

The Interaction Morawetz Estimate for Defocusing Schrödinger Equations on \mathbb{R}^2 ①

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September 2007.

Oberwolfach Talk

joint work w. M. Grillakis, N. Tzirakis.

simultaneous / independent [Planchon-Vega]

- describe new general estimate valid for all defocusing NLS equations on \mathbb{R}^2
- Application to mass supercritical equations.

outline

0. Generalized Virial identity ; $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ estimate.
1. [Fang-Grillakis] $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ with $T^{1/2}$ loss ; now $T^{1/3}$.
 - 2. Optimized Virial Weight Function
 - statement of new estimate
 - comparison with Bourgain's bilinear Strichartz
 - $L_t^4 L_x^8$ control
 - 3. Vector commutator proof
 - 4. Application : simplified proof of Nakanishi's scattering result for H^1
H^s-scattering extension

O. Generalized Virial Identity

Suppose $\phi: \mathbb{R}_t \times \mathbb{R}^d_x \rightarrow \mathbb{C}$ nicely solves $(i\partial_t + \Delta)\phi = N$.

Define the Morawetz Action with virial weight function $a: \mathbb{R}^d \rightarrow \mathbb{R}$:

$$M_a[\phi](t) = \int_{\mathbb{R}^d} \nabla a \cdot \Im(\bar{\phi} \nabla \phi)(t) dx.$$

local conservation of momentum \Rightarrow

Generalized Virial Identity:

$$\partial_t M_a[\phi] = \int_{\mathbb{R}^d} (-\Delta \Delta a) |\phi|^2 + 4a_{jk} \operatorname{Re}(\bar{\phi}_j \phi_k) + 2a_j \{N, \phi\}_p^j dx$$

where $\{N, \phi\}_p = \operatorname{Re}[N \nabla \bar{\phi} - g \nabla \bar{\phi}]$ is the momentum bracket.

$$\bullet \quad \{F'(|\phi|^2) \phi, \phi\}_p = -\nabla G(|\phi|^2) \text{ where } G(z) = z F'(z) - F(z).$$

$$\bullet \quad \{|\phi|^{2k} \phi, \phi\}_p = -\nabla \frac{k}{k+1} |\phi|^{2k+2}.$$

Idea of Morawetz Estimates:

cleverly choose the virial weight a so that

$$\bullet \quad \partial_t M_a = I + II \text{ with } I \geq 0, II \geq 0$$

$$\bullet \quad M_a|_0^T \leq C(\phi_0)$$

to obtain

$$\int_0^T I dt \leq C(\phi_0).$$

Implementing this idea seems to require:

- ∇a bounded
- a convex
- defocusing

} "⇒" $a \sim 1 \times 1$.

Application: Suppose $(i\partial_t + \Delta) u = F(|u|^2) u$ for $u: \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}$ with $F' \geq 0$. Form $\phi(t, x_1, x_2) = u(t, x_1) \bar{u}(t, x_2)$. Then ϕ satisfies defocusing NLS-type equation on $\mathbb{R}_t \times \mathbb{R}_x^6$. Choose $\alpha(x_1, x_2) = |x_1 - x_2| \rightarrow$

$$\int_0^T \int_{\mathbb{R}_x^3} |u(t, x)|^4 dx dt \leq \|u_0\|_{L_x^2}^3 \sup_{[0, T]} \|\nabla u(t)\|_{L_x^2} .$$

[CKSTT], [Hassell].

1. Improved [Fang - Grillakis] $L^4(R_t \times \mathbb{R}_x^2)$ estimate with $T^{1/3}$ loss

[CGT₂: IMRN] $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. $u: \mathbb{R}_t \times \mathbb{R}_x^2 \mapsto \mathbb{C}$ defocusing

form weight $\alpha(x_1, x_2) = f(|x_1 - x_2|)$ with f smooth convex s.t.

$$f(|x|) = \begin{cases} \frac{1}{2M} |x|^2 \left[1 - \log \frac{|x|}{M} \right] & |x| \leq \frac{M}{\sqrt{e}} \\ \infty & |x| > M. \end{cases}$$

$$-\Delta \Delta u = \frac{2\pi}{M} \delta_{\{x_1 = x_2\}} + \underbrace{\frac{1}{M} \sum_{\{x_1 - x_2 \geq \frac{M}{\sqrt{e}}\}}}_{\text{non-positive.}} \mathcal{O}\left(\frac{1}{|x_1 - x_2|^3}\right)$$

$$\begin{aligned} \text{LHS} &= \frac{1}{M} \int_0^T \int_{\mathbb{R}_x^2} |u(t, x)|^4 dx dt \leq \|u_0\|_{L_x^2}^3 \|Du\|_{L_T^\infty L_x^2} + \int_0^T \int_{\mathbb{R}_x^2} \frac{|u(x_1)|^2 |u(x_2)|^2}{|x_1 - x_2|^3} dx_1 dx_2 \\ &\leq c(\text{Mass, Energy}) + \frac{T}{M^3} \|u_0\|_{L_x^2}^4. \quad [\text{CRUDE}] \end{aligned}$$

$$\Rightarrow \boxed{\int_0^T \int_{\mathbb{R}_x^2} |u(t, x)|^4 dx dt \leq T^{\frac{1}{3}} \|u_0\|_{L_x^2}^3 \|Du\|_{L_T^\infty L_x^2} + T^{1/3} \|u_0\|_{L_x^2}^4.}$$

Remark: I-Method + Morawetz bootstrap \Rightarrow GWP in H^s , $s > \frac{2}{3}$
for NLS_2^+ .

Idea: Can we somehow absorb the non-positive term into the L_{tx}^4 term?
Is there a better weight function α ?

2. Optimized Virial Weight Function

$x \in \mathbb{R}^2$, $|x| = r$, Fix $r_0 > 0$.

Set $w(s) = \begin{cases} \frac{1}{s^3} & , s \geq 1 \\ 0 & \text{otherwise.} \end{cases}$; $w_{r_0}(r) = \frac{1}{r_0^3} w\left(\frac{r}{r_0}\right)$.

Set $\Delta a(r) = \frac{1}{r_0} \int_{r/r_0}^{\infty} s \log\left(\frac{r_0 s}{r}\right) w(s) ds \geq 0$.

Calculations show:

$$\cdot (-\Delta \Delta a)(|x|) = \frac{2\pi}{r_0} \delta_{\{|x|=0\}} - w_{r_0}(|x|).$$

$$\cdot \int_{\mathbb{R}^2} w_{r_0}(|x|) dx = \frac{2\pi}{r_0}$$

Virial identity \rightsquigarrow

$$\int_0^T \frac{2\pi}{r_0} \int_{\mathbb{R}_x^2} |u(t, x)|^4 dx dt - \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} w_{r_0}(|x_1 - x_2|) |u(t, x_1)|^2 |u(t, x_2)|^2 dx_1 dx_2 dt \leq M_a[u, u]$$

$$\int_0^T \left\{ \frac{\pi}{r_0} \int_{\mathbb{R}_{x_1}^2} |u(t, x_1)|^4 dx_1 - \int_{\mathbb{R}_{x_1}^2 \times \mathbb{R}_{x_2}^2} w_{r_0}(|x_1 - x_2|) |u(x_1)|^2 |u(x_2)|^2 dx_1 dx_2 + \frac{\pi}{r_0} \int_{\mathbb{R}_{x_2}^2} |u(x_2)|^4 dx_2 \right\} dt$$

$$\frac{1}{2} \int_{\mathbb{R}_{x_2}^2} w_{r_0}(|x_1 - x_2|) dx_2$$

$$\frac{1}{2} \int_{\mathbb{R}_{x_1}^2} w_{r_0}(|x_1 - x_2|) dx_1$$

PERFECT SQUARE!

$$\Rightarrow \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left\{ |u(t, x_1)|^2 - |u(t, x_2)|^2 \right\}^2 w_{r_0}(|x_1 - x_2|) dx_1 dx_2 dt \leq M_a[u, u] \left[\int_0^T \right]$$

Calculations show RHS uniformly bounded as $T \rightarrow 0 \Rightarrow$

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\{|u(t, x_1)|^2 - |u(t, x_2)|^2\}^2}{|x_1 - x_2|^3} dx_1 dx_2 dt \leq \sup_T \|u\|$$

equivalent to
 $B_2^{\frac{1}{2}, 2}$ norm.

Theorem [CGTz] Any finite energy defocusing Schrödinger Evolution satisfies

$$u: \mathbb{R}_t \times \mathbb{R}_x^2 \rightarrow \mathbb{C}$$

$$\left\| D^{\frac{1}{2}} |u|^2 \right\|_{L^2_{t,x}}^2 \leq \|u\|_{L_t^\infty L_x^2}^3 \| \nabla u \|_{L_t^\infty L_x^2}^3.$$

Remark:

$$\|u\|_{L_t^4 L_x^8}^2 \leq \|D^{\frac{1}{2}} |u|^2\|_{L_t^2 L_x^2}^2.$$

3. Vector Commutator Approach to proof

$$\rho(t, x) = \frac{1}{2} |U(t, x)|^2 \quad \hat{\rho}(t, x) = \text{Im}(\bar{U} \nabla U)(t, x).$$

Take $x \in \mathbb{R}^2$. $D^{-1}(13)^{-1} = 1x_1^{-1}$ so write $D^{-1}f(x) = \int_{\mathbb{R}^2} |x-y|^{-1} f(y) dy$.

$$m(t) := 2 \langle [D^{-1}; \vec{x}] \rho(t) | \vec{\rho}(t) \rangle$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{x_1 - x_2}{|x_1 - x_2|} \left\{ \rho(t, x_1) \vec{\rho}(t, x_2) - \rho(t, x_2) \vec{\rho}(t, x_1) \right\} dx_1 dx_2.$$

$$\vec{X} := [\vec{x}, D^{-1}] \quad \text{vector commutator operator.}$$

$$\vec{x} f = x D^{-1} f(x) - D^{-1}(x f) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|} f(y) dy.$$

A calculation shows

$$\begin{aligned} \partial_{x_j} \vec{X}^k f(x) &= \int_{\mathbb{R}^2} \left\{ \delta_j^k |x-y|^{-1} - \frac{(x^k - y^k)(x^j - y^j)}{|x-y|^3} \right\} f(y) dy \\ &= \left(D^{-1} \delta_j^k + [x^k; R_j] \right) f \end{aligned}$$

where

$$R_j = \partial_j D^{-1} \quad \text{has symbol } \frac{\tau_j}{|y|} \quad \text{and}$$

$$(R_j f)(x) = \int_{\mathbb{R}^2} \frac{x^j - y^j}{|x-y|^3} f(y) dy.$$

We have

$$M(t) = \langle \vec{x} \cdot \vec{p}(t) | \rho(t) \rangle$$

and calculate

$$\textcircled{*} \quad \dot{M}(t) = \langle \vec{x} \cdot \partial_t \vec{p}(t) | \rho(t) \rangle - \langle \vec{x} \partial_t \rho(t) | \vec{p}(t) \rangle$$

The conservation laws for cubic NLS may be expressed

derived from
Lagrangian formulation.

$$\partial_t \rho - \partial_j p^j = 0$$

$$\partial_t p_k - \partial_j \left\{ \sigma_k^j + \delta_k^j (-\Delta \rho + \frac{1}{2} \rho^2) \right\} = 0$$

where $\sigma_k^j = \rho^{-1} (p_j p_k + \partial_j \rho \partial_k \rho)$

substituting into $\textcircled{*}$ and rearranging via integration by parts reveals

$$\dot{M}(t) = S_1 + S_2 + S_3 + S_4$$

where

$$S_1 = \langle \rho^{-1} \partial_k \rho \partial^j \rho | (\partial_j \vec{x}^k) \rho \rangle$$

compressibility

$$S_2 = \langle \rho^{-1} p_k \rho^j | (\partial_j \vec{x}^k) \rho \rangle - \langle \rho^j | (\partial_j \vec{x}^k) p_k \rangle$$

convection

$$S_3 = \langle (-\Delta \rho) | (\partial_j \vec{x}^j) \rho \rangle$$

dispersion

$$S_4 = \frac{1}{2} \langle \rho^2 | (\partial_j \vec{x}^j) \rho \rangle$$

pressure

A calculation shows $\operatorname{div} \vec{X} = \partial_j \vec{X}^j = D^{-1}$.

Some manipulations show

$$S_1 \geq 0$$

$$S_2 \geq 0$$

$$S_3 = \langle -\Delta \rho | D^{-1} \rho \rangle = \langle D \rho | \rho \rangle = \| D^{\frac{1}{2}} \rho \|_{L^2}^2.$$

$$S_4 = \frac{1}{2} \langle D^{-1} \rho | \rho^2 \rangle \geq \langle D^{\frac{1-d}{2}} \rho^{\frac{3}{2}} | D^{\frac{1-d}{2}} \rho^{\frac{3}{2}} \rangle \geq 0.$$

Thus

$$\int_0^t \| D^{\frac{1}{2}} \rho \|_{L^2}^2 dt \leq m(t) - m(0) \leq \| u \|_{L_t^\infty L_x^2}^3 \| \nabla u \|_{L_t^\infty L_x^2}.$$

Remark: Since $\mathcal{F}^{-1}(|x|^{1-d}) \sim |x|^{-1}$ for $x \in \mathbb{R}^d$ we can modify the preceding discussion. Set

$\vec{X} = [\vec{x}; D^{1-d}]$ and revisit the calculations.

The final result involves an adjustment to S_3 :

$$S_3 = \langle -\Delta \rho | D^{1-d} \rho \rangle = \| D^{\frac{3-d}{2}} \rho \|_{L_x^2}^2$$

$$\hookrightarrow \| D^{\frac{3-d}{2}} |u|^2 \|_{L_{t,x}^2}^2 \leq \| u \|_{L_t^\infty L_x^2}^3 \| \nabla u \|_{L_t^\infty L_x^2}$$

is valid for all defocusing NLS equations on \mathbb{R}^d .

Remark Bourgain's bilinear Strichartz refinement reads in \mathbb{R}^{1+d}

$$\|u_{M_1} u_{M_2}\|_{L^2_{tx}} \leq \frac{\frac{M_1}{M_2}^{\frac{d-1}{2}}}{\frac{M_1}{M_2}^{\frac{1}{2}}} \|u_{M_1}(\cdot)\|_{L_x^2} \|u_{M_2}(\cdot)\|_{L_x^2}$$

which may be reexpressed as

$$\underbrace{\|M_2^{\frac{1}{2}} M_1^{\frac{2-d}{2}} u_{M_1} u_{M_2}\|_{L^2_{tx}}}_{D^{\frac{2-d}{2}}} \leq \|u_{M_1}(\cdot)\|_{H^{\frac{1}{2}}} \|u_{M_2}(\cdot)\|_{L_x^2}$$

Remark: The vector commutator approach may also be applied with the choice $\vec{X} = [x; K]$ where K is a self-adjoint operator. This flexibility is under study. More generally, $[\vec{x}(x); K] \dots$

4. Nakanishi: H^1 -scattering simplification + H^s extension

(4)

H^1 -simplification

$$NLS_p^+ (\mathbb{R}^2) \quad \begin{cases} i\partial_t u + \Delta u = +|u|^{p-1} u \\ u(0) = u_0 \in H^1(\mathbb{R}^2) \end{cases} \quad p > 1 + \frac{4}{2} = 3.$$

$$\|u\|_{S^1} := \sup_{\theta, r} \|e^{i\theta} u\|_{L_t^8 L_x^8} \quad i \cdot \frac{2}{8} + \frac{2}{r} = \frac{2}{2}.$$

$$\|u\|_{L_t^4 L_x^8} \leq C(u_0) \quad (\text{New input})$$

$$R_t = \bigcup_{j=1}^J I_j \quad \text{s.t.} \quad \|u\|_{L_{I_j}^4 L_x^8} \sim \delta \ll 1. \quad J \ll C(u_0).$$

Energy + Duhamel + Strichartz + Hölder $\Rightarrow \forall I_j$

$$\|u\|_{S_{I_j}^1} \lesssim \|u_0\|_{H^1} + \|u\|_{L_{I_j}^4 L_x^8}^\varepsilon \|u\|_{S_{I_j}^1}^{p-\varepsilon}$$

Better quantification of scattering size

H^s -extension

Specialize to $p-1 = 2K$, $K \in \mathbb{N}$, $K \geq 2$.

I-method + Morawetz bootstrap \Rightarrow

$NLS_{2K+1}^+ (\mathbb{R}^2)$ has GWP + Scattering for $u_0 \in H^s$ provided $1 > s > 1 - \frac{1}{4K-3}$.