Weak Turbulence for a 2D periodic Schrödinger equation

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The NLS Initial Value Problem

We consider the defocusing initial value problem:

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \text{ where } \ x \in \mathbb{T}^2, \, \mathbb{R}^2. \end{cases} \qquad (NLS(\mathbb{T}^2))$$

Smooth solution $u(x,t)$ exists globally and

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$

$$\text{Energy} = E(u) = \int (\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4)\, dx = E(0)$$

In particular if $f_t(\xi) = |\hat{u}(t,\xi)|^2$ then the area of the subgraph of $f_t(\xi)$ remains constant. On the other hand the shape of the subgraph may change in time, in particular in time most of the area may concentrate on very high.

Informal definition of weak turbulence

Definition

Weak turbulence is the phenomenon that describe the shifting over time of the mass of global solutions into increasingly high frequencies.

This shift is also called forward cascade.

- One way of measuring weak turbulence is to consider the function

$$g_s(t) = \|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,\xi)|^2 |\xi|^{2s} d\xi$$

  and prove that it griws for large times t.

- Weak turbulence is incompatible with scattering and complete integrability.

Explanation of second item

scattering: In this context scattering (at $+\infty$) means that for any global solution $u(t,x) \in H^s$ there exists $u_0^+ \in H^s$ such that, if $S(t)$ is the linear Schrödinger operator, then

$$\lim_{t \to +\infty} [u(t,x) - S(t)u_0^+(x)] = 0$$

in H^s sense. Since $\|S(t)u_0^+\|_{H^s} = \|u_0^+\|_{H^s}$, it follows that $g_s(t) = \|u(t)\|_{\dot{H}^s}^2$ will not grow.
complete integrability: For example the 1d equation

$$(i\partial_t + \Delta)u = -|u|^2 u$$

is integrable in the sense taht it admits infinitely many conservation laws. Combining them in the right way one gets that $g_s(t) = \|u(t)\|_{\dot{H}^s}^2 \leq C_s$ for all times.

Some numerical results

Some theoretical results

- Bourgain: (late 90's)
  For the periodic IVP $NLS(\mathbb{T}^2)$ one can prove
  $$g_s(t) = \|u(t)\|_{\dot{H}^s}^2 \leq C_s|t|^{4s}.$$
  The idea here is to improve the local estimate for $t \in [-1, 1]$
  $$\|u(t)\|_{H^s} \leq C_s\|u(0)\|_{H^s}, \quad \text{for } C_s >> 1$$
  to the better one
  $$\|u(t)\|_{H^s} \leq 1\|u(0)\|_{H^s} + C_s\|u(0)\|_{H^s}^{1-\delta} \quad \text{for } C_s >> 1,$$
  for some $\delta > 0$. This last one in fact gives
  $$\|u(t)\|_{H^s} \leq C_s|t|^{1/\delta}.$$
  For similar result see also Staffilani, Colliander-Delort-Kenig-Staffilani and a recent result of W.M. Wang.

More theoretical results

- Bourgain: (late 90's)
  Given $m, s >> 1$ there exists $\tilde{\Delta}$ such that a global solution $u(x,t)$ to the modified wave equation

$$(\partial_{tt} - \tilde{\Delta})u = u^p$$

such that

$$\|u(t)\|_{H^s} \sim |t|^m.$$

Conjecture

Solutions to dispersive equations on $\mathbb{R}^d$ DO NOT exhibit weak turbulace. Solutions to dispersive equations on $\mathbb{T}^d$ DO exhibit weak turbulence. In particular for NLS$(\mathbb{T}^2)$

$$g_s(t) = \|u(t)\|^2_{\dot{H}^s} \sim log(t).$$

Main Theorem

Theorem (Colliander-Keel-Staffilani-Takaoka-Tao)

Let $s > 1$, $k >> 1$ and $0 < \sigma < 1$ be given. Then there exists a global smooth solution $u(x,t)$ to

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \ where \ \ x \in \mathbb{T}^2, \end{cases} \qquad (NLS(\mathbb{T}^2))$$

and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma$$

and

$$g_s(t) = \|u(t)\|^2_{\dot{H}^s} \geq K.$$

Ingredients for the proof

- Reduction to a resonant problem
- Construction of a special finite set Δ of frequencies
- Reduction to a resonant, finite dimensional Toy Model
- Arnold diffusion for the Toy Model
- A perturbation lemma
- A scaling argument

Reduction to a resonant problem

We consider the gauge transformation

$$v(t,x) = e^{-i2Gt}u(t,x),$$

for $G \in \mathbb{R}$. If u solves $NLS(\mathbb{T}^2)$ above, then v solves the equation

$$(-i\partial_t + \Delta)v = (2G + v)|v|^2. \qquad ((NLS)_G)$$

We make the ansatz

$$v(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{i(\langle n,x\rangle + |n|^2 t)}.$$

Now the dynamics is all recast trough $a_n(t)$:

$$-i\partial_t a_n = 2Ga_n + \sum_{n_1-n_2+n_3=n} a_{n_1}\bar{a}_{n_2}a_{n_3}e^{i\omega_4 t}$$

where $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$.

The *FNLS* system

By choosing

$$G = -\|v(t)\|_{L^2}^2 = -\sum_k |a_k(t)|^2$$

which is constant from the conservation of the mass, one can rewrite the equation above as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3 \in \Gamma(n)} a_{n_1}\bar{a}_{n_2}a_{n_3}e^{i\omega_4 t}$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 \ / \ n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n\}.$$

From now on we will be refering to this system as the *FNLS* system, with the obvious connection with the original $NLS(\mathbb{T}^2)$ equation.

The *RFNLS* system

We define the set

$$\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) \,/\, \omega_4 = 0\}.$$

The geometric interpretation for this set is the following: If n_1, n_2, n_3 are in $\Gamma_{res}(n)$, then these four points represent the vertices of a rectangle in $\mathbb{Z}^2$.
We finally define the Resonant Truncation *RFNLS* to be the system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \bar{b}_{n_2} b_{n_3} e^{i\omega_4 t}.$$

It is time now to define the special set Δ of finite frequencies where we will define the initial data.

A special set of frequencies Δ

We call this set $\Delta \subset \mathbb{Z}^2$ and we ask for it several properties. In particular $\Delta = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_M$, that is M generations of nuclear families Δ_j. If $n_1, n_2, n_3, n_4 \in \Delta$, then they represent the vertices of a rectangle such that n_1 and n_3 are in Δ_j ("parents") and n_2, n_4 are in Δ_{j+1} ("children"). The interactions among these families follow the these rules:

- Existence and uniqueness of spouse and children: For any $1 \leq j < M$ and $n_1 \in \Delta_j$ there exist a unique $n_3 \in \Delta_j$ and n_2, $n_4 \in \Delta_{j+1}$ (up to permutations).
- Existence and uniqueness of siblings and parents: For any $1 < j \leq M$ and $n_2 \in \Delta_j$ there exist a unique $n_4 \in \Delta_j$ and n_1, $n_3 \in \Delta_{j-1}$ (up to permutations).
- No incest: The sibling at frequency n is never equal to its spouse.
- Faithfulness: Apart from nuclear families Δ does not contain other rectangles.

More properties for the set Δ

- Given $\sigma << 1$ and $K >> 1$, there exist M and $\Delta = \Delta_1 \cup .... \cup \Delta_M$ as above and
  a)

  $$\sum_{n \in \Delta_M} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n \in \Delta_1} |n|^{2s}.$$

  b) If $N = N(\sigma, K)$ is large enough then Δ consists of $M \times 2^{M-1}$ disjoint frequencies n with $|n| > N = N(\sigma, K)$ and the last frequency in Δ_M is of size $C(M)N$ with the first in Δ_1 is of size N. We call N the Inner Radius of Δ.

The system $RFNLS_\Delta$

The final propriety that we ask for the set Δ is that

- Δ is closed under resonant interactions:

$$n_1, n_2, n_3 \in \Delta \cap \Gamma_{res}(n) \implies n \in \Delta.$$

Remark

If Δ is closed under resonant interaction and if $b_n(0)$ has support in Δ, then the solution $b_n(t)$ of (RFNLS) on $[0, T]$ has also support in Δ. To see this one just uses Gronwall's estimate on $\sum_{n \notin \Delta} |b_n(t)|^2$.

We can then define the finite dimension resonant truncated system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1,n_2,n_3 \in \Delta \cap \Gamma_{res}(n)} b_{n_1} b_{n_2} b_{n_3}. \qquad ((RFNLS_\Delta))$$

The ODE system (Toy Model)

Remark

If we go back to (RFNLS) one can easily see that
$b_n(t) = b_m(t)$, *for any* $m, n \in \Delta_j$. *We call this the ''Intergenerational equality''.*

Using all these properties for Δ we can identify

$$(b_n(t))_{\{n \in \mathbb{Z}^2\}} = (b_j(t))_{\{j=1,\ldots,M\}}$$

and reduce $(RFNLS_\Delta)$ to the the system

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2 \bar{b_j}(t) - 2b_{j+1}(t)^2 \bar{b_j}(t), \qquad ((ODE))$$

with the boundary condition

$$b_0(t) = b_{M+1}(t) = 0. \qquad ((BC))$$

Conservation laws for the *ODE* system

The following are conserved quantities for (*ODE*)

$$\textit{Mass} \ \sum_j |b_j(t)|^2 = C_0$$

$$\textit{Momentum} \ \sum_j |b_j(t)|^2 \sum_{n\in\Delta_j} n = C_1,$$

and if

$$\textit{Kinetic Energy} = \sum_j |b_j(t)|^2 \sum_{n\in\Delta_j} |n|^2$$

$$\textit{Potential Energy} = \frac{1}{2}\sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2 |b_{j+1}(t)|^2,$$

then

$$\textit{Energy} = \textit{Kinetic Energy} + \textit{Potential Energy} = C_2.$$

Arnold Diffusion for *ODE*: the set up

Global well-posedness for *ODE* is not an issue. We define

$$\Sigma = \{x \in \mathbb{C}^M \ / \ |x|^2 = 1\} \text{ and } \ W(t) : \Sigma \to \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of *ODE*. It is easy to see that for any $b \in \Sigma$

$$\partial_t |b_j|^2 = 4\Re(i\bar{b_j}^2(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 \Longrightarrow b_j(t) = 0, \quad \text{for all} \quad t \in [0, T].$$

If moreover we define the torus

$$\Pi_j = \{(b_1, ..., b_M) \in \Sigma \, / \, |b_j| = 1, \ b_k = 0, \ k \neq j\}$$

then

$$W(t)\Pi_j = \Pi_j \ \text{ for all } j = 1, ..., M$$

(Π_j is invariant).

Arnold Diffusion for *ODE*

Theorem

(Arnold Diffusion)
Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of Π_3 and x_{M-2} within ϵ of Π_{M-2} and a time t such that

$$W(t)x_3 = x_{M-2}.$$

Remark

$W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated at mode $j = 3$ at some time t_0 and then arbitrarily concentrated at mode $j = M - 2$ at later time t.

Intuition

Consider $M = 2$. Then *ODE* is of the form

$$\begin{aligned} \partial_t b_1 &= -i|b_1|^2 b_1 + 2i\bar{b_1} b_2^2 \\ \partial_t b_2 &= -i|b_2|^2 b_2 + 2i\bar{b_2} b_1^2. \end{aligned}$$

This system has explicit solution

$$b_1(t) = \frac{e^{-it}}{\sqrt{1 + e^{2\sqrt{3}t}}}\omega \quad b_2(t) = \frac{e^{-it}}{\sqrt{1 + e^{-2\sqrt{3}t}}}\omega^2,$$

where $\omega = e^{2i\pi/3}$ (cube root of unity). Since

$$\lim_{t \to +\infty} |b_1(t)| = 0 \quad \text{and} \quad \lim_{t \to +\infty} |b_2(t)| = 1$$

and

$$\lim_{t \to -\infty} |b_2(t)| = 0 \quad \text{and} \quad \lim_{t \to -\infty} |b_1(t)| = 1.$$

More Intuition

It follows that $(b_1, b_2) \in \Pi_2$ at $t = +\infty$ and $(b_1, b_2) \in \Pi_1$ at $t = -\infty$. So with an infinite amount of time one can go from Π_1 to Π_2 and viceversa. A suitable perturbation of Π_i replacing the tori Π_i will be the key in proving diffusion in finite time:

A picture here?

A perturbation lemma

Lemma

Let $\Delta \subset \mathbb{Z}^2$ introduced above. Let $B >> 1$ and $\delta > 0$ small and fixed. Let $t \in [0, T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Delta)$ solving $RFNLS_\Delta$ such that

$$\|b(t)\|_{l^1} \lesssim B^{-1}.$$

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of RFNLS such that

$$a(0) = b(0), \quad \textit{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta},$$

for any $t \in [0, T]$.

Proof.

This is a standard perturbation lemma proved by checking that the "non resonant" part of the nonlinearity remains small enough. $\square$

Recasting the main theorem

With all the notations and reductions introduced we can now recast the main theorem in the following way:

Theorem

For any $0 < \sigma << 1$ *and* $K >> 1$ *there exists a complex sequence* (a_n) *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ *of* (*FNLS*) *and* $T > 0$ *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$

Scaling argument

In order to be able to use the Arnold Diffusion to move mass from lower frequencies to higher ones and start with a small data we ned to introduce scaling. Consider in $[0, t]$ the solution $b(t)$ of the system $RFNLS_{\Delta}$ with initial datum b_0. Then the rescaled function

$$b^{\lambda}(t) = \lambda^{-1} b(\frac{t}{\lambda^2})$$

solves the same system with datum $b_0^{\lambda} = \lambda^{-1} b_0$.
We then first pick the complex vector $b(0)$ that was found in the theorem on Arnold Diffusion. For simplicity let's assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} b_j^{\lambda}(0) & \text{for any} \quad n \in \Lambda_j \\ 0 & \text{otherwise .} \end{cases}$$

Estimating the size of $(a(0))$

By definition

$$\left(\sum_{n\in\Delta}|a_n(0)|^2|n|^{2s}\right)^{1/2} = \lambda^{-1}\left(\sum_{j=1}^{M}|b_j(0)|^2\left(\sum_{n\in\Delta_j}|n|^{2s}\right)\right)^{1/2} = \lambda^{-1}Q_3,$$

where the last equality follows from defining

$$\sum_{n\in\Delta_j}|n|^{2s} = Q_j,$$

and the definition of $a_n(0)$ given above. At this point we use the proprieties of the set Δ to estimate $Q_3 = C(M)N$, where N is the inner radius of Δ. We then conclude that

$$\left(\sum_{n\in\Delta}|a_n(0)|^2|n|^{2s}\right)^{1/2} = \lambda^{-1}C(M)N^s \sim \sigma.$$

Estimating the size of $(a(T))$

By using the perturbation lemma with $B = \lambda$ and $T = \lambda^2 t$ we have

$$\|a(T)\|_{H^s} \geq \|b^\lambda(T)\|_{H^s} - \|a(T) - b^\lambda(T)\|_{H^s} = I_1 - I_2.$$

We want $I_2 << 1$ and $I_1 > K$. For the first

$$I_2 \leq \|a(T) - b^\lambda(T)\|_{l^1(\mathbb{Z}^2)} \left(\sum_{n\in\Delta} |n|^{2s}\right)^{1/2} \lesssim \lambda^{-1-\delta} \left(\sum_{n\in\Delta} |n|^{2s}\right)^{1/2}.$$

As above

$$I_2 \lesssim \lambda^{-1-\delta} C(M) N^s$$

At this point we need to pick λ and N so that

$$\|a(0)\|_{H^s} = \lambda^{-1} C(M) N^s \sim \sigma \quad \text{and} \quad I_2 \lesssim \lambda^{-1-\delta} C(M) N^s << 1$$

and thanks to the presence of $\delta > 0$ this can be achieved by taking λ and N large enough.

Estimating I_1

It is important here that at time zero one starts with a fixed non zero datum, namely $\|a(0)\|_{H^s} = \|b^\lambda(0)\|_{H^s} \sim \sigma > 0$. In fact we will show that

$$I_1^2 = \|b^\lambda(T)\|_{H^s}^2 \geq \frac{K^2}{\sigma^2}\|b^\lambda(0)\|_{H^s}^2 \sim K^2.$$

If we define for $T = \lambda^2 t$

$$R = \frac{\sum_{n\in\Delta} |b_n^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{n\in\Delta} |b_n^\lambda(0)|^2 |n|^{2s}},$$

then we are reduce to showing that $R \gtrsim K^2/\sigma^2$. Now recall the notation

$$\Delta = \Delta_1 \cup ..... \cup \Delta_M \quad \text{and} \quad \sum_{n\in\Delta_j} |n|^{2s} = Q_j.$$

More on Estimating I_1

Using the fact that by the theorem on Arnold Diffusion (approximately) one obtains $b_j(T) = 1 - \epsilon$ if $j = M-2$ and $b_j(T) = \epsilon$ if $j \neq M-2$, it follows that

$$
\begin{aligned}
R &= \frac{\sum_{i=1}^{M} \sum_{n \in \Delta_i} |b_i^{\lambda}(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^{M} \sum_{n \in \Delta_i} |b_i^{\lambda}(0)|^2 |n|^{2s}} \\
&\geq \frac{Q_{M-2}(1-\epsilon)}{(1-\epsilon)Q_3 + \epsilon Q_1 + \dots . + \epsilon Q_M} \sim \frac{Q_{M-2}(1-\epsilon)}{Q_{M-2}\left[(1-\epsilon)\frac{Q_3}{Q_{M-2}} + \dots . + \epsilon\right]} \\
&\gtrsim \frac{(1-\epsilon)}{(1-\epsilon)\frac{Q_3}{Q_{M-2}}} = \frac{Q_{M-2}}{Q_3}
\end{aligned}
$$

and the conclusion follows from one the properties of the sets Δ_j:

$$
Q_{M-2} = \sum_{n \in \Delta_{M-2}} |n|^{2s} \gtrsim \frac{K^2}{\sigma^2} \sum_{n \in \Delta_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3 .
$$

Conclusions

Can one obtain a stronger result? We believe that by "concatenating" infinitely many solutions like the one described above one may be able to obtain a solution u for $NLS(\mathbb{T}^2)$ such that

$$\|u(t)\|_H^s \sim C_s \log(|t|), \quad \text{as} \quad t \to \pm\infty.$$