The I-method with a Morawetz bootstrap

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1. Generalized Virial Identities

Cauchy Problem for Lagrangian NLS

Consider the initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = \pm F'(|u|^2)u \\ u(0,x) = u_0(x) \end{cases} \qquad (NLS_F^{\pm}(\mathbb{R}^d))$$

Remarks:

- Assume $F' \geq 0$. The $+$ case is defocusing; $-$ is focusing.
- Generalized NLS with Lagrangian derivation.
- $U(1)$ solution symmetry: $u \longrightarrow e^{i\theta}u$.

Time Invariant Quantities

The following quantities do not change with time:

$$\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.$$

$$\text{Energy} = H[u(t)] = \frac{1}{2} \int_{R^2} |\nabla u(t)|^2 dx \pm F(|u(t)|^2) dx.$$

$\Longrightarrow$ a priori **conservation controls** (defocusing case):

$$\begin{aligned} \|u\|_{L_t^\infty L_x^2} &\leq \|u_0\|_{L^2} \\ \|\nabla u\|_{L_t^\infty L_x^2} &\leq E[u_0]. \end{aligned}$$

These are very useful bounds but do not give any decay in time.

Local Conservation Laws

We consider an even more general NLS equation.

- Suppose $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{C}$ solves **generalized NLS**

$$(i\partial_t + \Delta)\phi = \mathcal{N} \qquad (GNLS(\mathbb{R}^d))$$

  for $\mathcal{N} = \mathcal{N}(t, x, \phi) : [0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$. Assume ϕ is nice.
- Not necessarily Lagrangian; No $U(1)$ symmetry.
- Express mass & momentum (non)conservation for *GNLS*.

Write $\partial_{x_j}\phi = \partial_j\phi = \phi_j$.

Local mass/momentum (non)conservation

- mass density: $T_{00} = |\phi|^2$
- momentum density/mass current: $T_{0j} = T_{j0} = 2\Im(\overline{\phi}\phi_j)$
- (linear part of the) momentum current: $L_{jk} = L_{kj} = -\partial_j\partial_k|\phi|^2 + 4\Re(\overline{\phi_j}\phi_k)$
- mass bracket: $\{f, g\}_m = \Im(f\overline{g})$
- momentum bracket: $\{f, g\}_p^j = \Re(f\partial_j\overline{g} - g\partial_j\overline{f})$

Local mass (non)conservation identity:

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

Local momentum (non)conservation identity:

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j$$

Local mass/momentum (non)conservation

Consider $\mathcal{N} = F'(|\phi|^2)\phi$ for polynomial $F : \mathbb{R}^+ \to \mathbb{R}$.

- We calculate the mass bracket

$$\{F'(|\phi|^2)\phi, \phi\}_m = \Im(F'(|\phi|^2)\phi\overline{\phi}) = 0.$$

  Thus mass is conserved for these nonlinearities.

- We calculate the momentum bracket

$$\{F'(|\phi|^2)\phi, \phi\}_p^j = -\partial_j G(|\phi|^2)$$

  where $G(z) = zF'(z) - F(z) \sim F(z)$.
  Thus the momentum bracket contributes a divergence and momentum is conserved for these nonlinearities.

Generalized Virial Identity

Let $a : \mathbb{R}^d \to \mathbb{R}$ **(virial weight)**. Form the **virial potential**

$$V_a(t) = \int_{\mathbb{R}^d} a(x)|\phi(t,x)|^2 dx.$$

Form the **Morawetz action**

$$M_a(t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\Im(\overline{\phi}\nabla\phi) dx.$$

Conservation identities lead to the **generalized virial identities**

$$\partial_t V_a = M_a + \int_{\mathbb{R}^d} a(x)\{\mathcal{N},\phi\}_m(t,x)dx,$$

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi_j}\phi_k) + 2a_j\{\mathcal{N},\phi\}^j_p dx.$$

Remarks on Virial Identities

- The virial potential is a weighted average of the mass density against the virial weight a.
- The Morawetz action is a contraction of the momentum density against ∇a. Vector fields not arising as gradients could also be considered.
- Useful estimates emerge from monotonicity and boundedness of terms in the virial identities.
- Monotone quantities provide dynamical insights.
- **Idea of Morawetz Estimates:** Cleverly choose the weight function a so that $\partial_t M_a \geq 0$ but $M_a \leq C(\phi_0)$ to obtain spacetime control on ϕ. This strategy imposes various constraints on a which suggest choosing $a(x) = |x|$.

Variance Identity

[Glassey], [Vlasov-Petrischev-Talanov]

- Consider *GNLS* with $\mathcal{N} = \pm|u|^{4/d}u$. This is the L^2 critical focusing equation $NLS^{\pm}_{1+\frac{4}{d}}(\mathbb{R}^d)$.
- Choose $a(x) = |x|^2$. Calculations reveal that

$$\partial_t^2 \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx = 16H[u(t)].$$

- In the focusing case, we can consider initial data u_0 with $H[u_0] < 0$ and finite variance. Such data must blow up in finite time.

2. A Priori Spacetime Estimates

[Lin-Strauss] Morawetz identity

Consider $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. Choose $a(x) = |x|$. Observe that a is weakly convex, $\nabla a = \frac{x}{|x|}$ is bounded, and $-\Delta\Delta a = 4\pi\delta_0$. From monotonicity $\partial_t M_a \geq 0$ and the bound $|M_a| \leq \sqrt{H[u_0]}$ emerges the **Lin-Strauss Morawetz identity**

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^3} 4\pi\delta_0(x)|\phi(t,x)|^2 + (\geq 0) + 4\frac{G(|\phi|^2)}{|x|}dxdt.$$

This implies the spacetime control estimate (centered at $x = 0$)

$$(H[u_0])^{1/2}\|u_0\|_{L^2} \gtrsim \int_0^T \int_{\mathbb{R}^3} \frac{G(|\phi|^2)}{|x|}dxdt.$$

[Morawetz] Reward & Anchor. [Ginibre-Velo] H^1-Scattering.

[Bourgain] & [Grillakis] truncation

- Let χ_{B_R} denote a smooth cutoff adapted to $B_R = \{|x| < R\}$.
- Choose cutoff virial weight $a(x) = \chi_{B_R}(x)|x|$. and calculate

$$M_a\Big|_0^T \geq \int_0^T \int_{\mathbb{R}^3} 4\pi\delta_0(x)|\phi(t,x)|^2 + 4\int_0^T \int_{|x|<R/2} \frac{G(|\phi|^2)}{|x|}dxdt$$

- $|M_a\Big|_0^T| \leq R^{-1}TH[u_0] + RH[u_0] \implies$ choose $R \sim T^{1/2} \implies$

$$\int_0^T \int_{|x|<T^{1/2}} \frac{G(|\phi|^2)}{|x|}dx \lesssim T^{1/2}\|\nabla u\|^2_{L^\infty_{[0,T]}L^2_x}.$$

[Bourgain][Grillakis]: Energy critical bubbles sparse along time axis.

Averaging over [Lin-Strauss] center?

- Translation invariance? Weight $|x|^{-1}$ difficult in proofs.
- Recenter [L-S] at fixed $y \in \mathbb{R}^d$. Set $a(x) = |x - y|$.
- Recentered Morawetz action can be expressed

$$M_y[u](t) = \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|} 2\Im(u\nabla\overline{u})(t,x)dx.$$

- Monotonicity $\partial_t M_y[u] \geq 0$: mass is repelled from any $y \in \mathbb{R}^d$.
- Can we **average with respect to center** y and obtain new translation invariant spacetime control?
- Yes, if we average against the natural density $|u(t,y)|^2$.

Interaction Morawetz via Averaging

- Define the **Morawetz interaction potential**

$$M[u](t) = \int_{\mathbb{R}^d_y} |u(t,y)|^2 M_y[u](t) dy.$$

It is bounded: $\left|M[u](t)\right| \lesssim \|u(t)\|^3_{L^2_x} \|\nabla u(t)\|_{L^2_x}$. We calculate

$$\partial_t M[u] = \int_{\mathbb{R}^d_y} |u(t,y)|^2 \{\partial_t M_y[u]\} + \{\partial_t |u(y)|^2\} M_y[u] dy.$$

- Local conservation & [L-S] $\implies$ monotonicity:
$\exists\ I, II, III, IV$ such that $I, III \geq 0$ and $II + IV \geq 0$ and
$\partial_t M[u] = I + II + III + IV$. Integrating in time gives

$$\int_0^T \int_{\mathbb{R}^3} |u(t,x)|^4 dx dt \lesssim \|u(t)\|^3_{L^\infty_T L^2_x} \|\nabla u(t)\|_{L^\infty_T L^2_x}\ .$$

2-particle interaction Morawetz

(Hassell 04)

- Suppose ϕ_1, ϕ_2 are two solutions of $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$ with $F' \geq 0$ and $x \in \mathbb{R}^3$. The "2-particle" wave function

$$\Psi(t, x_1, x_2) = \phi_1(t, x_1)\phi_2(t, x_2)$$

  satisfies an NLS-type equation on $\mathbb{R}^{1+6}$

$$(i\partial_t + \Delta_1 + \Delta_2)\Psi = [F'(|\phi_1|^2) + F'(|\phi_2|^2)]\Psi.$$

- Note that $[F'(|\phi_1|^2) + F'(|\phi_2|^2)] \geq 0$ so defocusing.
- Reparametrize $\mathbb{R}^6$ using center-of-mass coordinates $(\overline{x}, y)$ with $\overline{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3$. Note that $y = 0$ corresponds to the diagonal $x_1 = x_2 = \overline{x}$. Apply the generalized virial identity with the **choice** $a(x_1, x_2) = |y|$. Dismissing terms with favorable signs, one obtains...

2-particle interaction Morawetz

$$\begin{aligned}
\|\nabla u\|_{L^\infty_{[0,T]}L^2_x}\|u_0\|^3_{L^2} &\geq \int_0^T\int_{\mathbb{R}^6}(-\Delta_6\Delta_6|y|)|\Psi(x_1,x_2)|^2dx_1dx_2dt \\
&\geq c\int_0^T\int_{\mathbb{R}^6}\delta_{\{y=0\}}(x_1,x_2)|\phi_1(x_1)\phi_2(x_2)|^2dx_1dx_2dt \\
&\geq c\int_0^T\int_{\mathbb{R}^3}|\phi_1(t,\overline{x})\phi_2(t,\overline{x})|^2d\overline{x}dt.
\end{aligned}$$

Specializing to $\phi_1 = \phi_2$ gives the **2-particle Morawetz estimate**

$$\int_0^T\int_{\mathbb{R}^3}|\phi(t,x)|^4dxdt \leq C\|\nabla u\|_{L^\infty_{[0,T]}L^2_x}\|u_0\|^3_{L^2_x}$$

valid uniformly for all defocusing NLS equations on $\mathbb{R}^3$.

"The" 2-particle Morawetz Estimate

Efforts to extend the $L^4(\mathbb{R}_t \times \mathbb{R}^3_x)$ interaction Morawetz to the $\mathbb{R}^2_x$ setting led to...

Theorem (C-Grillakis-Tzirakis & Planchon-Vega)

Finite energy solutions of any defocusing NLS$^+(\mathbb{R}^d)$ satisfy

$$\|D^{\frac{3-d}{2}}|u|^2\|^2_{L^2_{t,x}} \lesssim \|u_0\|^3_{L^2_x}\|\nabla u\|_{L^\infty_t L^2_x}.$$

- Independently & simultaneously by [Planchon-Vega].
- Gives simple proof of H^1-scattering in mass supercritical case. [Nakanishi]
- Simplified proof extends to H^s for certain $s < 1$.
- May play a role in resolving the L^2 scattering conjecture?

4-particle Morawetz Estimate

(Hassel-Tao) [C-Holmer-Visan-Zhang]

- $\mathbb{R}^4 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : x_i \in \mathbb{R}; i = 1, 2, 3, 4\}$
  $\overline{x} =$ **center of mass** $= \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$.
  Define $y = (x_1 - \overline{x}, x_2 - \overline{x}, x_3 - \overline{x}, x_4 - \overline{x})$.
  $\mathbb{R}^4 \ni \mathbf{x} = (x_1, x_2, x_3, x_4) \iff (\overline{x}, y) \in \mathbb{R} \times \mathbb{R}^3$
- The 4-particle wave function

$$\Psi(t, \mathbf{x}) = \prod_{i=1}^{4} \phi_1(t, x_i)$$

  satisfies a defocusing NLS equation on $\mathbb{R}^{1+4}$.
- Choice of virial weight $a(\mathbf{x}) = |y|$ spawns

$$\int_0^T \int_{\mathbb{R}} |u|^8 d\overline{x} dt \lesssim \|u\|^7_{L^\infty_T L^2_x} \|\nabla u\|_{L^\infty_T L^2_x}.$$

- How does this estimate generalize to other dimensions?

3. I-Method with Morawetz Bootstrap

Application: **Subcritical scattering** for certain $NLS_p^+(\mathbb{R}^d)$.
[CKSTT], [CHVZ], [Fang-Grillakis], [C-Grillakis-Tzirakis]

- 2-particle Morawetz is an $\dot{H}^{1/4}$-critical input.
  4-particle Morawetz is an $\dot{H}^{1/8}$-critical input.
- Scaling invariant H^s for $NLS_p^+(\mathbb{R}^d)$: $s_c = \frac{d}{2} - \frac{2}{p-1}$
- When $1/4 < s_c < 1$ $\exists$ $s_* \in (s_c, 1)$ and $\forall$ $s \geq s_*$ the H^s-solutions of $NLS_p^+(\mathbb{R}^d)$ scatter.
- We obtain scattering for certain energy subcritcal ($s_c < 1$) NLS for infinite energy data of subcritical regularity ($s_c < s_*$).
- The *critical scattering conjecture* corresponds to $s_c = s_*$. This is known (for general data) only for $s_c = 1$.

I-Method with Morawetz Bootstrap

Consider $NLS^+_{2k+1}(\mathbb{R})$ (nonlinearity $+|u|^{2k}u$) for $k = 3, 4, \ldots$.
Note that $s_c = \frac{1}{2} - \frac{1}{k}$.
Define

$$s_k = \frac{8k-16}{9k-14} < 1.$$

Theorem (C-Holmer-Visan-Zhang)

$\forall\ s > s_k$, $H^s(\mathbb{R}) \ni u_0 \longmapsto u$ solving $NLS^+_{2k+1}(\mathbb{R})$ is global in time and scatters: $\exists\ u_\pm \in H^s(\mathbb{R})$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_\pm\|_{H^s(\mathbb{R})} = 0.$$

Proof treats a family of equations; Wish that $s_k \to 1/2$ as $k \to \infty$.

Finite Energy Scattering

The 4-particle $L^8(\mathbb{R}_t \times \mathbb{R}_x)$ estimate may be reexpressed:

$$\int_0^T \int_{\mathbb{R}} |u|^8 d\overline{x} dt \lesssim \|u\|^6_{L^\infty_T L^2_x} \|u\|_{L^\infty_T \dot{H}^{1/2}_x},$$

$\implies \|u\|_{L^8_{t,x}} \lesssim \|u\|_{L^\infty_t H^1_x} \implies H^1$-scattering using some interpolation.

LWP: Data size versus Spacetime slabs

- Recall that, based on a Hölder-in-time step, subcritical local-in-time theory gives

$$T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s-s_c}}.$$

- Bootstrap toward scattering: Hölder-in-time is forbidden.

I-Method & Morawetz: Bootstrap Heuristic

Suppose we have almost conservation of modified energy.

1. RHS of 4-particle $\lesssim$ almost conserved modified energy
2. $\Longrightarrow$ $L^8_{t,x}$ controlled on long time interval $t \in [0, T]$
3. $\Longrightarrow$ spacetime slab decomposition:
   $[0, T] \times \mathbb{R} = \cup_{j=1}^{J} [t_j, t_{j+1}) \times \mathbb{R}$ such that

$$\|u\|_{L^8([t_j, t_{j+1}) \times \mathbb{R})} = \eta \sim \frac{1}{100}$$

4. $\Longrightarrow$ almost conserved modified energy on $[t_j, t_{j+1}]$
5. $\Longrightarrow$ RHS of 4-particle.... bootstrap loop!

Almost Conservation Used In Bootstrap

Lemma (Almost Conservation on Slab)

Let $H^s \ni u_0 \to u$ solve $NLS^+_{2k+1}(\mathbb{R}^d)$ with $s > s_k$. Suppose we have a spacetime slab $[t_-, t_+]$ on which

$$\|u\|_{L^8([t_-,t_+]\times\mathbb{R})} \lesssim \eta$$

and $\exists\ t_0 \in [t_-, t_+]$ such that $H[I_N u(t_0)] \leq 1$. Then for large N we have almost conservation:

$$\sup_{t\in[t_-,t_+]} H[I_N u(t)] = H[I_N u(t_0)] + O(N^{-1+}).$$

Rescaling and continuity arguments glue it all together.