

Maximal-in-time issues for NLS

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Texas Talk

RLM 6-107

$$NLS_p^\pm(\mathbb{R}^d) \begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d \end{cases}$$

+ defocus
- focusing.

$p=3$ physically ubiquitous.
 $p=5$ also physically relevant.
general p gives us a tool to probe nonlinearity.

$$u: (-T_+, T^+) \times \mathbb{R}^d \mapsto \mathbb{C}.$$

Q: What happens?

- Do solutions exist for short time?
- Do solutions exist globally in time? Do they blow up in finite time?
- How do solutions behave?

Conservation Laws

$\|u(t)\|_{L^2} = \|u_0\|$ "Mass"

$\text{Im} \int u(t) \nabla \bar{u}(t) dx = \text{Im} \int u_0 \nabla \bar{u}_0 dx$ "Momentum"

$H[u(t)] = \int |\nabla u(t)|^2 dx \pm \frac{2}{p+1} |u(t)|^{p+1} dx$ "Energy or Hamiltonian"

$$\partial_t |u|^2 = \nabla \cdot 2 \text{Im}(u \nabla \bar{u})$$

How is mass conserved?

Dilation Invariance

If $u: [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}$ solves $NLS_p^\pm(\mathbb{R}^d)$ then

$$u_\lambda(t, x) := \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) : [0, \lambda^2 T] \times \mathbb{R}^d \mapsto \mathbb{C}$$

also solves $NLS_p^\pm(\mathbb{R}^d)$.

Norms which are invariant w.r.t. dilation play a crucial role in the theory.

$$\|D_x^s u_\lambda\|_{L_x^q} = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1} + s - \frac{d}{q}} \|D_x^s u_0\|_{L_x^q}$$

specialize to $q=2$.

$$S_c = \frac{d}{2} - \frac{2}{p-1}$$

H^{S_c} is the L^2 -based Sobolev space which is critical or scaling invariant w.r.t. dilation.

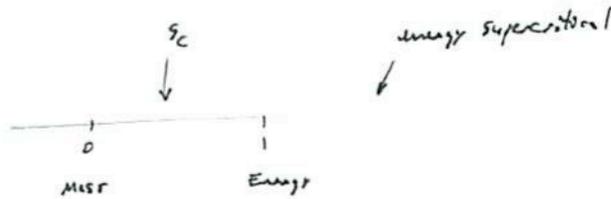
NLS_p[±](R^d)

Q: What happens?

Conservation laws

Dilation invariance

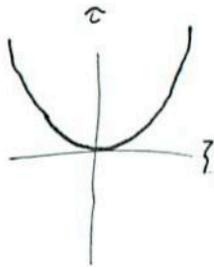
$$s_c = \frac{d}{2} - \frac{2}{p-1}$$



local-in-time theory.

Example NLS₃[±](R^d).

$$\begin{aligned} u_0 \mapsto e^{it\Delta} u_0(x) &= \int e^{ix\tau} e^{it|\tau|^2} \hat{u}_0(\tau) d\tau \\ &= \int e^{ix\tau} e^{it\tau^2} \hat{u}_0(\tau) \delta\{\tau = |\tau|^2\} d\tau d\tau. \end{aligned}$$



Tomas-Stein, Fourier Restriction phenomenon... → Strichartz Estimates

Strichartz Estimate

$$\|u\|_{L^4_{t,x}} \leq \|u|_{t=t_0}\|_{L^2_x} + \underbrace{\|(i\partial_x + \Delta)u\|}_{\pm |u|^2 u} \|u\|_{L^4_{t,x}}^{4/3}$$

More generally.

$$\|u\|_{L^4_{t,x}}^3 \rightarrow \text{Fixed point}$$

$$\|u\|_{L^p_t L^q_x} \leq \|u_0\|_{L^2_x} + \|(i\partial_x + \Delta)u\|_{L^p_t L^q_x}^{1/r}$$

LWP Theorem (Caizenare-Weissler)

∀ s ≥ s_c ≥ 0, ∀ u₀ ∈ H^s ∃ T = T(u₀) > 0 and ∃! u_s map

H^s ≥ u₀ ↦ u ∈ C([0, T]; H^s) ∩ (Strichartz)

taking initial data u₀ to the solution u of NLS_p[±](R^d).

Remarks T = \|u₀\|_{H^s}^{-2/(s-s_c)} if s > s_c

T = T(u₀) and not upon the norm \|u₀\|_{H^s} if s = s_c.

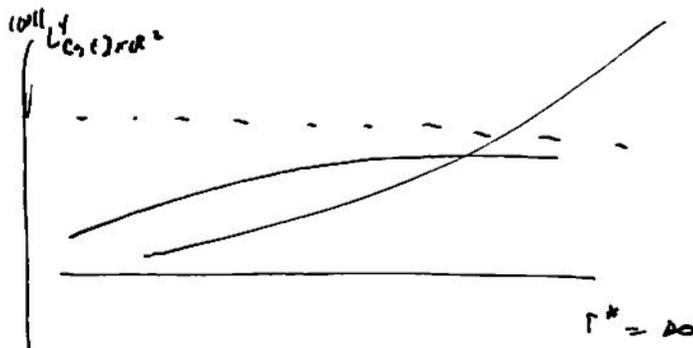
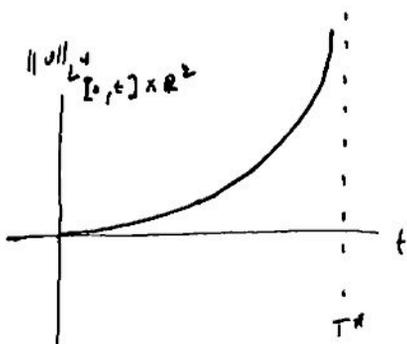
• Regularity persists but ~~the~~ the norms may grow.

• In H^s scale this result is optimal [KPV] NLS_p⁻(R^d)
[Christ-C-Tao] NLS_p[±](R^d).

Maximality Criterion

$NLS_p^\pm(\mathbb{R}^2)$

\exists maximal (forward) time interval of existence $[0, T^*)$



finite time blowup

infinite time blowup

infinite time bounded \Rightarrow scattering.

small data scattering

global conservation / subcritical regularity

spatially localized conservation laws
virial, Morawetz

Almost conservation laws / freq. localized cons.
B98 hi/lo I-method.

Critical techniques

- induction on energy; minimal blowups
- profile decompositions

\Rightarrow maximal-in-time knowledge of $NLS_p^\pm(\mathbb{R}^d)$.

2. mass critical case -

$$NLS_3^{\pm}(\mathbb{R}^2).$$

Recall the conservation laws: $\|u(t)\|_{L^2}$

$$H. = \int |\nabla u(t)|^2 \pm \frac{1}{2} |u(t)|^4 dx.$$

$$\leq \frac{1}{2} \|u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2}^2.$$

• Weinstein bound $S_{opt}^4 = \frac{2}{\|a\|_{L^2}^2} \rightarrow$ absorb.

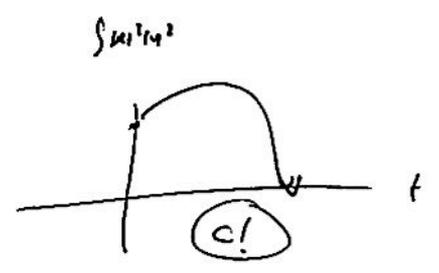
H^1 GWP. $-a + \Delta a = a^3$
 What is the fate of H^1 solutions?

• \exists explicit blowup solutions with mass $\|a\|_{L^2}$.

$$pc [e^{it} a] = S(t, x) = \frac{1}{|t|} e^{i \frac{|x|^2}{4t}} Q\left(\frac{x}{t}\right).$$

• \exists general blowup criterion.

$$\partial_t \int |x|^2 |u|^2 dx = 16 H[u_0].$$



Glassy's identity. Virial identity

Mass Concentration Results

Merle - Tsutsumi $H^1 \cap \{\text{radial}\} \ni v_0 \mapsto v \in NLS_3^-(\mathbb{R}^3)$, $T^* < \infty$

$$\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{\frac{1}{2}}} |v(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$

Bourgain: $L^2 \ni v_0 \mapsto v \in NLS_3^\pm(\mathbb{R}^3)$, $T^* < \infty$

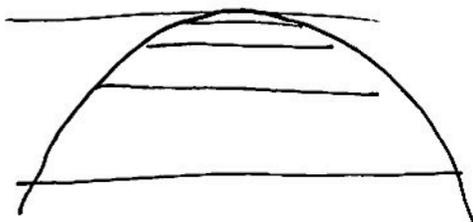
$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{\frac{1}{2}}} |v(t, x)|^2 dx \geq \|v_0\|_{L^1}^{-m}.$$

Tiny blowups?

• [CRSW] $\otimes S > \frac{9}{10}$. $H^s \cap \{\text{radial}\} \ni v_0 \mapsto v \in NLS_3^-(\mathbb{R}^3)$, $T^* < \infty$

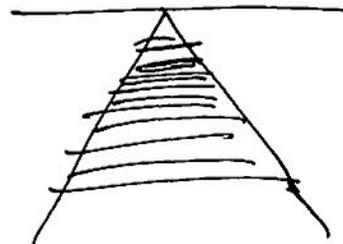
$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{\frac{3}{2}}} |v(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$

• [Chardonko].



Conjecture: $\|v_0\|_{L^2} < \infty$
 $\|v_0\|_{L^2} < \|v_0\|_{L^1}$

vs.



$v_0 \mapsto v \in NLS_3^+(\mathbb{R}^2)$
 $NLS_3^-(\mathbb{R}^2)$

$$\|v(t, x)\| = f(t).$$

$$f'(t_i)(t_{i+1} - t_i) \sim m_0.$$

$$(t_{i+1} - t_i)^{\frac{1}{2}} \sim \sqrt{\frac{m_0}{f'(t_i)}}$$

GWP / Scatters
 ..

Energy Critical Case

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u. \\ u(0) = \psi. \end{cases} \quad H^1\text{-critical}$$

Do \exists global-in-time classical solutions?

A1: Yes if $\|u_0\|_{H^1}$ small enough

A2: [B99] Yes if not H^1 -critical?

A3: [CKSTT03] Yes.

~~Induction Strategy. Minimal Energy blowup solutions.~~

Theorem: $H^1 \ni u_0 \mapsto u \in NLS_5^+(\mathbb{R}^3)$ is GWP and

$$\|u\|_{L_{t,x}^{10}} < C(u_0).$$

Strategy

$$M(E) = \sup_{I \subset \mathbb{C} \mathbb{R}_t, u_0: E[u_0] < E} \left\{ \|u\|_{L_{t,x}^{10}} \mid t \in I, x \in \mathbb{R}^3 \right\}.$$

$M(E) < \infty$ for E small enough.

Goal is to show $M(E) < \infty \forall E$.

Suppose not. $\exists E_{crit}$ s.t. $\forall E < E_{crit}$
we have it

Generalized Virial Identity.

Suppose ϕ satisfies

$$i\partial_t \phi + \Delta \phi = F(|\phi|^2) \phi, \quad x \in \mathbb{R}^d$$

Form

$$V_a[\phi](t) = \int a(x) |\phi|^2(t, x) dx.$$

$$\begin{aligned} \partial_t V_a[\phi] &= M_a[\phi] = \int a(x) \partial_t |\phi|^2 dx \\ &= \int a(x) \underbrace{\left\{ -2 \operatorname{Im}(\bar{\phi} \partial_t \phi) \right\}}_{\leftarrow \text{IBP}}(t, x) dx \end{aligned}$$

$$M_a[\phi] = \int a_j \underbrace{2 \operatorname{Im}(\bar{\phi} \phi_j)}_{\leftarrow \text{IBP}} dx.$$

$$\partial_t M_a[\phi] = \int a_j \underbrace{\partial_t 2 \operatorname{Im}(\bar{\phi} \phi_j)}_{\text{use eg. IBP}} dx$$

Generalized Virial Identity

$$\partial_t M_a = \int_{\mathbb{R}^d} (-\Delta \Delta a) |\phi|^2 + 4 a_{j,k} \operatorname{Re}(\phi_j \bar{\phi}_k) + \Delta a G(|\phi|^2) dx$$

where $G(z) = z F'(z) - F(z) \sim F(z).$

Example: $a(x) = |x|, \quad |\nabla a| < 1, \quad \Delta a = c \delta_0.$

$$C_E \geq M_a|_0^T \geq \int_0^T \int \frac{G(|\phi|^2)}{|x|} dx dt$$

Lin-Strouss
Morawetz

[Handwritten scribbles]

Interaction Morawetz.

(A. Hassel.)

Suppose $u_1 \in NLS_S^+(\mathbb{R}^3)$, $u_2 \in NLS_S^+(\mathbb{R}^3)$

Form $\phi(t, x_1, x_2) = u_1(t, x_1) u_2(t, x_2)$.

$$\text{Then } \left(i\partial_t \phi + \Delta_{\mathbb{R}^6} \phi \right) = \underbrace{\left(|u_1|^2 + |u_2|^2 \right)}_{\geq 0} \phi.$$

\hookrightarrow virial identity. What is a ?

$$(x_1, x_2) \in \mathbb{R}^6.$$

$$\bar{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3.$$

$$y = (x_1 - \bar{x}, x_2 - \bar{x}). \text{ "in } \mathbb{R}^3 \text{"}$$

$$a(x_1, x_2) = |y|^2.$$

$$M_a \Big|_0^T \geq \int_0^T \int_{\mathbb{R}^6} (-\Delta_y \Delta_y a) |\phi|^2 + (\geq 0).$$

$$= \int_0^T \int_{\mathbb{R}^6} \underbrace{(-\Delta_y \Delta_y a)}_{\delta_y = 0} |\phi|^2 dx^6 dt. \quad + (\geq 0).$$

$$= \int_0^T \int_{\mathbb{R}^3} |u(t, \bar{x}_1)|^2 |u(t, \bar{x}_2)|^2 d\bar{x}_y dt$$

$$= \|u(t)\|_{L^4}^4.$$