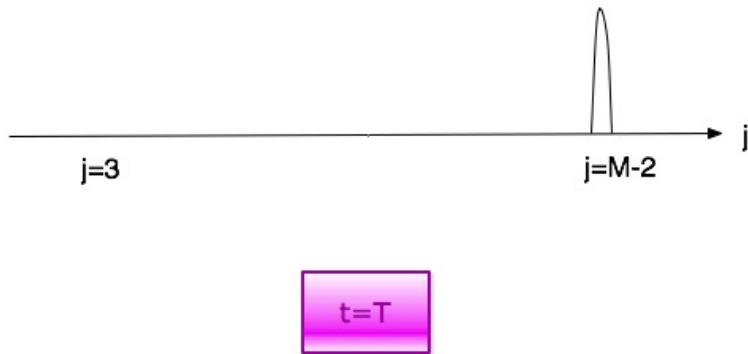
Using dynamical systems methods, we construct a Toy Model ODE evolution such that:



$$t=0$$

2. ARNOLD DIFFUSION FOR THE TOY MODEL ODE

Using dynamical systems methods, we construct a Toy Model ODE evolution such that:



A travelling wave through the generations.

PROPERTIES OF THE TOY MODEL *ODE*

- Solution of the Toy Model is a vector flow $t \rightarrow b(t) \in \mathbb{C}^M$

$$b(t) = (b_1(t), \dots, b_M(t)) \in \mathbb{C}^M; b_j = 0 \quad \forall j \leq 0, j \geq M + 1.$$

- Local Well-Posedness; Let $S(t)$ denote associated flowmap.

- Mass Conservation: $|b(t)|^2 = |b(0)|^2 \implies$

- Toy Model ODE is Globally Well-Posed.

- Invariance of the sphere: $\Sigma = \{x \in \mathbb{C}^M : |x|^2 = 1\}$

$$S(t)\Sigma = \Sigma.$$

PROPERTIES OF THE TOY MODEL *ODE*

- Support Conservation:

$$\begin{aligned}\partial_t |b_j|^2 &= 2\operatorname{Re}(\overline{b_j} \partial_t b_j) \\ &= 4\operatorname{Re}(i\overline{b_j}^2 [b_{j-1}^2 - b_{j+1}^2]) \\ &\leq C|b_j|^2.\end{aligned}$$

Thus, if $b_j(0) = 0$ then $b_j(t) = 0$ for all t .

- Invariance of coordinate tori:

$$\mathbb{T}_j = \{(b_1, \dots, b_M \in \Sigma) : |b_j| = 1, b_k = 0 \ \forall \ k \neq j\}$$

Mass Conservation $\implies S(T)\mathbb{T}_j = \mathbb{T}_j$.

Dynamics on the invariant tori is easy:

$$b_j(t) = e^{-i(t+\theta)}; b_k(t) = 0 \ \forall \ k \neq j.$$

EXPLICIT SLIDER SOLUTION

Consider $M = 2$. Then *ODE* is of the form

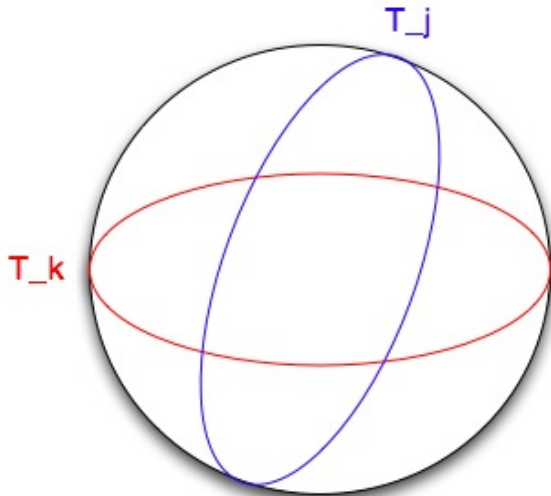
$$\begin{aligned}\partial_t b_1 &= -i|b_1|^2 b_1 + 2i\overline{b_1} b_2^2 \\ \partial_t b_2 &= -i|b_2|^2 b_2 + 2i\overline{b_2} b_1^2.\end{aligned}$$

Let $\omega = e^{2i\pi/3}$ (cube root of unity). This ODE has explicit solution

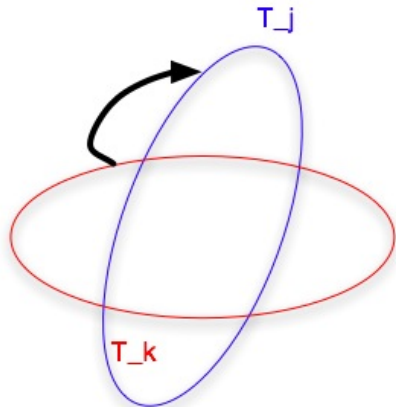
$$b_1(t) = \frac{e^{-it}}{\sqrt{1 + e^{2\sqrt{3}t}}} \omega, \quad b_2(t) = \frac{e^{-it}}{\sqrt{1 + e^{-2\sqrt{3}t}}} \omega^2.$$

- As $t \rightarrow -\infty$, $(b_1(t), b_2(t)) \rightarrow (e^{-it}\omega, 0) \in \mathbb{T}_1$.
- As $t \rightarrow +\infty$, $(b_1(t), b_2(t)) \rightarrow (0, e^{-it}\omega^2) \in \mathbb{T}_2$.

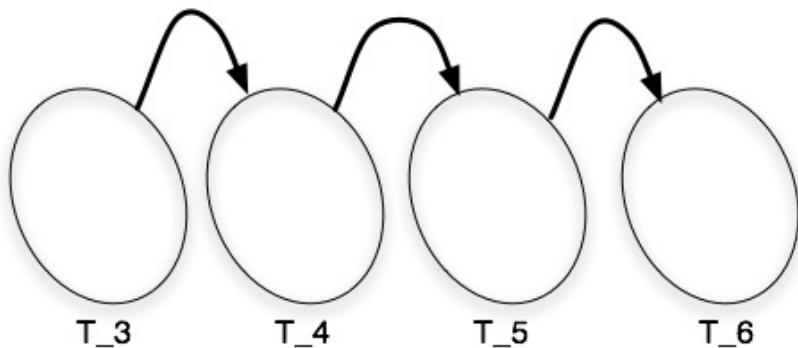
EXPLICIT SLIDER SOLUTION



TWO EXPLICIT SOLUTION FAMILIES



CONCATENATED SLIDERS: IDEA OF PROOF



ARNOLD DIFFUSION FOR TOY MODEL STATEMENT

THEOREM

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of \mathbb{T}_3 and x_{M-2} within ϵ of \mathbb{T}_{M-2} and a time t such that

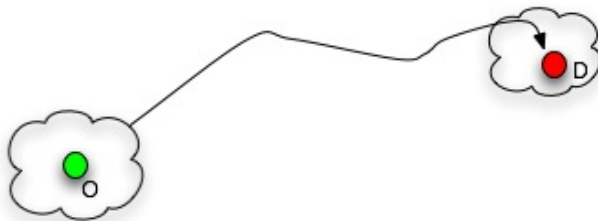
$$S(t)x_3 = x_{M-2}.$$

REMARK

$S(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time t_0 and then arbitrarily concentrated at mode $j = M - 2$ at later time t .

TARGETS AND COVERING

Let O, D denote points in our phase space Σ . Can we flow along $S(t)$ from *nearby* the origin point O to *nearby* the destination point D ? More generally, suppose O and D are subsets of Σ .



The notion of a **target** quantifies this question.

TARGETS

- Let \mathcal{M} denote a subset of Σ . Let d be a (pseudo)metric on Σ . Let $R > 0$ be a radius.
- The **Target** $(\mathcal{M}, d, R) := \{x \in \Sigma : d(x, M) < R\}$.
- Given $x, y \in \Sigma$.
We say x **hits** y if $y = S(t)x$ for some $t \geq 0$.

COVERING

Given an initial target (M_1, d_1, R_1) and a final target (M_2, d_2, R_2) .
We say (M_1, d_1, R_1) **can cover** (M_2, d_2, R_2) and write

$$(M_1, d_1, R_1) \implies (M_2, d_2, R_2)$$

if:

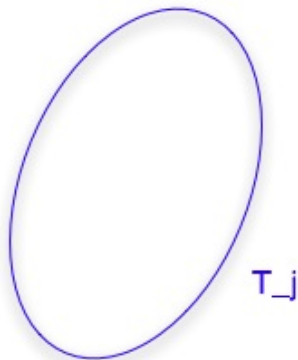
$\forall x_2 \in M_2 \exists x_1 \in M_1$ such that $\forall y_1 \in \Sigma$ with
 $d_1(x_1, y_1) < R_1 \exists y_2 \in \Sigma$ with $d(x_2, y_2) < R_2$ such that y_1 hits y_2 .

- The flowout of (M_1, d_1, R_1) is **surjective** onto (M_2, d_2, R_2) .
- Covering also includes a notion of stability.

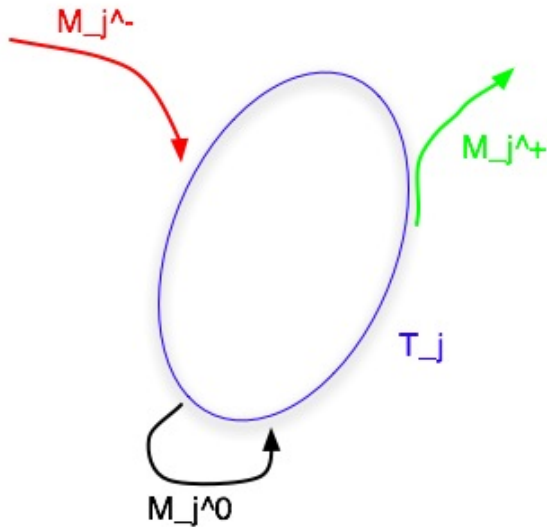
STRATEGY OF PROOF

- **Transitivity of Covering:** If $(M_1, d_1, r_1) \implies (M_2, d_2, r_2)$
and $(M_2, d_2, r_2) \implies (M_3, d_3, r_3)$
then $(M_1, d_1, r_1) \implies (M_3, d_3, r_3)$.
- $\forall j \in 3, \dots, M-2$ we define 3 targets close to \mathbb{T}_j :
 - Incoming Target (M_j^-, d_j^-, R_j^-)
 - Ricochet Target (M_j^0, d_j^0, R_j^0)
 - Outgoing Target (M_j^+, d_j^+, R_j^+)
- $\forall j = 3, \dots, M-2$ with appropriate $d_j^{-,0,+}, R_j^{-,0,+}$, prove:
 - $(M_j^-, d_j^-, R_j^-) \implies (M_j^0, d_j^0, R_j^0)$
 - $(M_j^0, d_j^0, R_j^0) \implies (M_j^+, d_j^+, R_j^+)$
 - $(M_j^+, d_j^+, R_j^+) \implies (M_{j+1}^+, d_{j+1}^+, R_{j+1}^+)$

TARGETS AROUND T_j



TARGETS AROUND T_j



4. CONSTRUCTION OF RESONANT SET Λ

The task is to **construct a finite set** $\Lambda \subset \mathbb{Z}^2$ satisfying the **properties** that led to the Toy Model ODE. We do this in two steps:

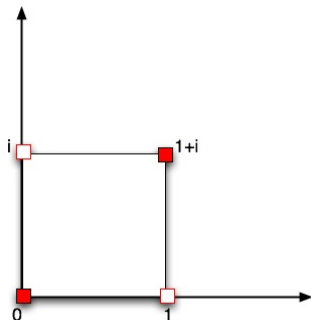
- 1 Build combinatorial model of Λ called $\Sigma \subset \mathbb{C}^{M-1}$.
- 2 Build a map $f : \mathbb{C}^{M-1} \rightarrow \mathbb{R}^2$ which gives

$$f(\Sigma) = \Lambda \subset \mathbb{Z}^2$$

satisfying the **properties**.

CONSTRUCTION OF COMBINATORIAL MODEL Σ

- Standard Unit Square: $S = \{0, 1, 1 + i, i\} \subset \mathbb{C}, S = S_1 \cup S_2$
where $S_1 = \{1, i\}$ and $S_2 = \{0, 1 + i\}$



- $\mathbb{Z}^2 \equiv \mathbb{Z}[i]; (n_1, n_2) \equiv n_1 + in_2$

CONSTRUCTION OF COMBINATORIAL MODEL Σ

- We define

$$\Sigma_j = \{(z_1, z_2, \dots, z_{M-1}) : z_1, \dots, z_{j-1} \in S_2, z_j, \dots, z_{M-1} \in S_1\}$$

with the properties

- $\Sigma_j = S_2^{j-1} \times S_1^{M-j} \subset \mathbb{C}^{M-1}$
 - $|\Sigma_j| = 2^{M-1}$
- Next, we define

$$\Sigma = \Sigma_1 \cup \dots \cup \Sigma_M.$$

- $|\Sigma| = M2^{M-1}$.
- Σ_j is called a generation.

COMBINATORIAL NUCLEAR FAMILY

- Consider the set $F = \{F_0, F_1, F_{1+i}, F_i\} \subset \Sigma$ defined by

$$F_w = (z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_n)$$

with $z_1, \dots, z_{j-1} \in S_2$ and $z_{j+1}, \dots, z_n \in S_2$ and $w \in S$.

- The elements $F_0, F_{1+i} \in \Sigma_{j+1}$ are called *children*.
- The elements F_1, F_i are called *parents*.
- The four element set F is called a **combinatorial nuclear family connecting the generations Σ_j and Σ_{j+1}** .
- $\forall j \exists 2^{M-2}$ combinatorial nuclear families connecting generations Σ_j and Σ_{j+1} .
- The set Σ satisfies
 - Existence and uniqueness of spouse and children (of sibling and parents).
 - Sibling is never also a spouse.

CONSTRUCTION OF THE PLACEMENT FUNCTION

We need to map $\Sigma \subset \mathbb{C}^{M-1}$ into the frequency lattice \mathbb{Z}^2 .

- We first define $f_1 : \Sigma_1 \rightarrow \mathbb{C}$.
- $\forall 1 \leq j \leq M$ and each combinatorial nuclear family F connecting generations Σ_j and Σ_{j+1} , we associate an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.
- Given f_1 and the angles of all the families, we define placement functions $f_j : \Sigma_j \rightarrow \mathbb{C}$ recursively by the rule: Suppose $f_j : \Sigma_j \rightarrow \mathbb{C}$ has been defined. We define $f_{j+1} : \Sigma_{j+1} \rightarrow \mathbb{C}$:

$$\begin{aligned}f_{j+1}(F_{1+i}) &= \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) + \frac{1 - e^{i\theta(F)}}{2} f_j(F_i) \\f_{j+1}(F_0) &= \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) - \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)\end{aligned}$$

for all combinatorial nuclear families connecting Σ_j to Σ_{j+1} .

THEOREM: GOOD PLACEMENT FUNCTION

Let $M \geq 2$, $s > 1$, and let N be a sufficiently large integer (depending on M). \exists an initial placement function $f_1 : \Sigma_1 \rightarrow \mathbb{C}$ and choices of angles $\theta(F)$ for each nuclear family F (and thus an associated complete placement function $f : \Sigma \rightarrow \mathbb{C}$) with the following properties:

- **(Non-degeneracy)** The function f is injective.
- **(Integrality)** We have $f(\Sigma) \subset \mathbb{Z}[i]$.
- **(Magnitude)** We have $C(M)^{-1}N \leq |f(x)| \leq C(M)N$ for all $x \in \Sigma$.
- **(Closure/Faithfulness)** If x_1, x_2, x_3 are distinct elements of Σ are such that $f(x_1), f(x_2), f(x_3)$ form a right-angled triangle, then x_1, x_2, x_3 belong to a combinatorial nuclear family.
- **(Wide Diaspora/Norm Explosion)** We have

$$\sum_{n \in f(\Sigma_M)} |n|^{2s} > \frac{1}{2} 2^{(s-1)(M-1)} \sum_{n \in f(\Sigma_1)} |n|^{2s}.$$