Resonant decompositions and the I-method for the cubic NLS on R^2

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1. Cubic NLS Initial Value Problem on $\mathbb{R}^2$

We consider the initial value problems:

$$\begin{cases} (i\partial_t + \Delta)u = \pm|u|^2 u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_3^{\pm}(\mathbb{R}^2))$$

The $+$ case is called defocusing; $-$ is focusing. $NLS_3^{\pm}$ is ubiquitous in physics. The solution has a dilation symmetry

$$u^{\lambda}(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1} y).$$

which is invariant in $L^2(\mathbb{R}^2)$. This problem is L^2-critical.

Time Invariant Quantities

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t)\nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2}\int_{R^2} |\nabla u(t)|^2 dx \pm \frac{1}{2}|u(t)|^4 dx.$$

- Mass is L^2; Momentum is close to $H^{1/2}$; Energy involves H^1.
- Dynamics on a sphere in L^2; focusing/defocusing energy.
- Local conservation laws express **how** quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Frequency Localizations?

LOCAL-IN-TIME THEORY FOR $NLS_3^{\pm}(\mathbb{R}^2)$

- $\forall\ u_0 \in L^2(\mathbb{R}^2)\ \exists\ T_{lwp}(u_0)$ determined by

$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0,T_{lwp}]\times\mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$

  $\exists$ unique $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$ solving $NLS_3^{+}(\mathbb{R}^2)$.
- $\forall\ u_0 \in H^s(\mathbb{R}^2), s > 0,\ T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$ and regularity persists: $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.
- Define the maximal forward existence time $T^*(u_0)$ by

$$\|u\|_{L^4_{tx}([0,T^*-\delta]\times\mathbb{R}^2)} < \infty$$

  for all $\delta > 0$ but diverges to ∞ as $\delta \searrow 0$.
- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

Global-in-time theory?

What is the ultimate fate of the local-in-time solutions?

L^2-critical Scattering Conjecture:

$L^2 \ni u_0 \longmapsto u$ solving $NLS_3^+(\mathbb{R}^2)$ is global-in-time and

$$\|u\|_{L^4_{t,x}} < A(u_0) < \infty.$$

Moreover, $\exists\ u_\pm \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \|e^{\pm it\Delta}u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

Same statement for focusing $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

Remarks:

- Known for small data $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$.
- Known for large radial data [Killip-Tao-Visan 07].

$NLS_3^{\pm}(\mathbb{R}^2)$: Present Status for General Data

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis05]
$s > \frac{2}{5}$	$H(Iu)$ & Interaction I-Morawetz	[CGTz07]
$s > \frac{4}{13}$?	resonant cut & I-Morawetz	[–?–]

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?

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H^1 Global Well-Posedness Scheme

Consider $NLS_3^{\pm}(\mathbb{R}^2)$ with finite energy data $u_0 \in H^1$.
Classical H^1-GWP Scheme relies on three inputs:

1. LWP lifetime dependence on data norm: $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.
2. Energy controls data norm: $\|u(t)\|_{H^1}^2 \lesssim H[u(t)] + \|u(t)\|_{L^2}^2$.
3. Conservation: $H[u(t)] + \|u(t)\|_{L^2}^2 \leq C(Energy, Mass)$.

Fix arbitrary time interval $[0, T]$. Break $[0, T]$ into subintervals of uniform size $c(Energy, Mass)$ + LWP iteration $\implies$ GWP.

For $u_0 \in H^s$ with $0 < s < 1$, we may have infinite energy. Classical persistence of regularity from LWP/Duhamel only gives

$$\sup_{t \in [0, T_{lwp}]} \|u(t)\|_{H^s} \lesssim 2\|u_0\|_{H^s}$$

and LWP iteration fails due to (possible) doubling. [Bourgain98]

2. Abstract I-method Scheme for H^s-GWP

Let $H^s \ni u_0 \longmapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A smoothing operator $I = I_N : H^s \longmapsto H^1$. The NLS evolution $u_0 \longmapsto u$ induces a smooth reference evolution $H^1 \ni Iu_0 \longmapsto Iu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.
- A modified energy $\widetilde{E}[Iu]$ built using the reference evolution.

We postpone how we actually choose these objects.

First Version of the I-method: $\widetilde{E} = H[Iu]$

For $s < 1, N \gg 1$ define smooth monotone $m : \mathbb{R}^2_\xi \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N. \end{cases}$$

The associated Fourier multiplier operator, $\widehat{(Iu)}(\xi) = m(\xi)\widehat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Set $\widetilde{E}[Iu(t)] = H[Iu(t)]$. Other choices of $\widetilde{E}$ are considered later.

AC LAW DECAY AND SOBOLEV GWP INDEX

1. **Modified LWP.** Initial v_0 s.t. $\|\nabla I v_0\|_{L^2} \sim 1$ has $T_{lwp} \sim 1$.
2. **Goal.** $\forall\ u_0 \in H^s$, $\forall\ T > 0$, construct $u : [0, T] \times \mathbb{R}^2 \to \mathbb{C}$.
3. $\iff$ **Dilated Goal.** Construct $u^\lambda : [0, \lambda^2 T] \times \mathbb{R}^2 \to \mathbb{C}$.
4. **Rescale Data.** $\|I \nabla u_0^\lambda\|_{L^2} \lesssim N^{1-s} \lambda^{-s} \|u_0\|_{H^s} \sim 1$ provided we choose $\lambda = \lambda(N) \sim N^{\frac{1-s}{s}} \iff N^{1-s}\lambda^{-s} \sim 1$.
5. **Almost Conservation Law.** $\|I \nabla u(t)\|_{L^2} \lesssim H[Iu(t)]$ and

$$\sup_{t \in [0, T_{lwp}]} H[Iu(t)] \leq H[Iu(0)] + N^{-\alpha}.$$

6. **Delay of Data Doubling.** Iterate modified LWP N^α steps with $T_{lwp} \sim 1$. We obtain rescaled solution for $t \in [0, N^\alpha]$.

$$\lambda^2(N) T < N^\alpha \iff T < N^{\alpha + \frac{2(s-1)}{s}} \text{ so } s > \frac{2}{2+\alpha} \text{ suffices.}$$

First Version of the I-method: $\widetilde{E} = H[Iu]$

A Fourier analysis established the almost conservation property of $\widetilde{E} = H[Iu]$ with $\alpha = \frac{3}{2}$ which led to...

Theorem (CKSTT:MRL02)

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$.

- Same result for $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Here Q is the ground state (unique positive solution of $-Q + \Delta Q = -Q^3$).
- Fourier analysis leading to $\alpha = \frac{3}{2}$ in fact gives $\alpha = 2$ for most frequency interactions.

L^2-critical in Weighted L^2 spaces

Based on PC transformation & inspired by [Bourgain98], we have:

Theorem (Blue-C:CPAA06)

For $s \geq 0$, if $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ is GWP for $H^s(\mathbb{R}^d)$ initial data then $NLS^+_{1+\frac{4}{d}}(\mathbb{R}^d)$ is GWP and scatters for data satisfying $\langle \cdot \rangle^s u_0(\cdot) \in L^2$. The same result applies to the focusing case provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

- Thus, GWP for L^2 data $\iff$ Scattering for L^2 data.
- H^s-GWP improvements imply weighted space improvements.
- PC transformation isometry in L^2-admissible Strichartz spaces.

Remarks

- The almost conservation property

$$\sup_{t \in [0, T_{lwp}]} \widetilde{E}[Iu(t)] \leq \widetilde{E}[Iu_0] + N^{-\alpha}$$

  leads to GWP for

$$s > s_\alpha = \frac{2}{2+\alpha}.$$

- The I-method is a *subcritical method*. To prove the Scattering Conjecture at $s = 0$ via the I-method would require $\alpha = +\infty$.
- The I-method *localizes the conserved density in frequency*. Similar ideas appear in recent critical scattering results.
- There is a *multilinear corrections algorithm* for defining new choices of $\widetilde{E}$ which should have a better AC property.

Focusing Case Below the Ground State Mass

- Modified LWP lifetime is controlled by $\|I\nabla u_0\|_{L^2}$.
- The GWP scheme progresses if $\|I\nabla u(t)\|_{L^2}^2 \lesssim H[Iu(t)]$.
- Weinstein's optimal Gagliardo-Nirenberg Inequality:

$$\|w\|_{L^4}^4 \leq \frac{2}{\|Q\|_{L^2}^2}\|w\|_{L^2}^2\|\nabla w\|_{L^2}^2.$$

- I has symbol m satisfying $|m| \leq 1$ so $\|If\|_{L^2} \leq \|f\|_{L^2}$. Thus,

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies \|Iu_0\|_{L^2} < \|Q\|_{L^2}.$$

- The required control then follows:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies \|I\nabla u(t)\|_{L^2}^2 \lesssim H[Iu(t)].$$

3. Multilinear Correction Terms

(Inspired by [Coifman-Meyer]; following [CKSTT:KdV])

1. For $k \in \mathbb{N}$, define the *convolution hypersurface*

$$\Sigma_k := \{(\xi_1, \ldots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \ldots + \xi_k = 0\} \subset (\mathbb{R}^2)^k.$$

2. For $M : \Sigma_k \to \mathbb{C}$ and $u_1, \ldots, u_k$ nice, define *k-linear functional*

$$\Lambda_k(M; u_1, \ldots, u_k) := c_k \, \Re \int_{\Sigma_k} M(\xi_1, \ldots, \xi_k) \widehat{u_1}(\xi_1) \ldots \widehat{u_k}(\xi_k).$$

3. For $k \in 2\mathbb{N}$ abbreviate $\Lambda_k(M; u) = \Lambda_k(M; u, \overline{u}, \ldots, \overline{u})$.
4. $\Lambda_k(M; u)$ invariant under interchange of even/odd arguments,

$$M(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) \mapsto \overline{M}(\xi_2, \xi_1, \ldots, \xi_k, \xi_{k-1}).$$

5. We can define a symmetrization rule via group orbit.

Examples

- $$\int_x u\bar{u}u\bar{u}dx = \int (\int e^{ix\cdot\xi_1}\widehat{u}(\xi_1)d\xi_1)\dots(\int e^{ix\cdot\xi_4}\widehat{\overline{u}}(\xi_4)d\xi_4)dx$$

  $$= \int_{\xi_1,\dots,\xi_4} \left[\int_x e^{ix\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}dx\right] \widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4)d\xi_{1,\dots,4}$$

  $$= \int_{\Sigma_4} \widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4) = \Lambda_4(1; u).$$

- $$\Lambda_2(-\xi_1\cdot\xi_2; u) = \|\nabla u\|_{L^2}^2.$$

Thus, $H[u] = \Lambda_2(-\xi_1\cdot\xi_2; u) \pm \Lambda_4(\frac{1}{2}; u)$.

Time Dependence of Multilinear Forms

Suppose u nicely solves $NLS_3^+(\mathbb{R}^2)$; M is time independent, symmetric. Calculations produce the *time differentiation formula*

$$\begin{aligned}\partial_t \Lambda_k(M; u(t)) &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}(ikX(M); u(t)) \\ &= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}([ikX(M)]_{sym}; u(t)).\end{aligned}$$

Here

$$\alpha_k(\xi_1, \dots, \xi_k) := -|\xi_1|^2 + |\xi_2|^2 - \dots - |\xi_{k-1}|^2 + |\xi_k|^2$$

(so $\alpha_2 = 0$ on Σ_2) and

$$X(M)(\xi_1, \dots, \xi_{k+2}) := M(\xi_{123}, \xi_4, \dots, \xi_{k+2}).$$

We use the notation $\xi_{ab} := \xi_a + \xi_b$, $\xi_{abc} := \xi_a + \xi_b + \xi_c$, etc.

AC Quantities via Multilinear Corrections

- Abbreviate $m(\xi_j)$ as m_j. Define σ_2 s.t. $\|I\nabla u\|_{L^2}^2 = \Lambda_2(\sigma_2; u)$:

$$\sigma_2(\xi_1, \xi_2) := -\frac{1}{2}\xi_1 m_1 \cdot \xi_2 m_2 = \frac{1}{2}|\xi_1|^2 m_1^2$$

- With $\tilde{\sigma}_4$ (symmetric, time independent) to be determined, set

$$\widetilde{E} := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u).$$

- Using the time differentiation formula, we calculate

$$\partial_t \widetilde{E} = \Lambda_4(\{i\tilde{\sigma}_4\alpha_4 - i2[X(\sigma_2)]_{sym}\}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}; u).$$

We'd like to define $\tilde{\sigma}_4$ to cancel away the Λ_4 contribution.

Small Divisor Problem

Resonant interactions obstruct the natural choice:

$$\tilde{\sigma}_4 =^{?} \frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4}.$$

On Σ_4, we can reexpress $\alpha_4 = -|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 + |\xi_4|^2$ as

$$\alpha_4 = -2\xi_{12} \cdot \xi_{14} = -2|\xi_{12}||\xi_{14}| \cos \angle(\xi_{12}, \xi_{14}),$$

and

$$[2iX(\sigma_2)]_{sym} = \frac{1}{4}(-m_1^2|\xi_1|^2 + m_2^2|\xi_2|^2 - m_3^2|\xi_3|^2 + m_4^2|\xi_4|^2).$$

When all the $m_j = 1$ (so $\max_j |\xi_j| < N$), $\tilde{\sigma}_4$ is well-defined.
However, α_4 can also vanish when ξ_{12} and ξ_{14} are orthogonal.

Remark: Integrable Systems Conjecture

For $NLS_3^+(\mathbb{R})$, the resonant obstruction disappears. Thus,

$$\widetilde{E}^1 = \Lambda_2(\sigma_2) + \Lambda_4(\tilde{\sigma}_4);$$

$$\partial_t \widetilde{E}^1 = -\Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}).$$

We can then define, with $\tilde{\sigma}_6$ to be determined,

$$\widetilde{E}^2 = \widetilde{E}^1 + \Lambda_6(\tilde{\sigma}_6);$$

$$\partial_t \widetilde{E}^2 = \Lambda_6(\{i\tilde{\sigma}_6\alpha_6 - [i4X(\tilde{\sigma}_4)]_{sym}\}) + \Lambda_8([i6X(\tilde{\sigma}_6)]_{sym}).$$

Let's define

$$\tilde{\sigma}_6 = \frac{[i4X(\tilde{\sigma}_4)]_{sym}}{i\alpha_6}.$$

Remark: Integrable Systems Conjecture

Thus, we formally obtain a continued-fraction-like algorithm.

$$\tilde{\sigma}_6 = \frac{\left[i4X \left(\frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4} \right) \right]_{sym}}{i\alpha_6},$$

$$\tilde{\sigma}_8 = \frac{\left[i6X \left(\frac{\left[i4X \left(\frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4} \right) \right]_{sym}}{i\alpha_6} \right) \right]_{sym}}{i\alpha_8}, \ldots.$$

Each step gains two derivatives but costs two more factors.

Conjecture: The multipliers $\tilde{\sigma}_6, \tilde{\sigma}_8, \ldots$ are well defined and lead to better AC properties. Same for other integrable systems.

4. Resonant Decomposition

We return to $NLS_3^+(\mathbb{R}^2)$.
Since the natural choice is not well-defined, we choose

$$\tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{sym}}{i\alpha_4}\ \chi_{\Omega_{nr}}$$

where the non-resonant set $\Omega_{nr} \subset \Sigma_4$ such that

$$\Omega_{nr} := \{\max_{1\leq j\leq 4} |\xi_j| \leq N\} \cup \{|\cos\angle(\xi_{12},\xi_{14})| \geq \theta_0\}.$$

Eventually, we choose θ_0 to balance the 4-linear and 6-linear contributions to the modified energy increment. We have

$$\partial_t \widetilde{E} = \Lambda_4(\{i\tilde{\sigma}_4\alpha_4 - i2[X(\sigma_2)]_{sym}\}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}; u).$$

The 4-linear contribution is constrained to the resonant set Ω_{nr}^C.

Improved Almost Conservation Property

Lemma

If $\|u_0\|_{L^2_x(\mathbb{R}^2)} \leq A$; $E(Iu_0) \leq 1$; *u is a nice solution of* $NLS_3^+(\mathbb{R}^2)$ *on a time interval* $[0, t_0]$, *then if* $t_0 = t_0(A)$ *is small enough,*

$$\left| \int_0^{t_0} \Lambda_4([-2iX(\sigma_2)]_{sym} + i\tilde{\sigma}_4\alpha_4; u(t))\ dt \right|$$

$$+ \left| \int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{sym}; u(t))\ dt \right|$$

$$\lesssim C(A)[N^{-2+} + \theta_0^{1/2} N^{-3/2+} + \theta_0^{-1} N^{-3+}].$$

The choice $\theta_0 = N^{-1}$ produces the AC property with $\alpha = 2$.

Overview and Delicate Case of Proof

- The 4-linear contribution has multiplier

$$([-2iX(\sigma_2)]_{sym} + i\tilde{\sigma}_4\alpha_4)(\xi) = [-2iX(\sigma_2)]_{sym}\chi_{\Omega_r}$$

where the *resonant set* $\Omega_r = \Omega_{nr}^C \subset \Sigma_4$,

$$\Omega_r := \{\max(|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|) > N; |\cos\angle(\xi_{12}, \xi_{14})| < \theta_0\}.$$

- We wish to bound the associated energy incremement

$$\int_0^{T_{lwp}} \Lambda_4([-2iX(\sigma_2)]_{sym}\chi_{\Omega_r}; u)dt.$$

- The 4 factors u are dyadically decomposed. The integral is studied case-by-case based on dyadic frequency sizes.
- On Σ_4, the two largest frequencies are comparable.

Overview and Delicate Case of Proof

- Let $|\xi_j| \sim N_j \in 2^{\mathbb{Z}}$. Symmetry properties and the Ω_r constraint allow to assume

$$N_1 \sim N_2 \gtrsim N, N_2 \gtrsim N_3 \gtrsim N_4 \gtrsim 1.$$

- For most cases, suffices to use enhanced [CKSTT:MRL] and

Lemma

$\forall\ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4$,

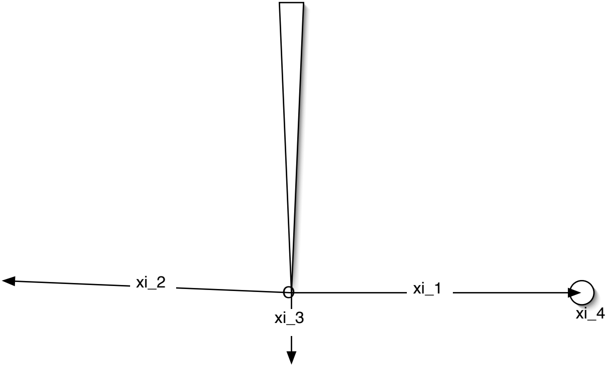
$$|[2iX(\sigma_2)]_{sym}| \lesssim \min(m_1, m_2, m_3, m_4)^2 |\xi_{12}||\xi_{14}|.$$

This follows from the mean value theorem.

Overview and Delicate Case of Proof

The most delicate case occurs in Ω_r and when

$$N_1 \sim N_2 \gg N, N_3 \gg N_4 \gtrsim 1.$$



Angle constraint in Ω_r gives better estimates based on two effects:

- Cancellation with $[X(\sigma_2)]_{sym}$,
- Angular refinement of Bilinear Strichartz.

We use a refinement exploiting spherical symmetry of m.

Lemma

Let $N_1, \ldots, N_4$ be in the delicate case with $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r$. Then

$$|[X(\sigma_2)]_{sym}| \lesssim m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2.$$

Angular Refinement of Bilinear Strichartz

Lemma (Angle Refined Bilinear Strichartz)

Let $0 < N_1 \leq N_2$ *and* $0 < \theta < \frac{1}{50}$*. Then for any* $v_1, v_2 \in X^{0,1/2+}$ *with spatial frequencies* N_1, N_2 *respectively, the spacetime function*

$$F(t,x) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(t(\tau_1+\tau_2)+x\cdot(\xi_1+\xi_2))} \times \chi_{\{|\cos\angle(\xi_1,\xi_2)|\leq\theta\}} \tilde{v}_1(\tau_1,\xi_1)\tilde{v}_2(\tau_2,\xi_2)\, d\xi_1 d\xi_2$$

obeys the bound

$$\|F\|_{L^2_{t,x}} \lesssim \theta^{1/2}\|v_1\|_{X^{0,1/2+}}\|v_2\|_{X^{0,1/2+}}.$$