

Rough Blowup Solutions of L^2 -Critical NLS

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P13 Talk ; 11 December 2007

Overview

1. Problem: NLS Blowup Dynamics?
2. The log-log regime
3. Dynamical Rescaled I-method Bootstrap.

1. NLS Blowup Dynamics?

$$\begin{cases} (i\partial_t + \Delta) u = -|u|^2 u \\ NLS_3^{-}(R^2) \\ u(0) = u_0. \end{cases}$$

What happens?

physical; fiber optics
canonical wave equation

• LWP Theory

$\exists u: [0, T_{\text{LWP}}] \times R^2 \rightarrow$ ① solving $NLS_3^{-}(R^2)$ with

$$T_{\text{LWP}} \sim \|u_0\|_{H^s}^{-\frac{2}{3}} \quad \text{provided } u_0 \in H^s, s > 0$$

$$T_{\text{LWP}} : \|e^{it\partial_x} u_0\|_{L^q([0, T_{\text{LWP}}] \times R^2)} < \infty \quad (\text{small enough})$$

(in particular for $u_0 \in L^2$) \exists Rough Solutions.
 $[0, T^*)$ maximal forward

• Dilatation Invariance

$$u_\lambda(t, x) = \left(\frac{1}{\lambda}\right) u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad \lambda > 0.$$

$u_\lambda : [0, \lambda^2 T] \times R^2 \rightarrow$ ② also solves if u does.

• Conservation Laws

$$\text{mass} \quad \|u(t)\|_{L^2}$$

$$\text{momentum} \quad \Im \int u \nabla \bar{u} \, dx = \underline{\mathcal{E}}[u]$$

$$\text{energy} \quad E[u(t)] = \int \frac{1}{2} |\nabla u(t)|^2 - \frac{1}{4} \|u(t)\|^4 \, dx.$$

Virial identity $\Rightarrow \exists$ many blowup solutions

$$\partial_t^2 \int |x|^2 |u(t, x)|^2 \, dx = 16 E[u(t)]$$

{Negative Energy} \cap {Finite Variance} $\ni u_0 \rightarrow u(t)$ explodes w. $T^* < \infty$.

Problem: Describe dynamics of NLS blowup.

Qualitative Properties?

• Scaling lower bound.

$T^* < \infty$, $\|v_0\|_{H^s}^{-\frac{2}{s}}$, dilation invariance

$$\Rightarrow \|D^s v(t)\|_{L^2} \gtrsim \frac{1}{(T^*-t)^{\frac{s}{2}}}.$$

Note: $\|D^s \frac{1}{\lambda} R(\frac{x}{\lambda})\|_{L^2} \sim \left(\frac{1}{\lambda}\right)^s$.

RK:

$\frac{1}{\lambda}$ is H^1 explosion rate

Thus, the size of the blowup core λ should shrink at least as fast as $(T^*-t)^{\frac{1}{2}}$:

but λ has meaning

$$\lambda(t) \leq (T^*-t)^{\frac{1}{2}}.$$

below H^1 .

• Mass Concentration Phenomena

$v_0 \in H^1$:

$$\liminf_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0| < (T^*-t)^{\frac{1}{2}}} |v(t, x)|^2 \geq \|v_0\|_{L^2}^2$$

{ [mild-Tsutsumi] [Nawa]

(I-method)

[CRSW]

$$(x^2 \rightarrow 0)$$

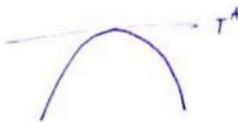
$v_0 \in H^s$:

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{(T^*-t)^{\frac{1}{2}}} \dots \geq \|v_0\|_{L^2}^2$$

$v_0 \in L^2$:

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0| < (T^*-t)^{\frac{1}{2}}} |v(t, x)|^2 \geq \|v_0\|_{L^2}^{-M}.$$

[Bourgain]



Remark: $E[I_n \nabla v(t)]$ systematically replaces $E[v]$ to relax H^1 theory of blowup to H^s theory?
e.g. GWP below $\|v_0\|_{L^2}$; orbital instability bounds

Explicit Blowup Solutions

$$p \in [e^{it} \alpha(x)] = \gamma(t, x) = \frac{1}{|t|} e^{i \frac{|x|^2}{4t} - \frac{i}{t}} \alpha\left(\frac{x}{t}\right).$$



[Bourgain-Wang]: \exists other solutions like these which leave L^2 residue at T^* .

$\lambda(t) \leq (T^*-t)$.
fast collapse!

stability of [BW] solutions is an open issue.
(Eduard Richards)

Numerical Simulations + Heuristics

$\lambda(t) \leq (T^*-t)$ solutions not observed!

$$\lambda(t) \sim \sqrt{\frac{T^*-t}{\log \log (T^*-t)}} \quad \text{Conjecture!}$$

2. The log-log Regime

[Pelinman] \exists log-log blowup solution of $NLS_3^-(\mathbb{R})$.

[Merle-Raphaël] series of papers:

- complete description (analogous to integrable cases) of blowup dynamic for data in an open region of energy space.
- motivates specific questions towards a partition of initial data space based on maximal-in-time behavior.

Merle-Raphaël Theorem

$NLS_3^-(\mathbb{R}^2)$, initial data u_0 . (also $N=1, 3, 4, 5$)

I. Existence and Geometrical Description of log-log regime.

Consider any initial data u_0 satisfying:

- small excess mass: \exists (universal, fixed) $\kappa^* > 0$ s.t.

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} \leq \|Q\|_{L^2} + \kappa^*$$

- Bounded kinetic energy: $u_0 \in H^1$

- Negative total energy: $E[u_0] < 0$.

The associated solution $u_0 \mapsto u(\cdot)$ will blow up in a finite time T^* with the following dynamics: (log-log regime)

- $\exists (\lambda(t), x(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}$ and residual profile $u^* \in L^2$ s.t.

$$u(t) = \frac{1}{\lambda(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \rightarrow u^* \text{ in } L^2.$$

- blowup point converges: $x(t) \rightarrow x(T^*) \in \mathbb{R}^2$ as $t \nearrow T^*$.

- blowup speed satisfies log-log law:

$$\lambda(t) \sqrt{\frac{\log |\log(T^* - t)|}{T^* - t}} \rightarrow \sqrt{2\pi} \text{ as } t \nearrow T^*.$$

II. H^1 -stability of log-log regime.

The set of initial data u_0 with small excess mass and $u_0 \in H^1$ which blows up in the log-log regime is open in H^1 .



III. Non-smoothness of residual profile:

$u^* \in L^2$ but $u^* \notin L^p \forall p > 2$.

motivates L^2 theory of blowup

Theorem (C-Raphaël) ($NLS_3^-(\mathbb{R}^2)$; only $N=2$)

Suppose $u_0 \in H^1$ evolves in the log-log blowup regime.

$\exists \varepsilon = \varepsilon(s, u_0)$ s.t. $\forall v_0 \in H^s$ with $\|u_0 - v_0\|_{H^s} < \varepsilon$ the associated solution $v_0 \mapsto v(\cdot)$ blows up in log-log regime.

(log-log regime is open in H^s). ($\Rightarrow \exists$ rough blowup solutions.)

partially relaxed.

$u_0 \in H^s$?
 L^2 ?

Stronger stability?

H^s ?

3. Dynamical Rescaled I-Method Bootstrap

Strategy of proof

- Isolate roles of energy conservation in [MR]
- Relax to almost conserved modified energy via I-method.

spawns various error terms

Near blowup time, log log blowup solutions admit a geometrical description:

$$v(t, x) = \frac{1}{\lambda(t)} (Q_{b(t)} + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}$$

A "deformation" of Q with exponentially small energy.
The parameter $b(t)$ is a "new" modulation parameter.

The log-log regime \iff

$$\left\{ \begin{array}{l} T^* < \infty, \frac{ds}{dt} = \frac{1}{\lambda^2} \\ b_s \sim e^{-\frac{c}{b}}, -\frac{\lambda s}{\lambda} \sim b \\ \int |\nabla \varepsilon|^2 dx \lesssim e^{-\frac{c}{b}} \end{array} \right.$$

H'

genesis of collaboration

The step that precedes this in [MR]:

$$\int |\nabla \varepsilon|^2 dx \lesssim e^{-\frac{c}{b}} + \lambda^2 |E[v(t)]|$$

\downarrow \downarrow conserved
FAST

RK: Non-conservation properties of $I_N v$ introduce various error terms requiring I-Method analysis.

As in [CRSW] H^s analog of [Merkle-Tsutsumi], observe we could allow for growth of $|E[v(t)]|$ and maintain the conclusion.

In fact: $\lambda \sim e^{-e^{\frac{c}{b}}} \ll e^{-\frac{c}{b}}$ **FAST**

We could tolerate growth slower than $\frac{1}{\lambda^2}$!!

I-Method Input: • $I_N v$ with $N \sim \frac{1}{\lambda^{1+}}$

• $E[I_{N(t)} v(t)] \lesssim \frac{1}{\lambda(t)^{2-\alpha(s)}}, \alpha(s) > 0.$

$$I_N v = I_N Q + I_N \varepsilon$$

$$= Q + I_N \varepsilon + \underbrace{(I_N - I) Q}_{\text{small}}$$

$$\hookrightarrow \int |\nabla I_N \varepsilon|^2 dx \lesssim e^{-\frac{c}{b}} + \lambda^{\alpha(s)} \lesssim e^{-\frac{c}{b}}$$

log log

• $I_N Q \approx Q$ since Q is smooth.

Trapped Dynamical Region

Suppose for $t \in [0, T^+] \subset [0, T^*)$ the solution admits geometric description

$$u(t) = \frac{1}{\lambda(t)} (Q_{b(t)} + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\varphi(t)}$$

where the parameters $t \mapsto (\lambda(t), b(t), x(t), \varphi(t))$ evolve to ensure certain orthogonality conditions.

Assume further that $\forall t \in [0, T^+]$

$$(1)' \quad b + \|\varepsilon\|_{L^2} \text{ control} \quad b(t) > 0, \quad b(t) + \|\varepsilon(t)\|_{L^2} \leq 10 \cdot (b(0) + \|\varepsilon(0)\|_{L^2})$$

$$(2)' \quad \text{scaling parameter } \lambda \text{ control and monotonicity} \quad \lambda(t) \leq \left(e^{-\frac{10}{100b(t)}} \right)^{\frac{5}{4}}$$

$$\forall 0 < t_1 \leq t_2 \leq T^+, \quad \lambda(t_2) \leq \frac{5}{2} \lambda(t_1)$$

Doubling time property: $\{t_k\}_{K_0}^{K^+} \subset [0, T^+]$ s.t. $\lambda(t_K) = \frac{1}{2^K}$

$$\text{Assume } t_{k+1} - t_k \leq \sqrt{K} \lambda^2(t_k)$$

$$(3)' \quad \|\varepsilon\|_{H^s} \text{ control of excess mass}$$

$$N(t) = \left(\frac{1}{\lambda(t)} \right)^{\frac{1}{\beta}} \quad ; \quad \beta = 1 - \frac{2s}{3} < 1$$

$$\|\mathcal{I}_{N(t)\lambda(t)} \nabla \varepsilon(t)\|_{L^2}^2 + \int |\varepsilon(t)|^2 e^{-|y|} dy \leq \left(\cdot e^{-\frac{c}{10b(t)}} \right)^{\frac{8}{3}}$$

We then prove $(1)', (2)', (3)' \Rightarrow [0, T^+] \rightsquigarrow [0, T^*)$.

$(1)', (2)', (3)' \Rightarrow (1)', (2)', (3)'$

\uparrow
 via the log-log analysis
 combined w. quantified
 dynamical rescaling of
 the I-METHOD.

I - Method Inputs

Proposition Assuming (1), (2), (3) on $[0, T^+]$ we have

$$| \mathbb{E} [I_{N(t)} u(t)] | \leq \left(\frac{1}{\lambda(t)} \right)^{2(1-\kappa_1)}$$

$$| \mathbb{E} [I_{N(t)}^* u(t)] | \leq \left(\frac{1}{\lambda(t)} \right)^{1-\kappa_1}$$

for some $\kappa_1 = \kappa_1(s) > 0$.

The proof involves a dynamical rescaling of the I-method. Note that here $N(t)$ is time dependent.

Recall the doubling times $\{\tau_k\}_{k=K_0}^{K^+} \subset [0, T^+]$ s.t. $\lambda(\tau_k) = \frac{1}{2^k}$.

From (2), we have $\tau_{k+1} - \tau_k \leq K \lambda^2(\tau_k)$.

Thus, $[\tau_k, \tau_{k+1}] = \bigcup_{j=1}^K [\tau_k^j, \tau_k^{j+1}]$ with $\tau_k^{j+1} - \tau_k^j \sim \left(\frac{1}{\|u(\tau_k)\|_H^s} \right)^{\frac{2}{s}}$.

We have at most K LWP time intervals $[\tau_k^j, \tau_k^{j+1}]$ inside $[\tau_k, \tau_{k+1}]$.

- Modified LWP $\Rightarrow \|\langle D \rangle I_{N(\tau^+)} u(\tau_k^j)\|_{L^2} \approx \|\langle D \rangle I_{N(\tau^+)} u(\tau_k^j)\|_{L^2}$.
[spacetime control]

- Terminal versus current smoothing:

$$\|\langle D \rangle I_{N(\tau^+)} u(\tau_k^j)\|_{L^2} \leq \left[\frac{N(\tau^+)}{N(\tau_k^j)} \right]^{1-s} \|\langle D \rangle I_{N(\tau_k^j)} u(\tau_k^j)\|_{L^2}.$$

- Rescaling at τ_k^j :

$$\begin{aligned} \|\nabla I_{N(\tau_k^j)} u(\tau_k^j)\|_{L^2} &\leq \frac{1}{\lambda(\tau_k^j)} \|\nabla I_{N(\tau_k^j)} \lambda(\tau_k^j) (Q_b + \varepsilon)(\tau_k^j)\|_{L^2} \\ &\lesssim \frac{1}{\lambda(\tau_k^j)} \quad (\text{using smoothness of } Q + (3)) \end{aligned}$$

Almost Conservation Lemma

$$\left| \int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{d\tau} E [I_{N(\tau_+)} v(\tau)] d\tau \right| \leq \frac{\| \langle D \rangle I_{N(\tau^+)} v \|_{S_{[\tau_k^j, \tau_k^{j+1}]}^3}}{N(\tau^+)}$$

$$\left| \int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{d\tau} [I_{N(\tau^+)} v(\tau)] d\tau \right| \leq \frac{\| \langle D \rangle I_{N(\tau^+)} v \|_{S_{[\tau_k^j, \tau_k^{j+1}]}^3}}{(N(\tau^+))^2}.$$

(implied constants depend upon $\|v\|_{L^2}$)