

# RECENT PROGRESS ON CUBIC NLS ON $\mathbb{R}^2$

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# NONLINEAR SCHRÖDINGER INITIAL VALUE PROBLEM

Consider the defocusing initial value problem  $NLS_3^+(\mathbb{R}^2)$ :

$$\begin{cases} (i\partial_t + \Delta)u = +|u|^2 u \\ u(0, x) = u_0(x). \end{cases}$$

## Time Invariant Quantities

$$\text{Mass} = \|u(t)\|_{L_x^2}$$

$$\text{Momentum} = 2\Im \int_{\mathbb{R}^2} \bar{u}(t) \nabla u(t) dx$$

$$\text{Hamiltonian} = H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx + \frac{1}{2} |u(t)|^4 dx$$

## Dilation Invariance ( $L^2(\mathbb{R}^2)$ critical)

$$u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2}\tau, \lambda^{-1}y)$$

# LOCAL-IN-TIME THEORY

[Cazenave-Weissler 90]

- $\forall u_0 \in L^2(\mathbb{R}^2) \exists T_{lwp}(u_0)$  determined by

$$\|e^{it\Delta} u_0\|_{L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100} \text{ such that}$$

$\exists$  unique  $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$  solving  $NLS_3^+(\mathbb{R}^2)$ .

- $\forall u_0 \in H^s(\mathbb{R}^2), s > 0, T_{lwp} \sim \|u_0\|_{H^s}^{-\frac{2}{s}}$  and regularity persists:  $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$ .
- Define the **maximal forward existence time**  $T^*(u_0)$  by

$$\|u\|_{L^4_{tx}([0, T^* - \delta] \times \mathbb{R}^2)} < \infty$$

for all  $\delta > 0$  but diverges to  $\infty$  as  $\delta \searrow 0$ .

- $\exists$  **small data scattering threshold**  $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.$$

# GLOBAL-IN-TIME THEORY?

What is the ultimate fate of the local-in-time solutions?

**Defocusing Scattering Conjecture:**  $L^2 \ni u_0 \longmapsto u$  solving  $NLS_3^+(\mathbb{R}^2)$  is global-in-time and

$$\|u\|_{L^4_{t,x}} < C(u_0).$$

Moreover,  $\exists u_\pm \in L^2(\mathbb{R}^2)$  such that

$$\lim_{t \rightarrow \pm\infty} \|e^{\pm it\Delta} u_\pm - u(t)\|_{L^2(\mathbb{R}^2)} = 0.$$

**Remarks:**

- Known for small data  $\|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0$ .
- Known for defocusing  $L^2(\mathbb{R}^d)$ -critical  $NLS_{1+\frac{4}{d}}^+(\mathbb{R}^d)$  for large radial data,  $d \geq 3$ . [Tao-Visan-Zhang 06]
- GWP for  $L^2$  data  $\iff$  Scattering for  $L^2$  data.

# MAIN RESULT

## THEOREM (C-GRILLAKIS-TZIRAKIS 07)

$NLS_3^+(\mathbb{R}^2)$  is globally well-posed for data in  $H^s(\mathbb{R}^2)$  for  $\frac{2}{5} < s < 1$ .  
Moreover, the solution satisfies

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s(\mathbb{R}^2)} \leq C(1 + T)^{\frac{3s(1-s)}{2(5s-2)}}.$$

regularity	idea	reference
$s > \frac{2}{3}$	high/low frequency decomposition	[Bourgain 98]
$s > \frac{4}{7}$	$H(Iu)$	[CKSTT 02]
$s > \frac{1}{2}$	resonant cut of 2nd energy	[CKSTT 07]
$s \geq \frac{1}{2}$	$H(Iu)$ & Interaction Morawetz	[Fang-Grillakis 05]
$s > \frac{4}{13}?$	resonant cut & $I$ -Morawetz	[—]

## NEW IDEAS IN THE PROOF

Proof follows [FG 05] ([CKSTT 04] scheme) with two new inputs:

- 1 Interaction Morawetz estimate of [FG 05]:

$$\|u\|_{L^4([0,T]\times\mathbb{R}^2)}^4 \lesssim T^{\frac{1}{2}} \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \|u_0\|_{L^2(\mathbb{R}^2)}^3$$

versus the improvement

$$\|u\|_{L^4([0,T]\times\mathbb{R}^2)}^4 \lesssim T^{\frac{1}{3}} \left( \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \|u_0\|_{L^2(\mathbb{R}^2)}^3 + \|u_0\|_{L^2(\mathbb{R}^2)}^4 \right).$$

- 2 Interaction Morawetz for a regularized reference evolution  $Iu$ :

$$\begin{aligned} \|Iu\|_{L^4([0,T]\times\mathbb{R}^2)}^4 &\lesssim T^{\frac{1}{3}} \left[ \sup_{t \in [0, T]} \|\nabla Iu(t)\|_{L^2(\mathbb{R}^2)} \|Iu_0\|_{L^2(\mathbb{R}^2)}^3 \right. \\ &\quad \left. + \|Iu_0\|_{L^2(\mathbb{R}^2)}^4 + \text{Error}(T) \right] \end{aligned}$$

# MAIN INGREDIENTS IN PROOF

## ■ **$I$ -method/Almost Conservation of $H[lu]$**

- Finite energy reference evolution  $lu$ ,  $I = I_N : H^s \rightarrow H^1$  is a smoothing operator of order  $1 - s$ .
- Local well-posedness of  $I(NLS_3^+(\mathbb{R}^2))$  initial value problem.
- Hamiltonian increment quantification:

$$|H[lu](T_{lwp}) - H[lu](0)| \lesssim N^{-\alpha} \|lu(0)\|_{H^1(\mathbb{R}^2)}^4.$$

## ■ **Interaction Morawetz Estimate**

- Improved interaction Morawetz; Morawetz for  $I(NLS_3^+(\mathbb{R}^2))$ .
- Morawetz error increment quantification:

$$|\text{Error}(T_{lwp}) - \text{Error}(0)| \lesssim N^{-\beta} \|lu(0)\|_{H^1}^4.$$

## ■ **Bootstrap argument**

# $I$ -METHOD/ALMOST CONSERVATION OF $H[Iu]$

For  $s < 1$ ,  $N \gg 1$  define smooth monotone  $m : \mathbb{R}_\xi^2 \rightarrow \mathbb{R}^+$  s.t.

$$m(\xi) = \begin{cases} 1 & \text{for } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N. \end{cases}$$

The associated Fourier multiplier operator,  $\widehat{(Iu)}(\xi) = m(\xi)\widehat{u}(\xi)$ , satisfies  $I : H^s \rightarrow H^1$ . Note that

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s} \|u\|_{H^s}.$$

**Idea of the  $I$ -method:**  $NLS_3^+$  evolution  $u_0 \mapsto u(t)$  induces finite energy reference evolution  $Iu_0 \mapsto Iu(t)$ . (Almost) conservation of energy  $H(Iu(t))$  provides control on  $\|u(t)\|_{H^s}$ .

# MODIFIED LOCAL WELL-POSEDNESS

$Iu$  satisfies the initial value problem  $I(NLS_3^+(\mathbb{R}^2))$ :

$$\begin{cases} (i\partial_t + \Delta)Iu = +I(|u|^2 u) \\ Iu(0, x) = Iu_0(x). \end{cases}$$

- $(\widehat{\langle D \rangle u})(\xi) := (1 + |\xi|^2)^{1/2} \widehat{u}(\xi).$
- An ordered pair  $(q, r)$  is **admissible** if  $\frac{2}{q} + \frac{2}{r} = \frac{2}{2}$  and  $2 < q \leq \infty.$
- $\mu([0, T]) := \int_0^T \int_{\mathbb{R}^2} |Iu(t, x)|^4 dx dt.$

Classical arguments establish...

# MODIFIED LOCAL WELL-POSEDNESS

## LEMMA

If  $\mu([0, T]) < \mu_0$  (universal constant) then  $\forall s > 0$  the initial value problem  $I(NLS_3^+(\mathbb{R}^2))$  is locally well-posed and

$$Z_I([0, T]) := \sup_{(q,r)\text{admissible}} \|\langle D \rangle Iu\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^2)} \lesssim \|\langle D \rangle Iu_0\|_{L^2}.$$

Define  $T_{lwp}$  by the condition  $\mu([0, T_{lwp}]) = \mu_0$ . Thus, the local theory gives spacetime control on  $Iu$  on the slab  $[0, T_{lwp}] \times \mathbb{R}^2$ . A Fourier analysis of the expression

$$\int_0^{T_{lwp}} \partial_t H[Iu(t)] dt = \Re \int_0^{T_{lwp}} \int_{\mathbb{R}^2} (\partial_t Iu) [ |Iu|^2 Iu - I(|u|^2 u)] dx dt$$

permits proving....

# ALMOST CONSERVATION OF MODIFIED ENERGY

## LEMMA

If  $H^s \ni u_0 \rightarrow u(t)$  solves  $NLS_3^+(\mathbb{R}^2)$  with  $\frac{1}{2} > s > \frac{1}{3}$  then

$$\begin{aligned} \sup_{t \in [0, T]} H[I_N u(t)] \leq & \quad H[I_N u(0)] + C N^{-\frac{3}{2}+} [Z_I([0, T])]^4 \\ & + C N^{-2+} [Z_I([0, T])]^6. \end{aligned}$$

This quantifies the increment in  $H[Iu]$  over  $t \in [0, T]$ .

In particular, when  $\|I_N \langle D \rangle u(0)\|_{L_x^2} \leq 1$  we have

$$\sup_{t \in [0, T_{lw}]} H[I_N u(t)] \leq H[I_N u(0)] + C N^{-\alpha}$$

for  $\alpha = \frac{3}{2} -$ .

# INTERACTION MORAWETZ: LOCAL CONSERVATION

Suppose  $\phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{C}$  solves **generalized NLS**

$$(i\partial_t + \Delta)\phi = \mathcal{N}$$

for some  $\mathcal{N} = \mathcal{N}(t, x, u) : [0, T] \times \mathbb{R}^d \times \mathbb{C} \rightarrow \mathbb{C}$ . Assume  $\phi$  is nice.

We introduce notation to compactly express mass and momentum  
(non)conservation for solutions of generalized NLS.

Write  $\partial_{x_j}\phi = \partial_j\phi = \phi_j$ .

# LOCAL MASS/MOMENTUM (NON)CONSERVATION

- mass density:  $T_{00} = |\phi|^2$
- momentum density/mass current:  
 $T_{0j} = T_{j0} = 2\Im(\bar{\phi}\phi_j)$
- (linear part of the) momentum current:  
 $L_{jk} = L_{kj} = -\partial_j\partial_k|\phi|^2 + 4\Re(\bar{\phi}_j\phi_k)$
- mass bracket:  $\{f, g\}_m = \Im(f\bar{g})$
- momentum bracket:  $\{f, g\}_p^j = \Re(f\partial_j\bar{g} - g\partial_j\bar{f})$

**Local mass (non)conservation identity:**

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

**Local momentum (non)conservation identity:**

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j$$

## LOCAL MASS/MOMENTUM (NON)CONSERVATION

Consider  $\mathcal{N} = F'(|\phi|^2)\phi$  for polynomial  $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ .

- We calculate the mass bracket

$$\{F'(|\phi|^2)\phi, \phi\}_m = \Im(F'(|\phi|^2)\phi\bar{\phi}) = 0.$$

Thus mass is conserved for these nonlinearities.

- We calculate the momentum bracket

$$\{F'(|\phi|^2)\phi, \phi\}_p^j = -\partial_j G(|\phi|^2)$$

where  $G(z) = zF'(z) - F(z) \sim F(z)$ .

Thus the momentum bracket contributes a divergence and momentum is conserved for these nonlinearities.

# GENERALIZED VIRIAL IDENTITY

Suppose  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ . Form the **Morawetz Action**

$$M_a(t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\Im(\bar{\phi}\nabla\phi) dx.$$

Conservation identities lead to the **generalized virial identity**

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\bar{\phi}_j\phi_k) + 2a_j\{\mathcal{N}, \phi\}_p^j dx.$$

**Idea of Morawetz Estimates:** Cleverly choose the weight function  $a$  so that  $\partial_t M_a \geq 0$  but  $M_a \leq C(\phi_0)$  to obtain spacetime control on  $\phi$ . This strategy imposes various constraints on  $a$  which suggest choosing  $a(x) = |x|$ .

## EXAMPLE: [LIN-STRAUSS 78] MORAWETZ IDENTITY

Consider  $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$  with  $F' \geq 0$  and  $x \in \mathbb{R}^3$ . Choose  $a(x) = |x|$ . Observe that  $a$  is weakly convex,  $\nabla a = \frac{x}{|x|}$  is bounded, and  $-\Delta \Delta a = 4\pi \delta_0$ . One gets the **Lin-Strauss Morawetz identity**

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^3} 4\pi \delta_0(x) |\phi(t, x)|^2 + (\geq 0) + 4 \frac{G(|\phi|^2)}{|x|} dx dt$$

which implies the spacetime control estimate

$$(H[u_0])^{1/2} \|u_0\|_{L^2} \gtrsim \int_0^T \int_{\mathbb{R}^3} \frac{G(|\phi|^2)}{|x|} dx dt.$$

## EXAMPLE: $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ INTERACTION MORAWETZ

[CKSTT 04] (Hassell 04)

- Suppose  $\phi_1, \phi_2$  are two solutions of  $(i\partial_t + \Delta)\phi = F'(|\phi|^2)\phi$  with  $F' \geq 0$  and  $x \in \mathbb{R}^3$ . The “2-particle” wave function

$$\Psi(t, x_1, x_2) = \phi_1(t, x_1)\phi_2(t, x_2)$$

satisfies an NLS-type equation on  $\mathbb{R}^{1+6}$

$$(i\partial_t + \Delta_1 + \Delta_2)\Psi = [F'(|\phi_1|^2) + F'(|\phi_2|^2)]\Psi.$$

- Note that  $[F'(|\phi_1|^2) + F'(|\phi_2|^2)] \geq 0$  so defocusing.
- Reparametrize  $\mathbb{R}^6$  using center-of-mass coordinates  $(\bar{x}, y)$  with  $\bar{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3$ . Note that  $y = 0$  corresponds to the diagonal  $x_1 = x_2 = \bar{x}$ . Apply the generalized virial identity with the **choice**  $a(x_1, x_2) = |y|$ . Dismissing terms with favorable signs, one obtains...

## EXAMPLE: $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ INTERACTION MORAWETZ

$$\begin{aligned}
\|\nabla u\|_{L_{[0,T]}^\infty L_x^2} \|u_0\|_{L^2}^3 &\geq \int_0^T \int_{\mathbb{R}^6} (-\Delta_6 \Delta_6 |y|) |\Psi(x_1, x_2)|^2 dx_1 dx_2 dt \\
&\geq c \int_0^T \int_{\mathbb{R}^6} \delta_{\{y=0\}}(x_1, x_2) |\phi_1(x_1) \phi_2(x_2)|^2 dx_1 dx_2 dt \\
&\geq c \int_0^T \int_{\mathbb{R}^3} |\phi_1(t, \bar{x}) \phi_2(t, \bar{x})|^2 d\bar{x} dt.
\end{aligned}$$

Specializing to  $\phi_1 = \phi_2$  gives the **interaction Morawetz estimate**

$$\int_0^T \int_{\mathbb{R}^3} |\phi(t, x)|^4 dx dt \leq C \|\nabla u\|_{L_{[0,T]}^\infty L_x^2} \|u_0\|_{L_x^2}^3$$

valid uniformly for all defocusing NLS equations on  $\mathbb{R}^3$ .

# $L^4(\mathbb{R}_t \times \mathbb{R}_x^2)$ INTERACTION MORAWETZ

- The “ $\mathbb{R}^3$  miracle” underpinning the  $L^4(\mathbb{R}^{1+3})$  estimate is  $-\Delta\Delta|y| = 4\pi\delta_0$ . For  $\mathbb{R}_x^2$ , we’d like to replace  $|y|$  with  $a(x) = |x|^2 \log|x|$  but this violates  $|\nabla a(x)| \leq C$ .
- Inspired by [FG 05], define smooth convex  $f$  satisfying

$$f(|x|) = \begin{cases} \frac{1}{2M}|x|^2(1 - \log \frac{|x|}{M}) & \text{for } |x| < \frac{M}{\sqrt{e}} \\ 100|x| & \text{for } |x| > M \end{cases}$$

where  $M$  is a large parameter we will later choose.

- Choose  $a(x_1, x_2) = f(|x_1 - x_2|)$  in the “2-particle” virial identity. A calculation shows that

$$-\Delta\Delta a = \frac{2\pi}{M} \delta_{\{x_1=x_2\}} + \mathbf{1}_{\{|x_1-x_2|>\frac{M}{\sqrt{e}}\}} O\left(\frac{1}{|x_1 - x_2|^3}\right).$$

# $L^4(\mathbb{R}_t \times \mathbb{R}_x^2)$ INTERACTION MORAWETZ

Inserting this into the virial identity, dismissing terms with favorable signs, and collapsing to  $u_1 = u_2$  produces

$$\begin{aligned} \frac{1}{M} \int_0^T \int_{\mathbb{R}^2} |u(x)|^4 dx dt &\lesssim M_a|_0^T + \int_0^T \int_{|x_1 - x_2| > \frac{M}{\sqrt{\epsilon}}} \frac{|u(x_1)|^2 u(x_2)|^2}{|x_1 - x_2|^3} dx_1 dx_2 dt \\ &\lesssim \|\nabla u\|_{L_{[0,T]}^\infty L_x^2} \|u_0\|_{L_x^2}^3 + \frac{T}{M^3} \|u_0\|_{L_x^2}^4. \end{aligned}$$

Multiplying through by  $M$  and balancing terms by choosing  $M \sim T^{1/3}$  gives the **improved interaction Morawetz estimate**

$$\int_0^T \int_{\mathbb{R}^2} |u(t, x)|^4 dx dt \lesssim T^{\frac{1}{3}} \|\nabla u\|_{L_{[0,T]}^\infty L_x^2} \|u_0\|_{L_x^2}^3 + T^{\frac{1}{3}} \|u_0\|_{L_x^2}^4.$$

# INTERACTION MORAWETZ FOR $Iu$ : ERROR TERM.

- Reexpress  $I(NLS_3^+)$  evolution equation

$$(i\partial_t + \Delta)Iu = |Iu|^2 Iu + [I(|u|^2 u) - |Iu|^2 Iu].$$

- First term contributes positive term to Morawetz identity.
- $[ \dots ]$  produces error. Commutator makes it small.
- Form  $\Psi(t, x_1, x_2) = Iu_1(t, x_1)Iu_2(t, x_2)$  and ...

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} |Iu(x)|^4 dx dt &\lesssim T^{\frac{1}{3}} \|\nabla Iu\|_{L_{[0,T]}^\infty L_x^2} \|Iu_0\|_{L_x^2}^3 + T^{\frac{1}{3}} \|Iu_0\|_{L_x^2}^4 \\ &+ T^{\frac{1}{3}} \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_{error}, Iu_1(x_1)Iu_2(x_2)\}_p dx_1 dx_2 dt \right| \end{aligned}$$

with  $\mathcal{N}_{error} = [I(|u_1|^2 u_1) - |Iu_1|^2 Iu_1]Iu_2 + (1 \leftrightarrow 2)$ .

# MORAWETZ ERROR INCREMENT QUANTIFICATION

## LEMMA

If  $H^s \ni u_0 \rightarrow u$  solves  $NLS_3^+(\mathbb{R}^2)$  with  $\frac{1}{2} > s > 0$  then

$$\left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_{\text{error}}, Iu_1(x_1)Iu_2(x_2)\}_p dx_1 dx_2 dt \right| \lesssim N^{-1+}[Z_I([0, T])]^6.$$

The proof imitates the **almost conservation analysis**.

In particular, when  $\|I_N \langle D \rangle u(0)\|_{L_x^2} \leq 1$  we have

$$\left| \int_0^{T_{\text{lwp}}} \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_{\text{error}}, Iu_1(x_1)Iu_2(x_2)\}_p dx_1 dx_2 dt \right| \lesssim N^{-\beta}$$

for  $\beta = 1-$ .

## BOOTSTRAP ARGUMENT: PRELIMINARIES

- Fix a huge time interval  $[0, T_0]$ . Consider a global-in-time solution  $C_0^\infty(\mathbb{R}^2) \ni u_0 \rightarrow u$ . We will prove

$$\sup_{t \in [0, T_0]} \|u(t)\|_{H^s} \leq CT_0^{\theta(s)}$$

for  $s > \frac{2}{5}$  with  $C$  independent of extra  $C_0^\infty$  assumptions.

- Rescale initial data so that

$$\|Iu_0^\lambda\|_{H^1} = O(1) \iff \lambda \sim N^{\frac{1-s}{s}}.$$

- We will choose  $N = N(T_0)$  and show

$$\sup_{t \in [0, \lambda^2 T_0]} \|Iu^\lambda(t)\|_{H^1} \leq O(1)$$

This unravels to prove the polynomial bound above.

## BOOTSTRAP ARGUMENT: SETUP

- With  $\alpha = \frac{3}{2} -$ ,  $\beta = 1 -$  and  $\gamma = \frac{1}{3}$ , define the set

$$S_K = \{t \in [0, \lambda^2 T_0] : \|Iu^\lambda\|_{L^4([0,t] \times \mathbb{R}^2)}^4 \leq KN^{\alpha-\beta} t^\gamma\}$$

Here  $K$  is a large constant to be chosen.

- Claim:**  $S_K = [0, \lambda^2 T_0]$ . **Assume not.** Since  $\|Iu^\lambda\|_{L^4([0,t] \times \mathbb{R}^2)}$  is continuous with  $t$ ,  $\exists T \in (0, \lambda^2 T_0)$  such that

$$\|Iu^\lambda\|_{L^4([0,T] \times \mathbb{R}^2)}^4 = KN^{\alpha-\beta} T^\gamma.$$

- Cut  $[0, T]$  into disjoint intervals  $J_k, k = 1, \dots, L$  such that

$$\int_{J_k} \int_{\mathbb{R}^2} |Iu^\lambda|^4 dx dt \leq \mu_0.$$

Each  $J_k$  is like  $[0, T_{lwp}]$ . Also,  $L \sim \frac{KN^{\alpha-\beta} T^\gamma}{\mu_0}$ .

# BOOTSTRAP: MODIFIED HAMILTONIAN CONTROL

- Almost Conservation Lemma gives

$$\sup_{t \in J_1} H[Iu^\lambda(t)] \leq H[Iu^\lambda(0)] + N^{-\alpha} \lesssim O(1)$$

- Accumulating increments and recalling rescaling gives

$$\sup_{t \in [0, T]} H[Iu^\lambda(t)] \leq O(1) + \frac{L}{N^\alpha} \leq O(1) \text{ if } L < N^\alpha.$$

- Recalling  $L, \lambda$ , using  $T < \lambda^2 T_0$ , choose  $N = N(T_0)$  s.t.

$$T_0^\gamma \frac{K}{\mu_0} = N^{\frac{(\beta+2\gamma)s-2\gamma}{s}} \implies L < N^\alpha.$$

- Since  $T_0$  is big, this requires  $s > \frac{2\gamma}{\beta+2\gamma} = \frac{2}{5}$ .

## BOOTSTRAP: CONCLUSION

- Recall the interaction Morawetz estimate for  $Iu^\lambda$

$$\int_0^T \int_{\mathbb{R}^2} |Iu^\lambda(x)|^4 dx dt \lesssim T^{\frac{1}{3}} \|\nabla Iu^\lambda\|_{L_{[0,T]}^\infty L_x^2} \|Iu_0^\lambda\|_{L_x^2}^3 + T^{\frac{1}{3}} \|Iu_0^\lambda\|_{L_x^2}^4$$
$$+ T^{\frac{1}{3}} \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_{\text{error}}, Iu_1^\lambda(x_1) Iu_2^\lambda(x_2)\}_p dx_1 dx_2 dt \right|$$

- On each  $J_k$  the Morawetz error term contributes at most  $N^{-\beta}$

$$\left| \int_{J_k} \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_{\text{error}}, Iu_1^\lambda(x_1) Iu_2^\lambda(x_2)\}_p dx_1 dx_2 dt \right| \lesssim N^{-\beta}.$$

- Accumulating the increments over  $N^\alpha$  steps proves

$$\|Iu^\lambda\|_{L^4([0,T] \times \mathbb{R}^2)}^4 \lesssim N^{\alpha-\beta} T^{\frac{1}{3}}.$$

Contradiction for  $K$  larger than the implied constants.

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