Resonant decompositions and the \( I \)-method for the cubic NLS on \( R^2 \)

J. Colliander

University of Toronto

Séminaire EDP, Université de Paris-Sud 11, Orsay
1 Cubic NLS on $\mathbb{R}^2$

2 The $I$-method

3 Multilinear Corrections

4 Resonant Decomposition
1. **Cubic NLS Initial Value Problem on $\mathbb{R}^2$**

We consider the initial value problems:

\[
\begin{cases}
(i\partial_t + \Delta)u = \pm |u|^2 u \\
u(0, x) = u_0(x).
\end{cases}
\] (\textit{NLS}_3^{\pm}(\mathbb{R}^2))

The $+$ case is called \textbf{defocusing}; $-$ is \textbf{focusing}. \textit{NLS}_3^{\pm} is ubiquitous in physics. The solution has a dilation symmetry

\[u^\lambda(\tau, y) = \lambda^{-1}u(\lambda^{-2}\tau, \lambda^{-1}y).\]

which is invariant in $L^2(\mathbb{R}^2)$. This problem is \textit{L}^2-\textit{critical}.
**Time Invariant Quantities**

\[
\text{Mass} = \int_{\mathbb{R}^d} |u(t,x)|^2 dx.
\]

\[
\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx.
\]

\[
\text{Energy} = H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx \pm \frac{1}{2} |u(t)|^4 dx.
\]

- Mass is $L^2$; Momentum is close to $H^{1/2}$; Energy involves $H^1$.
- Dynamics on a sphere in $L^2$; **focusing/defocusing** energy.
- Local conservation laws express **how** quantity is conserved:
  e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im (\overline{u} \nabla u)$. Frequency Localizations?
**Local-in-time theory for \( NLS_{3}^{\pm}(\mathbb{R}^{2}) \)**

- \( \forall \ u_{0} \in L^{2}(\mathbb{R}^{2}) \) \( \exists \ T_{lwp}(u_{0}) \) determined by
  \[
  \| e^{it\Delta} u_{0} \|_{L_{tx}^{4}([0, T_{lwp}] \times \mathbb{R}^{2})} < \frac{1}{100}
  \]
  such that
  \( \exists \) unique \( u \in C([0, T_{lwp}]; L^{2}) \cap L_{tx}^{4}([0, T_{lwp}] \times \mathbb{R}^{2}) \) solving \( NLS_{3}^{\pm}(\mathbb{R}^{2}) \).

- \( \forall \ u_{0} \in H^{s}(\mathbb{R}^{2}), s > 0, T_{lwp} \sim \| u_{0} \|_{H^{s}}^{-\frac{2}{s}} \) and regularity persists: \( u \in C([0, T_{lwp}]; H^{s}(\mathbb{R}^{2})) \).

- Define the maximal forward existence time \( T^{*}(u_{0}) \) by
  \[
  \| u \|_{L_{tx}^{4}([0, T^{*} - \delta] \times \mathbb{R}^{2})} < \infty
  \]
  for all \( \delta > 0 \) but diverges to \( \infty \) as \( \delta \downarrow 0 \).

- \( \exists \) small data scattering threshold \( \mu_{0} > 0 \)
  \[
  \| u_{0} \|_{L^{2}} < \mu_{0} \implies \| u \|_{L_{tx}^{4}\left(\mathbb{R} \times \mathbb{R}^{2}\right)} < 2\mu_{0}.
  \]
GLOBAL-IN-TIME THEORY?

What is the ultimate fate of the local-in-time solutions?

**L^2-critical Scattering Conjecture:**

\[ L^2 \ni u_0 \mapsto u \text{ solving } NLS^+_3(\mathbb{R}^2) \text{ is global-in-time and} \]

\[ \| u \|_{L^4_{t,x}} < A(u_0) < \infty. \]

Moreover, \( \exists u_\pm \in L^2(\mathbb{R}^2) \) such that

\[ \lim_{t \to \pm \infty} \| e^{\pm it\Delta} u_\pm - u(t) \|_{L^2(\mathbb{R}^2)} = 0. \]

Same statement for focusing \( NLS^-_3(\mathbb{R}^2) \) if \( \| u_0 \|_{L^2} < \| Q \|_{L^2}. \)

**Remarks:**

- Known for small data \( \| u_0 \|_{L^2(\mathbb{R}^2)} < \mu_0. \)
- Known for large radial data [Killip-Tao-Visan 07].
**$NLS_3^\pm(\mathbb{R}^2)$: Present Status**

<table>
<thead>
<tr>
<th>regularity</th>
<th>idea</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s &gt; \frac{2}{3}$</td>
<td>high/low frequency decomposition $H(|u|)$</td>
<td>[Bourgain98]</td>
</tr>
<tr>
<td>$s &gt; \frac{4}{7}$</td>
<td>resonant cut of 2nd energy $H(|u|)$ &amp; Interaction Morawetz</td>
<td>[CKSTT02]</td>
</tr>
<tr>
<td>$s &gt; \frac{1}{2}$</td>
<td>$H(|u|)$ &amp; Interaction $I$-Morawetz</td>
<td>[CKSTT07]</td>
</tr>
<tr>
<td>$s &gt; \frac{2}{5}$</td>
<td>$H(|u|)$ &amp; Interaction $I$-Morawetz</td>
<td>[Fang-Grillakis05]</td>
</tr>
<tr>
<td>$s &gt; \frac{4}{13}$?</td>
<td>resonant cut &amp; $I$-Morawetz</td>
<td>[CGTz07]</td>
</tr>
</tbody>
</table>

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?
**NLS\(_3^\pm(\mathbb{R}^2)\): Present Status**

<table>
<thead>
<tr>
<th>regularity</th>
<th>idea</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s &gt; \frac{2}{3})</td>
<td>high/low frequency decomposition (H(Iu))</td>
<td>[Bourgain98]</td>
</tr>
<tr>
<td>(s &gt; \frac{4}{7})</td>
<td>resonant cut of 2nd energy (H(Iu))</td>
<td>[CKSTT02]</td>
</tr>
<tr>
<td>(s &gt; \frac{1}{2})</td>
<td>(H(Iu)) &amp; Interaction Morawetz</td>
<td>[CKSTT07]</td>
</tr>
<tr>
<td>(s &gt; \frac{1}{2} + \frac{2}{5})</td>
<td>(H(Iu)) &amp; Interaction I-Morawetz</td>
<td>[Fang-Grillakis05]</td>
</tr>
<tr>
<td>(s &gt; \frac{4}{13})</td>
<td>resonant cut &amp; I-Morawetz</td>
<td>[CGTz07]</td>
</tr>
</tbody>
</table>

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume \(\|u_0\|_{L^2} < \|Q\|_{L^2}\).
- Unify theory of focusing-under-ground-state and defocusing?
Consider $NLS_3^\pm (\mathbb{R}^2)$ with finite energy data $u_0 \in H^1$. Classical $H^1$-GWP Scheme relies on three inputs:

1. **LWP lifetime dependence** on data norm: $T_{lwp} \sim \| u_0 \|_{H^s}^{-2/s}$.
2. **Energy controls data norm**: $\| u(t) \|_{H^1}^2 \lesssim H[u(t)] + \| u(t) \|_{L^2}^2$.
3. **Conservation**: $H[u(t)] + \| u(t) \|_{L^2}^2 \leq C(Energy, Mass)$.

Fix arbitrary time interval $[0, T]$. Break $[0, T]$ into subintervals of uniform size $c(Energy, Mass) +$ LWP iteration $\implies$ GWP.

For $u_0 \in H^s$ with $0 < s < 1$, we may have infinite energy. Classical persistence of regularity from LWP/Duhamel only gives

$$\sup_{t \in [0, T_{lwp}]} \| u(t) \|_{H^s} \lesssim 2\| u_0 \|_{H^s}$$

and LWP iteration fails due to (possible) doubling. [Bourgain98]
2. Abstract \textit{l}-method Scheme for $H^s$-GWP

Let $H^s \ni u_0 \mapsto u$ solve \textit{NLS} for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A \textbf{smoothing operator} $I = I_N : H^s \hookrightarrow H^1$. The \textit{NLS} evolution $u_0 \mapsto u$ induces a \textbf{smooth reference evolution} $H^1 \ni Iu_0 \mapsto Iu$ solving $I(\text{\textit{NLS}})$ equation on $[0, T_{lwp}]$.

- A \textbf{modified energy} $\tilde{E}[Iu]$ built using the reference evolution.

We postpone how we actually choose these objects.
**Abstract l-method Scheme for $H^s$-GWP**

We want $I_N$ and $\tilde{E}$ chosen to give a progressive $H^s$-GWP scheme:

1. **Lifetime dependence on data norm:** $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$. ✓
2. **$\tilde{E}$ controls data norm:** $\exists \ t_g \in \left[\frac{1}{2} T_{lwp}, T_{lwp}\right]$ s.t.
   $$\|u(t_g)\|_{H^s}^2 \lesssim \tilde{E}[lu(t_g)] + \|u(t_g)\|_{L^2}^2.$$
3. **Almost Conservation of Modified Energy:**
   $$\sup_{t \in [0, T_{lwp}]} \tilde{E}[lu(t)] \leq \tilde{E}[lu_0] + N^{-\alpha}.$$

The scheme advances over $K$ uniform sized time steps of length $O(\tilde{E}[u_0]^{-1/s})$ until the modified energy doubles

$$KN^{-\alpha} \sim \tilde{E}[lu_0].$$

This extends to solution for $t \in [0, N^\alpha E[lu_0]^{1-\frac{1}{s}}]$ which contains $[0, T]$ for large enough $N$ provided $s > s_\alpha$ with $s_\alpha < 1$. 
First Version of the $I$-method: $\tilde{E} = H[lu]$  

For $s < 1$, $N \gg 1$ define smooth monotone $m : \mathbb{R}_\xi^2 \to \mathbb{R}^+$ s.t. 

$$m(\xi) = \begin{cases} 
1 & \text{for } |\xi| < N \\
\left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N.
\end{cases}$$

The associated Fourier multiplier operator, $(\widehat{lu})(\xi) = m(\xi)\hat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have 

$$\|u\|_{H^s} \lesssim \|lu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$  

Set $\tilde{E}[lu(t)] = H[lu(t)]$. Other choices of $\tilde{E}$ are considered later.
**AC Law Decay and Sobolev GWP index**

1. **Modified LWP.** Initial \( v_0 \) s.t. \( \| \nabla lv_0 \|_{L^2} \sim 1 \) has \( T_{lwp} \sim 1 \).

2. **Goal.** \( \forall u_0 \in H^s, \forall T > 0, \) construct \( u : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{C} \).

3. \( \iff \) **Dilated Goal.** Construct \( u^\lambda : [0, \lambda^2 T] \times \mathbb{R}^2 \rightarrow \mathbb{C} \).

4. **Rescale Data.** \( \| l \nabla u_0^\lambda \|_{L^2} \lesssim N^{1-s} \lambda^{-s} \| u_0 \|_{H^s} \sim 1 \) provided we choose \( \lambda = \lambda(N) \sim N^{\frac{1-s}{s}} \iff N^{1-s} \lambda^{-s} \sim 1 \).

5. **Almost Conservation Law.** \( \| l \nabla u(t) \|_{L^2} \lesssim H[l u(t)] \) and

\[
\sup_{t \in [0, T_{lwp}]} H[l u(t)] \leq H[l u(0)] + N^{-\alpha}.
\]

6. **Delay of Data Doubling.** Iterate modified LWP \( N^\alpha \) steps with \( T_{lwp} \sim 1 \). We obtain rescaled solution for \( t \in [0, N^\alpha] \).

\[
\lambda^2(N) T < N^\alpha \iff T < N^\alpha + \frac{2(s-1)}{s} \text{ so } s > \frac{2}{2 + \alpha} \text{ suffices.}
\]
First Version of the I-method: $\tilde{E} = H[\|u\|]$ 

A Fourier analysis established the almost conservation property of $\tilde{E} = H[\|u\|]$ with $\alpha = \frac{3}{2}$ which led to...

**Theorem (CKSTT:MRL02)**

$NLS^+_3(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$.

Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$.

- Same result for $NLS^-_3(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Here $Q$ is the ground state (unique positive solution of $-Q + \Delta Q = Q^3$).
- Fourier analysis leading to $\alpha = \frac{3}{2}$ in fact gives $\alpha = 2$ for most frequency interactions.
**L²-critical in Weighted L² spaces**

Based on PC transformation & inspired by [Bourgain98], we have:

**Theorem (Blue-C:CPAA06)**

For $s \geq 0$, if $\text{NLS}^+_1\left(\mathbb{R}^d\right)$ is GWP for $H^s(\mathbb{R}^d)$ initial data then $\text{NLS}^+_1\left(\mathbb{R}^d\right)$ is GWP and scatters for data satisfying

$\langle \cdot \rangle^s u_0(\cdot) \in L^2$. The same result applies to the focusing case provided $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

- Thus, GWP for $L^2$ data $\iff$ Scattering for $L^2$ data.
- $H^s$-GWP improvements imply weighted space improvements.
- PC transformation isometry in $L^2$-admissible Strichartz spaces.
Remarks

- The almost conservation property

\[ \sup_{t \in [0, T_{lwp}]} \tilde{E}[Iu(t)] \leq \tilde{E}[Iu_0] + N^{-\alpha} \]

leads to GWP for

\[ s > s_\alpha = \frac{2}{2 + \alpha}. \]

- The \( I \)-method is a subcritical method. To prove the Scattering Conjecture at \( s = 0 \) via the \( I \)-method would require \( \alpha = +\infty \).

- The \( I \)-method localizes the conserved density in frequency. Similar ideas appear in recent critical scattering results.

- There is a multilinear corrections algorithm for defining new choices of \( \tilde{E} \) which should have a better AC property.
3. **Multilinear Correction Terms**

(Inspired by [Coifman-Meyer]; following [CKSTT:KdV])

1. For $k \in \mathbb{N}$, define the *convolution hypersurface*

   $$\Sigma_k := \{ (\xi_1, \ldots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \ldots + \xi_k = 0 \} \subset (\mathbb{R}^2)^k.$$ 

2. For $M: \Sigma_k \to \mathbb{C}$ and $u_1, \ldots, u_k$ nice, define *$k$-linear functional*

   $$\Lambda_k(M; u_1, \ldots, u_k) := c_k \Re \int_{\Sigma_k} M(\xi_1, \ldots, \xi_k) \hat{u}_1(\xi_1) \ldots \hat{u}_k(\xi_k).$$

3. For $k \in 2\mathbb{N}$ abbreviate $\Lambda_k(M; u) = \Lambda_k(M; u, \bar{u}, \ldots, \bar{u})$.

4. $\Lambda_k(M; u)$ invariant under interchange of even/odd arguments,

   $$M(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) \mapsto \overline{M}(\xi_2, \xi_1, \ldots, \xi_k, \xi_{k-1}).$$

5. We can define a symmetrization rule via group orbit.
\begin{equation}
\int_{x} u\bar{u}u\bar{u}dx = \int \left( \int e^{ix\cdot\xi_1} \hat{u}(\xi_1) d\xi_1 \right) \cdots \left( \int e^{ix\cdot\xi_4} \hat{u}(\xi_4) d\xi_4 \right) dx
\end{equation}

\begin{equation}
= \int_{\xi_1, \ldots, \xi_4} \left[ \int_{x} e^{ix\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} d\xi_1 \int_{x} e^{ix\cdot(\xi_2+\xi_3+\xi_4)} d\xi_1 \right] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) d\xi_1, \ldots, 4
\end{equation}

\begin{equation}
= \int_{\Sigma_4} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) = \Lambda_4(1; u).
\end{equation}

\begin{equation}
\Lambda_2(-\xi_1 \cdot \xi_2; u) = \|\nabla u\|_{L^2}^2.
\end{equation}

Thus, \( H[u] = \Lambda_2(-\xi_1 \cdot \xi_2; u) \pm \Lambda_4(\frac{1}{2}; u). \)
Suppose $u$ nicely solves $NLS_3^+(\mathbb{R}^2)$; $M$ is time independent, symmetric. Calculations produce the time differentiation formula

$$
\partial_t \Lambda_k(M; u(t)) = \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}(ikX(M); u(t)) \\
= \Lambda_k(iM\alpha_k; u(t)) - \Lambda_{k+2}([ikX(M])_{\text{sym}}; u(t)).
$$

Here

$$\alpha_k(\xi_1, \ldots, \xi_k) := -|\xi_1|^2 + |\xi_2|^2 - \ldots - |\xi_{k-1}|^2 + |\xi_k|^2
$$

(so $\alpha_2 = 0$ on $\Sigma_2$) and

$$X(M)(\xi_1, \ldots, \xi_{k+2}) := M(\xi_{123}, \xi_4, \ldots, \xi_{k+2}).$$

We use the notation $\xi_{ab} := \xi_a + \xi_b$, $\xi_{abc} := \xi_a + \xi_b + \xi_c$, etc.
Abbreviate $m(\xi_j)$ as $m_j$. Define $\sigma_2$ s.t. $\|I \nabla u\|_{L^2}^2 = \Lambda_2(\sigma_2; u)$:

$$\sigma_2(\xi_1, \xi_2) := -\frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2 = \frac{1}{2} |\xi_1|^2 m_1^2$$

With $\tilde{\sigma}_4$ (symmetric, time independent) to be determined, set

$$\tilde{E} := \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u).$$

Using the time differentiation formula, we calculate

$$\partial_t \tilde{E} = \Lambda_4(\{i\tilde{\sigma}_4 \alpha_4 - i2[X(\sigma_2)]_{sym}\}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{sym}; u).$$

We’d like to define $\tilde{\sigma}_4$ to cancel away the $\Lambda_4$ contribution.
Resonant interactions obstruct the natural choice:

\[ \tilde{\sigma}_4 = \frac{[2iX(\sigma_2)]_{\text{sym}}}{i\alpha_4}. \]

On \( \Sigma_4 \), we can reexpress \( \alpha_4 = -|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 + |\xi_4|^2 \) as

\[ \alpha_4 = -2\xi_{12} \cdot \xi_{14} = -2|\xi_{12}| |\xi_{14}| \cos \angle(\xi_{12}, \xi_{14}), \]

and

\[ [2iX(\sigma_2)]_{\text{sym}} = \frac{1}{4} (-m_1^2 |\xi_1|^2 + m_2^2 |\xi_2|^2 - m_3^2 |\xi_3|^2 + m_4^2 |\xi_4|^2). \]

When all the \( m_j = 1 \) (so \( \max_j |\xi_j| < N \)), \( \tilde{\sigma}_4 \) is well-defined. However, \( \alpha_4 \) can also vanish when \( \xi_{12} \) and \( \xi_{14} \) are orthogonal.
Remark: Integrable Systems Conjecture

For $\text{NLS}_3^+(\mathbb{R})$, the resonant obstruction disappears. Thus,

\[ \tilde{E}^1 = \Lambda_2(\sigma_2) + \Lambda_4(\tilde{\sigma}_4); \]

\[ \partial_t \tilde{E}^1 = -\Lambda_6([i4X(\tilde{\sigma}_4)]_{\text{sym}}). \]

We can then define, with $\tilde{\sigma}_6$ to be determined,

\[ \tilde{E}^2 = \tilde{E}^1 + \Lambda_6(\tilde{\sigma}_6); \]

\[ \partial_t \tilde{E}^2 = \Lambda_6(\{i\tilde{\sigma}_6\alpha_6 - [i4X(\tilde{\sigma}_4)]_{\text{sym}}\}) + \Lambda_8([i6X(\tilde{\sigma}_6)]_{\text{sym}}). \]

**Conjecture:** The multipliers $\tilde{\sigma}_6, \tilde{\sigma}_8, \ldots$ are well defined and lead to better AC properties. Same for other integrable systems.
4. Resonant Decomposition

We return to $NLS_3^+(\mathbb{R}^2)$.
Since the natural choice is not well-defined, we choose

$$\tilde{\sigma}_4 := \frac{[2iX(\sigma_2)]_{\text{sym}}}{i\alpha_4} \chi_{\Omega_{nr}}$$

where the non-resonant set $\Omega_{nr} \subset \Sigma_4$ such that

$$\Omega_{nr} := \{ \max_{1 \leq j \leq 4} |\xi_j| \leq N \} \cup \{ |\cos \angle(\xi_{12}, \xi_{14})| \geq \theta_0 \}.$$

Eventually, we choose $\theta_0$ to balance the 4-linear and 6-linear contributions to the modified energy increment. We have

$$\partial_t \tilde{E} = \Lambda_4(\{ i\tilde{\sigma}_4 \alpha_4 - i2[X(\sigma_2)]_{\text{sym}} \}; u) - \Lambda_6([i4X(\tilde{\sigma}_4)]_{\text{sym}}; u).$$

The 4-linear contribution is constrained to the resonant set $\Omega_{nr}^C$. 
**Improved Almost Conservation Property**

**Lemma**

If $\|u_0\|_{L^2_x(\mathbb{R}^2)} \leq A$; $E(\dot{u}_0) \leq 1$; $u$ is a nice solution of $NLS_3^+(\mathbb{R}^2)$ on a time interval $[0, t_0]$, then if $t_0 = t_0(A)$ is small enough,

$\left| \int_0^{t_0} \Lambda_4([\mathcal{L}_{-2iX}(\sigma_2)]_{sym} + i\tilde{\sigma}_4 \alpha_4; u(t)) \, dt \right|$

$+ \left| \int_0^{t_0} \Lambda_6([4iX(\tilde{\sigma}_4)]_{sym}; u(t)) \, dt \right| \lesssim C(A)[N^{-2} + \theta_0^{1/2} N^{-3/2} + \theta_0^{-1} N^{-3}]$.

The choice $\theta_0 = N^{-1}$ produces the AC property with $\alpha = 2$.
The 4-linear contribution has multiplier

$$([-2iX(\sigma_2)]_{\text{sym}} + i\tilde{\sigma}_4\alpha_4)(\xi) = [-2iX(\sigma_2)]_{\text{sym}}\chi_{\Omega_r}$$

where the resonant set $\Omega_r = \Omega_{nr}^C \subset \Sigma_4$,

$$\Omega_r := \{\max(|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|) > N; |\cos \angle(\xi_{12}, \xi_{14})| < \theta_0\}.$$

We wish to bound the associated energy increment

$$\int_0^{T_{\text{lwp}}} \Lambda_4([-2iX(\sigma_2)]_{\text{sym}}\chi_{\Omega_r}; u)dt.$$

The 4 factors $u$ are dyadically decomposed. The integral is studied case-by-case based on dyadic frequency sizes.

On $\Sigma_4$, the two largest frequencies are comparable.
Let $|\xi_j| \sim N_j \in 2\mathbb{Z}$. Symmetry properties and the $\Omega_r$ constraint allow to assume

$$N_1 \sim N_2 \gtrsim N, N_2 \gtrsim N_3 \gtrsim N_3 \gtrsim 1.$$ 

For most cases, suffices to use enhanced [CKSTT:MRL] and

**Lemma**

$$\forall \ (\xi_1, \xi_2, \xi_3, \xi_4) \in \Sigma_4,$$

$$|[2iX(\sigma_2)]_{sym}| \lesssim \min(m_1, m_2, m_3, m_4)^2 |\xi_{12}| |\xi_{14}|.$$ 

This follows from the mean value theorem.
The most delicate case occurs when

\[ N_1 \sim N_2 \gg N, N_3 \gg N_4 \gtrsim 1. \]

We use a refinement exploiting spherical symmetry of \( m \).

**Lemma**

Let \( N_1, \ldots, N_4 \) be in the delicate case with \( (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r \).

Then

\[ ||X(\sigma_2)||_{\text{sym}} \lesssim m(N_1)^2 N_1 N_3 \theta_0 + m(N_3)^2 N_3^2. \]

Combining this lemma with angular enhancements of the [CKSTT:MRL] analysis completes the proof. What are these enhancements?
**Lemma (Angle Refined Bilinear Strichartz)**

Let $0 < N_1 \leq N_2$ and $0 < \theta < \frac{1}{50}$. Then for any $v_1, v_2 \in X^{0,1/2+}$ with spatial frequencies $N_1, N_2$ respectively, the spacetime function

$$F(t,x) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(t(\tau_1+\tau_2)+x \cdot (\xi_1+\xi_2))} \times \chi_{\{ | \cos \angle(\xi_1,\xi_2) \leq \theta \}} \tilde{v}_1(\tau_1,\xi_1) \tilde{v}_2(\tau_2,\xi_2) \ d\xi_1 d\xi_2$$

obeys the bound

$$\| F \|_{L^2_{t,x}} \lesssim \theta^{1/2} \| v_1 \|_{X^{0,1/2+}} \| v_2 \|_{X^{0,1/2+}}.$$