Low Regularity Aspects of NLS Blowup

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1. Blowup Solutions Exist

We consider the Cauchy problem for L^2 critical focusing NLS:

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^-(\mathbb{R}^2))$$

The solution has an L^2-invariant dilation symmetry

$$u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

Time invariant conserved quantities:

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx - \frac{1}{2} |u(t)|^4 dx.$$

$NLS_3^-(\mathbb{R}^2)$ H^1-GWP THEORY

- Weinstein's H^1-GWP mass threshold for $NLS_3^-(\mathbb{R}^2)$:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \longmapsto u,\, T^* = \infty,$$

based on optimal Gagliardo-Nirenberg inequality on $\mathbb{R}^2$

$$\|u\|_{L^4}^4 \leq \left[\frac{2}{\|Q\|_{L^2}^2}\right] \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

- Q is the ground state solution to $-Q + \Delta Q = -Q^3$.
- The ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$ is

$$u(t,x) = e^{it} Q(x).$$

Pseudoconformal Symmetry

- Pseudoconformal transformation:
$$\mathcal{PC}[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^{d/2}} e^{\frac{i|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right),$$
- $\mathcal{PC}$ is L^2-critical *NLS* solution symmetry:
Suppose $0 < t_1 < t_2 < \infty$. If
$$u : [t_1, t_2] \times \mathbb{R}^2_x \to \mathbb{C} \text{ solves } NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$$
then
$$\mathcal{PC}[u] = v : [-t_1^{-1}, -t_2^{-1}]_\tau \times \mathbb{R}^2_y \to \mathbb{C}$$
solves
$$i\partial_\tau v + \Delta_y v = \pm|v|^{4/d} v.$$
- $\mathcal{PC}$ is an L^2-Strichartz isometry:
If $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ then
$$\|\mathcal{PC}[u]\|_{L^q_\tau L^r_y([-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d)} = \|u\|_{L^q_t L^r_x([t_1, t_2] \times \mathbb{R}^d)}.$$

Explicit Blowup Solutions

- The *pseudoconformal* image of ground state soliton $e^{it}Q(x)$,

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}},$$

  is an explicit blowup solution.

- S has minimal mass:

$$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in S is conically concentrated into a point.

- Minimal mass H^1 blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

Many non-explicit Blowup Solutions

- Suppose $a : \mathbb{R}^2 \to \mathbb{R}$. Form virial weight

$$V_a = \int_{\mathbb{R}^2} a(x)|u|^2(t,x)dx$$

and

$$\partial_t V_a = M_a(t) = \int_{\mathbb{R}^2} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi)dx.$$

Conservation identities lead to the generalized virial identity

$$\partial_t^2 V_a = \partial_t M_a = \int_{\mathbb{R}^2} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) - a_{jj}|u|^4 dx.$$

- Choosing $a(x) = |x|^2$ produces the variance identity

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2|u(t,x)|^2 dx = 16H[u_0].$$

- $H[u_0] < 0, \int |x|^2|u_0(x)|^2 dx < \infty$ blows up.
- How do these solutions blow up?

Mass Concentration Property: H^1 theory

H^1 Theory of Mass Concentration

- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\liminf_{t \nearrow T^*} \int_{|x|<(T^*-t)^{1/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

  [Merle-Tsutsumi]
- H^1 blowups parabolically concentrate at least the ground state mass. Explicit blowups S concentrate mass much faster.
- Fantastic recent progress on the H^1 blowup theory. [Merle-Raphaël]

Mass Concentration Property: L^2 Theory

L^2 Theory of Mass Concentration

- $L^2 \ni u_0 \longmapsto u$, $T^* < \infty$ implies

$$\limsup_{t \nearrow T^*} \sup_{cubes\ I, side(I) \leq (T^* - t)^{1/2}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

  [Bourgain]

  L^2 blowups parabolically concentrate some mass.

- For large L^2 data, do there exist tiny concentrations?
- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
- Upgrading $\limsup$ into $\liminf$ appears challenging.

L^2 Critical Case: Conjectures/Questions

Consider focusing $NLS_3^-(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass**

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies^{???} u_0 \longmapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$

  (Also, L^2 solutions of $NLS_3^+(\mathbb{R}^2)$ satisfy$^{???}$ $\|u\|_{L^4_{tx}} < \infty$.)
- **Minimal Mass Blowup Characterization**

  $$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \longmapsto u, T^* < \infty \implies^{???} u = S,$$

  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in H^s for $s < 1$.
- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.
- **Are there any general upper bounds?** $\limsup$ **vs.** $\liminf$ **?**

L^2 Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

  H^s-blowup solutions concentrate ground state mass. [C-Raynor-C.Sulem-Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies \exists\ t_n \nearrow T^*$ s.t. $u(t_n) \to Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence). For H^s blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an H^s analog of an H^1 result of [Weinstein] which preceded the minimal H^1 blowup solution characterization.

Progress Toward Conjectures

- Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

$$\limsup_{t\nearrow T^*} \sup_{cubes\ I, side(I)\leq(T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}} \int_I |u(t,x)|^2 dx \geq \|u_0\|_{L^2}^{-M}.$$

[C-Roudenko]

2. Ground State Mass Concentration for H^s

Theorem (C-Raynor-Sulem-Wright)

For $0.86 \sim \frac{1}{5}(1+\sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty \implies$

$$\limsup_{t \nearrow T^*} \int_{|x|<(T^*-t)^{s/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

- $\{radial\}$ removed by concentration compactness. [Tzirakis] $NLS_5^-(\mathbb{R})$
- Higher dimension generalization $NLS_{1+\frac{4}{d}}^-(\mathbb{R}^d)$. [Visan-Zhang]

Ground State Mass Concentration for H^1

Recall [Merle-Tsutsumi]. $H^1 \cap \{radial\} \ni u_0 \longmapsto u$ with $T^* < \infty$.

- **Rescalings (weakly) converge to asymptotic profile.**

  Consider $\{u(t_n, \cdot)\}_{n\in\mathbb{N}} = \{u_n(\cdot)\}_{n\in\mathbb{N}}$ along $t_n \nearrow T^*$. Form

  $$v_n(\cdot) = \lambda_n^{-1} u_n(\lambda_n^{-1}(\cdot))$$

  with $\lambda_n = \|\nabla u_n\|_{L^2} \gtrsim (T^* - t_n)^{-1/2}$ so that $\|\nabla v_n\|_{L^2} = 1$.
  Thus, $\exists\ v \in H^1$ such that $v_n \rightharpoonup v$ in H^1 along subsequence.
- **Compactness and energy of rescaled asymptotic object.**

  Radial & Rellich Compactness $\implies$ $v_n \to v$ strongly in L^4.
  $|E[v_n]| = \lambda_n^{-2} |E[u(t_n)]| \to 0 \implies E[v] \leq 0$.
- $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$; **undo scaling.**

Ground State Mass Concentration for H^1

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Ground State Mass Concentration for H^s

We imitate the [Merle-Tsutsumi] result using modified energy.

- **Blowup Parameter:**

$$\lambda(t) = \|u(t)\|_{H^s};\ \Lambda(t) = \sup_{\tau \in [0,t]} \lambda(\tau).$$

- **Modified Blowup Parameter:**

$$\sigma(t) = \|I\langle\nabla\rangle u(t)\|_{L^2};\ \Sigma(t) = \sup_{\tau \in [0,t]} \sigma(\tau).$$

Recall,

$$\|f\|_{H^s} \leq \|I\langle\nabla\rangle f\|_{L^2} \leq N^{1-s}\|f\|_{H^s}.$$

Thus, $E[v] \leq 0 \implies \|v\|_{L^2} \geq \|Q\|_{L^2}$.

Ground State Mass Concentration for H^s

Lemma (Modified Kinetic $\gg$ Modified Total Energy)

$\forall\ s > 0.86$ if $H^s \ni u_0 \longmapsto u$ on maximal $[0, T^*)$ then $\forall T < T^*\ \exists\ N = N(T)$ such that

$$|E[I_{N(T)}u(T)]| \leq C_0 \Lambda(T)^{p(s)}$$

with $p(s) < 2$ and $C_0 = C_0(s, T^*, \|u_0\|_{H^s})$.

- Modified Kinetic Energy $\gg$ Modified Total Energy.
- $N(T) = C\Lambda(T)^{\frac{p(s)}{2(1-s)}}$.
- Proof based on almost conservation; multilinear analysis.

Ground State Mass Concentration for H^s

1. **Rescale by modified kinetic energy.**
   Choose any *maximizing sequence* $t_n \nearrow T^*$ satisfying $\|u(t_n)\|_{H^s} = \Lambda(t_n)$. Define $v_n(y) = \sigma_n^{-1} I_{N(t_n)} u(t_n, \sigma_n^{-1} y)$ where $N(t_n)$ is as in the Lemma.
2. **Weak convergence and L^4 compactness.**
   Rescaling $\implies$ $\|\nabla v_n\|_{H^1} \to 1$ so $\exists\ v \in H^1$ s.t. $v_n \rightharpoonup v$ along subsequence. Radial & Rellich $\implies$ $v_n \to v$ strongly L^4.
3. **Energy of asymptotic object.**
   $|E[v_n]| = \sigma_n^{-2} |E[I_N u_n]| \leq \sigma_n^{-2} \Lambda^{p(s)}(t_n) \leq (\Lambda(t_n))^{p(s)-2} \to 0.$
4. **Undo the rescaling.**
   Unravelling scaling using lower bound $\sigma_n \gtrsim (T^* - t_n)^{-s/2}$ completes proof.

3. Concentration & Strichartz Explosion

- Ground state soliton $u(t,x) = e^{it} Q(x)$ satisfies

$$\|u\|_{L^4([j,j+1]_t \times \mathbb{R}^2_x)} = \eta = O(1), \ \forall\, j \in \mathbb{N}.$$

- L^4-isometry & explicit $S = \mathcal{PC}[e^{it} Q] \sim |\tau|^{-1} Q(\tau^{-1} y) e^{i\cdots}$,

$$\|S\|_{L^4([-\frac{1}{j}, -\frac{1}{j+1}]_\tau \times \mathbb{R}^2_y)} = \eta, \ \forall\, j \in \mathbb{N}.$$

- Thus, $\|S\|_{L^4([-1,t] \times \mathbb{R}^2)} \sim \frac{1}{|t|}$; Mass concentrated in $|y| \lesssim |t|$.
- Contrast with [Merle-Tsutsumi], [Bourgain] Concentration: $\|u\|_{L^4([-1,t]\times\mathbb{R}^2)} \nearrow \infty \implies$ Mass concentrated in $|y| \lesssim |t|^{1/2}$.
- Observation?
  Strichartz explosion rate $= f$(concentration window size).

Heuristic: Window size & L^4 Explosion

- When $\|u\|_{L^4([t_n,t_{n+1}]\times\mathbb{R}^2)} \sim \eta$ [Bourgain] essentially shows parabolic concentration: $\exists\ t_n^* \in [t_n, t_{n+1}]$ and $x_0 \in \mathbb{R}^2$ where

$$\int_{|x-x_0|\lesssim|t_{n+1}-t_n|^{1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}.$$

- In [C-Roudenko], we observe (overstated!):

$$\|u\|_{L^4_{[0,T^*-t]\times\mathbb{R}^2}} := f(T^*-t) \nearrow \infty \text{ as } t \nearrow T^*$$

$$\Updownarrow$$

$$\sup_{x_0\in\mathbb{R}^2} \int_{|x-x_0|\lesssim[-\partial_t f(T^*-t)]^{-1/2}} |u(t,x)|^2 dx \gtrsim \|u_0\|_{L^2}^{-M}$$

- Why? By first order Taylor approximation, we have $\eta \sim f(T^*-t_{n+1}) - f(T^*-t_n) \sim [-\partial_t f(T^*-t_n)](t_{n+1}-t_n)$.

Strichartz Explosion $\implies$ Tight Window

Theorem (C-Roudenko)

Suppose $T^* < \infty$ *and* $\|u\|_{L^{\frac{2(d+2)}{d}}([0,t]\times\mathbb{R}^d)} \gtrsim (T^*-t)^{-\beta}$. *Then*

$$\limsup_{t\nearrow T^*} \sup_{\substack{\text{cubes } J\in\mathbb{R}^d:\\ l(J)<(T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |u(t,x)|^2\,dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

Furthermore, $\forall\ t\in(0,T^*)\ \exists$ *a cube* $\tau(t)\subseteq\mathbb{R}^d_\xi$ *of size* $l(\tau(t)) \gtrsim (T^*-t)^{-(\frac{1}{2}+\frac{\beta}{2})}$ *such that*

$$\limsup_{t\nearrow T^*} \sup_{\substack{\text{cubes } J\in\mathbb{R}^d:\\ l(J)<(T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}}} \int_J |P_{\tau(t)}u(t,x)|^2\,dx \geq \|u_0\|_{L^2}^{-c(d)}.$$

Remarks on Proof (follow [Bourgain])

- Decompose $[0, T^*)$ into $\bigcup [t_n, t_{n+1})$ on which

$$\|u\|_{L^4([t_n, t_{n+1}] \times \mathbb{R}^2)} = \eta \sim \frac{1}{100}.$$

- For $t \in [t_n, t_{n+1})$, we have $u \sim e^{i(t-t_n)\Delta} u(t_n)$.
- Strichartz Refinements and the conditions

$$\|f\|_{L^2} < \|u_0\|_{L^2}; \ \|e^{it\Delta} f\|_{L^4} > \eta$$

  spawn a spacetime tube decomposition of $e^{it\Delta} f$.
- $\exists$ **concentration time** $t_n^* \in [t_n, t_{n+1}) \ \forall \ n$.
  Thus, proof is more refined than the $\limsup$ claim.
- Taylor expansion connects $(t_{n+1} - t_n)$ with $T^* - t_n$.

Thickened Time Interval of Concentration

Lemma (Frequency localized waves persist)

Let $f \in L^2_x(\mathbb{R}^d)$ *and spt* $\hat{f} \subset [0,1]^d$ *and suppose*

$$\int_{[0,1]^d} |f(x)|^2 \, dx \geq c_1 > 0.$$

Then for $|t| < c(c_1, \|f\|_{L^2})$ *concentration persists*

$$\int_{[0,1]^d} |e^{it\Delta} f(x)|^2 \, dx \geq \frac{c_1}{2}.$$

- Frequency localization in conclusion shows concentration persists for t in an interval containing t_n^* of size $(T^* - t)^{1+\beta}$.
- Thickened concentration interval may not cover $[t_n, t_{n+1}]$.

Tight Window $\implies$ Strichartz Explosion

Let $F(t) = \|u\|^4_{L^4([0,t]\times\mathbb{R}^2)}$ and $P_{L(t)} = P_{\{|\xi|\leq L(t)\}}$.

Lemma (Pointwise Derivative Lower Bound)

Suppose $\exists\ \alpha \geq \frac{1}{2}, \epsilon > 0$ *such that*

$$\limsup_{t\nearrow T^*} \sup_{\substack{\text{cubes } J\subset\mathbb{R}^d:\\ l(J)<(T^*-t)^\alpha}} \int_J |P_{L(t)}u(t,x)|^2\,dx \geq \epsilon.$$

Then $\exists\ t_n \nearrow T^*$ *such that*

$$F'(t_n) \gtrsim (T^*-t_n)^{-2\alpha}.$$

On thickened concentration time intervals, we integrate the derivative lower bound get a Strichartz lower bound.

Cautious Remark Concerning $\liminf$

- Consider $NLS_3^-(\mathbb{R}^2)$ posed at time $t = -\epsilon$ with data

$$\phi_\epsilon(x) = e^{i\epsilon^{-1}|x|^2} e^{i\epsilon^{-1}} Q(x).$$

- Dilated explicit solution which blows up at $t = 0 = T^*$!
- The parabolic scale related to distance to blowup time is $\sqrt{\epsilon}$. For τ a cube of side $\sqrt{\epsilon}$, observe that ϕ_ϵ is non-concentrated

$$\int_\tau |\phi_\epsilon|^2 dx \lesssim \epsilon.$$

- Consider data $(1-\delta)\phi_\epsilon$....
  Phase oscillations violently influence L^2 blowup behavior.