Weak Turbulence for a 2D Periodic Schrödinger Equation

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We consider the defocusing initial value problem:

\[
\begin{align*}
\left\{ \begin{array}{l}
(-i\partial_t + \Delta) u &= |u|^2 u \\
u(0, x) &= u_0(x), \text{ where } x \in \mathbb{T}^2, \mathbb{R}^2.
\end{array} \right. \\
\end{align*}
\]

(Symbols $NLS(\mathbb{T}^2)$)

Smooth solution $u(x, t)$ exists globally and

\[
\begin{align*}
\text{Mass} &= M(u) = \|u(t)\|^2 = M(0) \\
\text{Energy} &= E(u) = \int (\frac{1}{2}|\nabla u(t, x)|^2 + \frac{1}{4}|u(x, t)|^4) \, dx = E(0)
\end{align*}
\]

In particular if $f_t(\xi) = |\hat{u}(t, \xi)|^2$ then the area of the subgraph of $f_t(\xi)$ remains constant. On the other hand the shape of the subgraph may change in time, in particular in time most of the area may concentrate on very high.
Informal definition of weak turbulence

Definition

Weak turbulence is the phenomenon that describe the shifting over time of the mass of global solutions into increasingly high frequencies.

This shift is also called forward cascade.

- One way of measuring weak turbulence is to consider the function

\[ g_s(t) = \| u(t) \|_{\dot{H}^s}^2 = \int |\hat{u}(t, \xi)|^2 |\xi|^{2s} d\xi \]

and prove that it grows for large times \( t \).

- Weak turbulence is incompatible with scattering and complete integrability.
**Explanation of second item**

**scattering:** In this context scattering (at $+\infty$) means that for any global solution $u(t, x) \in H^s$ there exists $u_0^+ \in H^s$ such that, if $S(t)$ is the linear Schrödinger operator, then

$$\lim_{t \to +\infty} [u(t, x) - S(t)u_0^+(x)] = 0$$

in $H^s$ sense. Since $\|S(t)u_0^+\|_{H^s} = \|u_0^+\|_{H^s}$, it follows that $g_s(t) = \|u(t)\|^2_{\dot{H}^s}$ will not grow.

**complete integrability:** For example the 1d equation

$$(i\partial_t + \Delta)u = -|u|^2u$$

is integrable in the sense that it admits infinitely many conservation laws. Combining them in the right way one gets that $g_s(t) = \|u(t)\|^2_{\dot{H}^s} \leq C_s$ for all times.
Some numerical results
**Some theoretical results**

- **Bourgain**: (late 90’s)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove
  \[ g_s(t) = \|u(t)\|_{\dot{H}^s}^2 \leq C_s |t|^{4s}. \]
  The idea here is to improve the local estimate for $t \in [-1, 1]$
  \[ \|u(t)\|_{H^s} \leq C_s \|u(0)\|_{H^s}, \quad \text{for } C_s >> 1 \]
  to the better one
  \[ \|u(t)\|_{H^s} \leq 1 \|u(0)\|_{H^s} + C_s \|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s >> 1, \]
  for some $\delta > 0$. This last one in fact gives
  \[ \|u(t)\|_{H^s} \leq C_s |t|^{1/\delta}. \]
  For similar result see also **Staffilani**, **Colliander-Delort-Kenig-Staffilani** and a recent result of **W.M. Wang**.

More theoretical results

- **Bourgain:** (late 90’s)
  Given $m, s \gg 1$ there exists $\tilde{\Delta}$ such that a global solution $u(x, t)$ to the modified wave equation

\[(\partial_{tt} - \tilde{\Delta})u = u^p\]

such that

\[\|u(t)\|_{H^s} \sim |t|^m.\]

**Conjecture**

*Solutions to dispersive equations on $\mathbb{R}^d$ DO NOT exhibit weak turbulence. Solutions to dispersive equations on $\mathbb{T}^d$ DO exhibit weak turbulence.* In particular for *NLS$(\mathbb{T}^2)$*

\[g_s(t) = \|u(t)\|_{H^s}^2 \sim \log(t).\]
Main Theorem

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $k >> 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x, t)$ to

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2u \\ u(0, x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \end{cases} \quad (\text{NLS} (\mathbb{T}^2))$$

and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$g_s(t) = \|u(t)\|_{H^s}^2 \geq K.$$
Ingredients for the proof

- Reduction to a resonant problem
- Construction of a special finite set $\Delta$ of frequencies
- Reduction to a resonant, finite dimensional Toy Model
- Arnold diffusion for the Toy Model
- A perturbation lemma
- A scaling argument
Reduction to a resonant problem

We consider the gauge transformation

\[ v(t, x) = e^{-i2Gt} u(t, x), \]

for \( G \in \mathbb{R} \). If \( u \) solves \( \text{NLS}(\mathbb{T}^2) \) above, then \( v \) solves the equation

\[ (-i\partial_t + \Delta)v = (2G + v)|v|^2. \tag{((\text{NLS})_G)} \]

We make the ansatz

\[ v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(\langle n,x \rangle + |n|^2t)}. \]

Now the dynamics is all recast through \( a_n(t) \):

\[ -i\partial_t a_n = 2Ga_n + \sum_{n_1-n_2+n_3=n} a_{n_1} \bar{a}_{n_2} a_{n_3} e^{i\omega_4 t} \]

where \( \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2. \)
The \textit{FNLS} system

By choosing
\[ G = -\|v(t)\|_{L^2}^2 = -\sum_k |a_k(t)|^2 \]
which is constant from the conservation of the mass, one can rewrite the equation above as
\[ -i\partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \bar{a}_{n_2} a_{n_3} e^{i\omega_4 t} \]
where
\[ \Gamma(n) = \{ n_1, n_2, n_3 \in \mathbb{Z}^2 / n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n \}. \]

From now on we will be referring to this system as the \textit{FNLS} system, with the obvious connection with the original \textit{NLS}(\mathbb{T}^2) equation.
The RFNLS system

We define the set

$$\Gamma_{\text{res}}(n) = \{n_1, n_2, n_3 \in \Gamma(n) / \omega_4 = 0\}.$$

The geometric interpretation for this set is the following: If $n_1, n_2, n_3$ are in $\Gamma_{\text{res}}(n)$, then these four points represent the vertices of a rectangle in $\mathbb{Z}^2$.

We finally define the Resonant Truncation RFNLS to be the system

$$-i \partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma_{\text{res}}(n)} b_{n_1} \overline{b}_{n_2} b_{n_3} e^{i\omega_4 t}.$$

It is time now to define the special set $\Delta$ of finite frequencies where we will define the initial data.
A special set of frequencies $\Delta$

We call this set $\Delta \subset \mathbb{Z}^2$ and we ask for it several properties. In particular $\Delta = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_M$, that is $M$ generations of nuclear families $\Delta_j$. If $n_1, n_2, n_3, n_4 \in \Delta$, then they represent the vertices of a rectangle such that $n_1$ and $n_3$ are in $\Delta_j$ ("parents") and $n_2, n_4$ are in $\Delta_{j+1}$ ("children"). The interactions among these families follow these rules:

- **Existence and uniqueness of spouse and children:** For any $1 \leq j < M$ and $n_1 \in \Delta_j$ there exist a unique $n_3 \in \Delta_j$ and $n_2, n_4 \in \Delta_{j+1}$ (up to permutations).

- **Existence and uniqueness of siblings and parents:** For any $1 < j \leq M$ and $n_2 \in \Delta_j$ there exist a unique $n_4 \in \Delta_j$ and $n_1, n_3 \in \Delta_{j-1}$ (up to permutations).

- **No incest:** The sibling at frequency $n$ is never equal to its spouse.

- **Faithfulness:** Apart from nuclear families $\Delta$ does not contain other rectangles.
More properties for the set $\Delta$

- Given $\sigma << 1$ and $K >> 1$, there exist $M$ and $\Delta = \Delta_1 \cup \ldots \cup \Delta_M$ as above and

  a) 
  \[
  \sum_{n \in \Delta_M} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n \in \Delta_1} |n|^{2s}.
  \]

  b) If $N = N(\sigma, K)$ is large enough then $\Delta$ consists of $M \times 2^{M-1}$ disjoint frequencies $n$ with $|n| > N = N(\sigma, K)$ and the last frequency in $\Delta_M$ is of size $C(M)N$ with the first in $\Delta_1$ is of size $N$. We call $N$ the Inner Radius of $\Delta$. 
The system $RFNLS_\Delta$

The final propriety that we ask for the set $\Delta$ is that

- $\Delta$ is closed under resonant interactions:

$$n_1, n_2, n_3 \in \Delta \cap \Gamma_{res}(n) \implies n \in \Delta.$$  

**Remark**

*If $\Delta$ is closed under resonant interaction and if $b_n(0)$ has support in $\Delta$, then the solution $b_n(t)$ of $(RFNLS)$ on $[0, T]$ has also support in $\Delta$. To see this one just uses Gronwall’s estimate on $\sum_{n \notin \Delta} |b_n(t)|^2$. *

We can then define the finite dimension resonant truncated system

$$-i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Delta \cap \Gamma_{res}(n)} b_{n_1} b_{n_2} b_{n_3}. \quad (RFNLS_\Delta)$$
The ODE system (Toy Model)

**Remark**

If we go back to \((RFNLS)\) one can easily see that 
\[ b_n(t) = b_m(t), \quad \text{for any } m, n \in \Delta_j. \]
We call this the "Intergenerational equality".

Using all these properties for \(\Delta\) we can identify

\[
(b_n(t))_{\{n \in \mathbb{Z}^2\}} = (b_j(t))_{\{j=1,\ldots,M\}}
\]

and reduce \((RFNLS_\Delta)\) to the system

\[
-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2b_j(t) - 2b_{j+1}(t)^2b_j(t),
\]

\((ODE)\)

with the boundary condition

\[
b_0(t) = b_{M+1}(t) = 0.
\]

\((BC)\)
Conservation laws for the ODE system

The following are conserved quantities for (ODE)

\[ \text{Mass} \sum_j |b_j(t)|^2 = C_0 \]

\[ \text{Momentum} \sum_j |b_j(t)|^2 \sum_{n \in \Delta_j} n = C_1, \]

and if

\[ \text{Kinetic Energy} = \sum_j |b_j(t)|^2 \sum_{n \in \Delta_j} |n|^2 \]

\[ \text{Potential Energy} = \frac{1}{2} \sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2, \]

then

\[ \text{Energy} = \text{Kinetic Energy} + \text{Potential Energy} = C_2. \]
Arnold Diffusion for ODE: the set up

Global well-posedness for ODE is not an issue. We define

$$\Sigma = \{ x \in \mathbb{C}^M / |x|^2 = 1 \} \text{ and } W(t) : \Sigma \rightarrow \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of ODE. It is easy to see that for any $b \in \Sigma$

$$\partial_t |b_j|^2 = 4\Re(i\bar{b_j}^2(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 \implies b_j(t) = 0, \text{ for all } t \in [0, T].$$

If moreover we define the torus

$$\Pi_j = \{(b_1, \ldots, b_M) \in \Sigma / |b_j| = 1, b_k = 0, k \neq j\}$$

then

$$W(t)\Pi_j = \Pi_j \text{ for all } j = 1, \ldots, M$$

($\Pi_j$ is invariant).
Arnold Diffusion for ODE

**Theorem**

(Arnold Diffusion)

Let $M \geq 6$. Given $\epsilon > 0$ there exist $x_3$ within $\epsilon$ of $\Pi_3$ and $x_{M-2}$ within $\epsilon$ of $\Pi_{M-2}$ and a time $t$ such that

$$W(t)x_3 = x_{M-2}.$$ 

**Remark**

$W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time $t_0$ and then arbitrarily concentrated at mode $j = M - 2$ at later time $t$. 
Consider $M = 2$. Then \textit{ODE} is of the form

\[
\begin{align*}
\partial_t b_1 &= -i|b_1|^2 b_1 + 2i\bar{b}_1 b_2^2 \\
\partial_t b_2 &= -i|b_2|^2 b_2 + 2i\bar{b}_2 b_1^2.
\end{align*}
\]

This system has explicit solution

\[
\begin{align*}
b_1(t) &= \frac{e^{-it}}{\sqrt{1 + e^{2\sqrt{3}t}}} \omega \\
b_2(t) &= \frac{e^{-it}}{\sqrt{1 + e^{-2\sqrt{3}t}}} \omega^2,
\end{align*}
\]

where $\omega = e^{2i\pi/3}$ (cube root of unity). Since

\[
\lim_{t \to +\infty} |b_1(t)| = 0 \quad \text{and} \quad \lim_{t \to +\infty} |b_2(t)| = 1
\]

and

\[
\lim_{t \to -\infty} |b_2(t)| = 0 \quad \text{and} \quad \lim_{t \to -\infty} |b_1(t)| = 1.
\]
It follows that \((b_1, b_2) \in \Pi_2\) at \(t = +\infty\) and \((b_1, b_2) \in \Pi_1\) at \(t = -\infty\). So with an infinite amount of time one can go from \(\Pi_1\) to \(\Pi_2\) and vice versa. A suitable perturbation of \(\Pi_i\) replacing the tori \(\Pi_i\) will be the key in proving diffusion in finite time: A picture here?
**A perturbation lemma**

**Lemma**

Let $\Delta \subset \mathbb{Z}^2$ introduced above. Let $B \gg 1$ and $\delta > 0$ small and fixed. Let $t \in [0, T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Delta)$ solving $\text{RFNLS}_\Delta$ such that

$$\|b(t)\|_{l^1} \lesssim B^{-1}.$$ 

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of $\text{RFNLS}$ such that

$$a(0) = b(0), \quad \text{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta},$$

for any $t \in [0, T]$.

**Proof.**

This is a standard perturbation lemma proved by checking that the ”non resonant” part of the nonlinearity remains small enough. $\Box$
Recasting the main theorem

With all the notations and reductions introduced we can now recast the main theorem in the following way:

**Theorem**

For any $0 < \sigma \ll 1$ and $K \gg 1$ there exists a complex sequence $(a_n)$ such that

$$\left( \sum_{n \in \mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ of (FNLS) and $T > 0$ such that

$$\left( \sum_{n \in \mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$
In order to be able to use the Arnold Diffusion to move mass from lower frequencies to higher ones and start with a small data we need to introduce scaling. Consider in $[0, t]$ the solution $b(t)$ of the system $RFNLS_{\Delta}$ with initial datum $b_0$. Then the rescaled function

$$b^\lambda(t) = \lambda^{-1} b\left(\frac{t}{\lambda^2}\right)$$

solves the same system with datum $b^\lambda_0 = \lambda^{-1} b_0$.

We then first pick the complex vector $b(0)$ that was found in the theorem on Arnold Diffusion. For simplicity let’s assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} 
  b^\lambda_j(0) & \text{for any } n \in \Lambda_j \\
  0 & \text{otherwise}
\end{cases}.$$
ESTIMATING THE SIZE OF \((a(0))\)

By definition

\[
\left( \sum_{n \in \Delta} |a_n(0)|^2 |n|^{2s} \right)^{1/2} = \lambda^{-1} \left( \sum_{j=1}^{M} |b_j(0)|^2 \left( \sum_{n \in \Delta_j} |n|^{2s} \right) \right)^{1/2} = \lambda^{-1} Q_3,
\]

where the last equality follows from defining

\[
\sum_{n \in \Delta_j} |n|^{2s} = Q_j,
\]

and the definition of \(a_n(0)\) given above. At this point we use the proprieties of the set \(\Delta\) to estimate \(Q_3 = C(M)N\), where \(N\) is the inner radius of \(\Delta\). We then conclude that

\[
\left( \sum_{n \in \Delta} |a_n(0)|^2 |n|^{2s} \right)^{1/2} = \lambda^{-1} C(M)N^s \sim \sigma.
\]
By using the perturbation lemma with \( B = \lambda \) and \( T = \lambda^2 t \) we have

\[
\|a(T)\|_{H^s} \geq \|b^\lambda(T)\|_{H^s} - \|a(T) - b^\lambda(T)\|_{H^s} = I_1 - I_2.
\]

We want \( I_2 << 1 \) and \( I_1 > K \). For the first

\[
l_2 \leq \|a(T) - b^\lambda(T)\|_{l^1(\mathbb{Z}^2)} \left( \sum_{n \in \Delta} |n|^{2s} \right)^{1/2} \lesssim \lambda^{-1-\delta} \left( \sum_{n \in \Delta} |n|^{2s} \right)^{1/2}.
\]

As above

\[ l_2 \lesssim \lambda^{-1-\delta} C(M) N^s \]

At this point we need to pick \( \lambda \) and \( N \) so that

\[
\|a(0)\|_{H^s} = \lambda^{-1} C(M) N^s \sim \sigma \quad \text{and} \quad l_2 \lesssim \lambda^{-1-\delta} C(M) N^s \ll 1
\]

and thanks to the presence of \( \delta > 0 \) this can be achieved by taking \( \lambda \) and \( N \) large enough.
**ESTIMATING $I_1$**

It is important here that at time zero one starts with a fixed nonzero datum, namely $\|a(0)\|_{H^s} = \|b^{\lambda}(0)\|_{H^s} \sim \sigma > 0$. In fact we will show that

$$I_1^2 = \|b^{\lambda}(T)\|_{H^s}^2 \geq \frac{K^2}{\sigma^2} \|b^{\lambda}(0)\|_{H^s}^2 \sim K^2.$$  

If we define for $T = \lambda^2 t$

$$R = \frac{\sum_{n \in \Delta} \left| b_n^{\lambda}(\lambda^2 t) \right|^2 |n|^{2s}}{\sum_{n \in \Delta} \left| b_n^{\lambda}(0) \right|^2 |n|^{2s}},$$

then we are reduced to showing that $R \gtrsim K^2 / \sigma^2$. Now recall the notation

$$\Delta = \Delta_1 \cup \ldots \cup \Delta_M \quad \text{and} \quad \sum_{n \in \Delta_j} |n|^{2s} = Q_j.$$
More on Estimating $I_1$

Using the fact that by the theorem on Arnold Diffusion (approximately) one obtains $b_j(T) = 1 - \epsilon$ if $j = M - 2$ and $b_j(T) = \epsilon$ if $j \neq M - 2$, it follows that

$$
R = \frac{\sum_{i=1}^M \sum_{n \in \Delta_i} |b_i^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^M \sum_{n \in \Delta_i} |b_i^\lambda(0)|^2 |n|^{2s}} \\
\geq \frac{Q_{M-2}(1 - \epsilon)}{(1 - \epsilon)Q_3 + \epsilon Q_1 + \ldots + \epsilon Q_M} \sim \frac{Q_{M-2}(1 - \epsilon)}{Q_{M-2} \left[ (1 - \epsilon) \frac{Q_3}{Q_{M-2}} + \ldots + \epsilon \right]}
$$

and the conclusion follows from one the properties of the sets $\Delta_j$:

$$
Q_{M-2} = \sum_{n \in \Delta_{M-2}} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n \in \Delta_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3.
$$
Can one obtain a stronger result? We believe that by "concatenating" infinitely many solutions like the one described above one may be able to obtain a solution \( u \) for \( NLS(\mathbb{T}^2) \) such that

\[
\|u(t)\|_{H^s} \sim C_s \log(|t|), \quad \text{as} \quad t \to \pm \infty.
\]