

# Weak Turbulence for Periodic NLS

James Colliander

**Abstract** This paper summarizes a talk given in the PDE Session at the 2006 International Congress on Mathematical Physics about joint work with M. Keel, G. Staffilani, H. Takaoka and T. Tao. We build new smooth solutions of the cubic defocussing nonlinear Schrödinger equation on the two dimensional torus which are weakly turbulent: given any  $\delta \ll 1$ ,  $K \gg 1$ ,  $s > 1$ , we construct smooth initial data  $u_0$  in the Sobolev space  $H^s$  with  $\|u_0\|_{H^s} < \delta$ , so that the corresponding time evolution  $u$  satisfies  $\|u(T)\|_{H^s} > K$  at some time  $T$ .

## 1 Introduction

This note describes aspects of joint work with M. Keel, G. Staffilani, H. Takaoka and T. Tao appearing in [3]. We study the initial value problem for the cubic defocussing nonlinear Schrödinger (*NLS*) equation

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u \\ u(0, x) = u_0(x) \end{cases} \quad (1)$$

where  $u(t, x)$  is a  $\mathbb{C}$ -valued function with  $x \in \mathbb{T}^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2$ . Smooth solutions of (1) satisfy energy conservation,

$$E[u](t) = \int_{\mathbb{T}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 dx(t) = E[u_0] \quad (2)$$

and mass conservation,

---

James Colliander

Department of Mathematics, University of Toronto, Bahen Centre, Toronto, Ontario M5S 3G3, Canada, e-mail: colliand@math.toronto.edu

$$\int_{\mathbb{T}^2} |u|^2 dx(t) = \int_{\mathbb{T}^2} |u_0|^2 dx, \quad (3)$$

for all  $t > 0$ . The local well-posedness result of Bourgain [1] for data  $u_0 \in H^s(\mathbb{T}^2)$ ,  $s > 0$ , and these conservation laws imply the existence of a unique global smooth solution to (1) evolving from smooth initial data.

The main result of the paper is the construction of solutions to (1) with arbitrarily large growth in high Sobolev norms:

**Theorem 1.** *Let  $1 < s$ ,  $K \gg 1$ , and  $0 < \delta \ll 1$  be given parameters. Then there exists a global smooth solution  $u(t, x)$  to (1) and a time  $T > 0$  with*

$$\|u(0)\|_{H^s} \leq \delta$$

and

$$\|u(T)\|_{H^s} \geq K.$$

Using the conservation laws, we have an  $H^1$ -stability property near zero,

$$\left( \limsup_{|t| \rightarrow \infty} \left[ \sup_{\|u_0\|_{H^1} \leq \delta} \|u(t)\|_{H^1} \right] \right) \leq C\delta.$$

Theorem 1 implies a different behavior in the range  $s > 1$ . Since  $\delta$  may be chosen to be arbitrarily small and  $K$  may be chosen arbitrarily large, we observe that (1) is strongly unstable in  $H^s$  near zero for all  $s > 1$ :

$$\inf_{\delta > 0} \left( \limsup_{|t| \rightarrow \infty} \left[ \sup_{\|u_0\|_{H^s} \leq \delta} \|u(t)\|_{H^s} \right] \right) = \infty. \quad (4)$$

It remains an open question [2] whether there exist solutions of (1) which satisfy  $\limsup_{|t| \rightarrow \infty} \|u(t)\|_{H^s} = \infty$ . Theorem 1 is also motivated by an effective (but not entirely rigorous) statistical description of the cascade toward high frequencies known as *weak turbulence theory* (see for example [4]). The relationship of Theorem 1 to previous literature is discussed in more detail in [3].

We overview the rest of this note and highlight some of the objects appearing in the proof of Theorem 1. Section 2 recasts the *NLS* equation as an equivalent infinite system of ordinary differential equations (ODEs) and introduces a resonant truncation  $R\mathcal{F}\text{NLS}$  of that ODE system. Section 3 imagines a finite set  $\Lambda$  in the frequency lattice  $\mathbb{Z}^2$  satisfying conditions which reduce the resonant truncation  $R\mathcal{F}\text{NLS}$  to the key object in the proof: the finite dimensional *Toy Model ODE System*. In Sect. 4, the Toy Model System is shown to have a solution with a particular dynamics which drives the cascade of energy toward higher frequencies in *NLS*. Section 5 briefly describes the construction of the special resonant set  $\Lambda$ .

## 93 2 NLS as an Infinite System of ODEs

94  
95 Equation (1) may be gauge transformed into  
96

$$97 \quad (-i\partial_t + \Delta)v = (G + |v|^2)v \quad (5)$$

98 by writing  $v(t, x) = e^{iGt}u(t, x)$ ,  $G \in \mathbb{R}$ . The constant  $G$  will soon be chosen to  
99 cancel away part of the nonlinearity. Motivated by the explicit solution formula for  
100 the linear Schrödinger equation, we make the *ansatz*  
101

$$102 \quad 103 \quad v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n \cdot x + |n|^2 t)} \quad (6)$$

104 and the dynamics are recast in terms of the Fourier coefficients  $\{a_n(t)\}_{n \in \mathbb{Z}^2}$ . A cal-  
105 culation shows that (5) (which is equivalent to (1)) transforms into an infinite ODE  
106 system  
107

$$108 \quad 109 \quad -i\partial_t a_n = G a_n + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t}, \quad (7)$$

110 where  $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$ . Some manipulations with the sum appearing  
111 in (7) and the choice  $G = -2\|u(t)\|_{L^2}^2$  cancels away certain nonlinear interactions  
112 and recasts (1) into an ODE system ( $\mathcal{FNLS}$ )  
113

$$114 \quad 115 \quad -i\partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t}, \quad (8)$$

116 where  
117

$$118 \quad 119 \quad \Gamma(n) = \{(n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}. \quad (9)$$

120 The set  $\Gamma(n)$  consists of frequency triples which contribute to the dynamics of the  
121 Fourier coefficient  $a_n$ . Among all triples in  $\Gamma(n)$ , we expect those in  
122

$$123 \quad 124 \quad \Gamma_{res}(n) = \{(n_1, n_2, n_3) \in \Gamma(n) : \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0\}. \quad (10)$$

125 will have the most influence on the dynamics of  $a_n$ . Heuristically, the phase fac-  
126 tor  $e^{i\omega_4 t}$  oscillates when  $\omega_4 \neq 0$  so the time integrated contribution of these non-  
127 resonant interactions to  $a_n$  should be small compared to those in  $\Gamma_{res}(n)$ . The defin-  
128 ing property for  $\Gamma_{res}(n)$  has a geometric interpretation:  $(n_1, n_2, n_3) \in \Gamma_{res}(n) \iff$   
129  $(n_1, n_2, n_3, n)$  form four corners of a non-degenerate rectangle with the segment  
130  $[n_2, n]$  forming one diagonal and  $[n_1, n_3]$  the other.  
131

132 We define the *resonant truncation R $\mathcal{FNLS}$*  of  $\mathcal{FNLS}$  by writing  
133

$$134 \quad 135 \quad -i\partial_t r_n = -r_n |r_n|^2 + \sum_{(n_1, n_2, n_3) \in \Gamma_{res}(n)} r_{n_1} \overline{r_{n_2}} r_{n_3}. \quad (11)$$

139 Our proof Theorem 1 constructs initial data  $\{r_n(0)\}_{n \in \mathbb{Z}^2}$  such that:

- 140 • The evolution  $r_n(0) \mapsto r_n(t)$  satisfying  $R\mathcal{F}NLS$  satisfies the conclusions of  
 141 Theorem 1.
- 142 • The evolution  $r_n(0) = a_n(0) \mapsto a_n(t)$  satisfying  $\mathcal{F}NLS$  is well-approximated  
 143 by  $r_n(t)$ .

144 The approximation step is standard and involves making the heuristic ideas about  
 145 the non-resonant interactions rigorous using a non-stationary phase analysis. Building  
 146 the data  $\{r_n(0)\}_{n \in \mathbb{Z}^2}$  which evolves along  $R\mathcal{F}NLS$  from low toward high fre-  
 147 quencies as in the statement of Theorem 1 is more intricate and is outlined in what  
 148 follows.

### 152 3 Conditions on a Finite Set $\Lambda \subset \mathbb{Z}^2$

154 The initial data  $\{r_n(0)\}_{n \in \mathbb{Z}^2}$  that we construct will satisfy  $r_n(0) = 0$  unless  $n \in$   
 155  $\Lambda \subset \mathbb{Z}^2$  where  $\Lambda$  is a specially designed finite set of lattice points. The set  $\Lambda$  and  
 156 the data  $\{r_n(0)\}$  will be constructed to satisfy a list of conditions which lead to a  
 157 simplification of  $R\mathcal{F}NLS$  which we call the *Toy Model ODE System*.

158 Imagine we can build a finite set  $\Lambda \subset \mathbb{Z}^2$  and choose initial data  $\{r_n(0)\}_{n \in \Lambda}$  sat-  
 159 isfying the following properties. For some integer  $N$  (eventually chosen to depend  
 160 upon the parameters  $s, \delta, K$  appearing in Theorem 1), the set  $\Lambda$  breaks up into  $N$   
 161 disjoint *generations*  $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_N$ . Each generation is comprised of nuclear  
 162 families. A *nuclear family* is a rectangle  $(n_1, n_2, n_3, n_4)$  where the frequencies<sup>1</sup>  
 163  $n_1, n_3$  (known as the “parents”) live in a generation  $\Lambda_j$ , and the frequencies  $n_2, n_4$   
 164 (known as the “children”) live in the next generation  $\Lambda_{j+1}$ . Suppose further that the  
 165 following conditions hold true:

- 166 1. *Initial Data Support*: The initial data  $r_n(0)$  is entirely supported in  $\Lambda$  (i.e.  $r_n(0) = 0$  whenever  $n \notin \Lambda$ ).
- 167 2. *Closure*: Whenever  $(n_1, n_2, n_3, n_4)$  is a rectangle in  $\mathbb{Z}^2$  such that three of the  
 168 corners lie in  $\Lambda$ , then the fourth corner must lie in  $\Lambda$ . In other words,  $(n_1, n_2,$   
 $n_3) \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda$ .
- 169 3. *∃! Spouse & Children*:  $\forall 1 \leq j < N$  and  $\forall n_1 \in \Lambda_j \exists$  a unique nuclear family  
 $170 (n_1, n_2, n_3, n_4)$  (up to trivial permutations) such that  $n_1$  is a parent of this family.  
 $171$  In particular each  $n_1 \in \Lambda_j$  has a unique spouse  $n_3 \in \Lambda_j$  and has two unique  
 $172$  children  $n_2, n_4 \in \Lambda_{j+1}$ .
- 173 4. *∃! Sibling & Parents*:  $\forall 1 \leq j < N$  and  $\forall n_2 \in \Lambda_{j+1} \exists$  a unique nuclear family  
 $174 (n_1, n_2, n_3, n_4)$  (up to trivial permutations) such that  $n_2$  is a child of this family.  
 $175$  In particular each  $n_2 \in \Lambda_{j+1}$  has a unique sibling  $n_4 \in \Lambda_{j+1}$  and two unique  
 $176$  parents  $n_1, n_3 \in \Lambda_j$ .
- 177 5. *Nondegeneracy*: The sibling of a frequency  $n$  is never equal to its spouse.
- 178 6. *Faithfulness*: Besides nuclear families,  $\Lambda$  contains no other rectangles.

---

182 <sup>1</sup> Note that if  $(n_1, n_2, n_3, n_4)$  is a nuclear family, then so is  $(n_1, n_4, n_3, n_2)$ ,  $(n_3, n_2, n_1, n_4)$ , and  
 183  $(n_3, n_4, n_1, n_2)$ ; we shall call these the *trivial permutations* of the nuclear family.

- 185 7. *Intragenerational Equality*: The function  $n \mapsto r_n(0)$  is constant on each genera-  
 186 tion  $\Lambda_j$ . Thus  $1 \leq j \leq N$  and  $n, n' \in \Lambda_j$  imply  $r_n(0) = r_{n'}(0)$ .  
 187 8. *Norm Explosion*:  $\sum_{n \in \Lambda_{N-2}} |n|^{2s} \gtrsim \frac{K^2}{\delta^2} \sum_{n \in \Lambda_3} |n|^{2s}$ .  
 188 9. *Inner Radius*: For large enough fixed  $\mathcal{R}$ ,  $\Lambda \cap \{n \in \mathbb{Z}^2 : |n| < \mathcal{R}\} = \emptyset$ . In other  
 189 words,  $\Lambda$  is supported far from the frequency origin.  
 190

191 Simple arguments based on the Gronwall inequality show that the initial data  
 192 support and intragenerational equality conditions propagate under  $R\mathcal{F}NLS$ . In other  
 193 words, for all times  $t$ , the solution  $\{r_n(t)\}_{n \in \mathbb{Z}^2}$  of  $R\mathcal{F}NLS$  emerging from data sat-  
 194 isfying the conditions above will satisfy  $r_n(t) = 0$  for all  $n \notin \Lambda$  and  $r_n(t) = r_{n'}(t)$   
 195 for  $n, n' \in \Lambda_j$ . Propagation of support implies that  $R\mathcal{F}NLS$  collapses to an ODE  
 196 indexed by  $n \in \Lambda$ . Propagation of intragenerational equality means that, for each  
 197 fixed  $t$ , the function  $n \mapsto r_n(t)$  is constant for  $n \in \Lambda_j$ . We can therefore introduce  
 198  $b_j(t) = r_n(t)$ ,  $n \in \Lambda_j$  and collapse further to

$$199 -i \partial_t b_j(t) = -|b_j(t)|^2 b_j(t) + 2b_{j-1}(t)^2 \overline{b_j}(t) + 2b_{j+1}(t)^2 \overline{b_j}(t), \quad (12)$$

200 which we call the *Toy Model ODE System*.

## 205 4 Arnold Diffusion for the Toy Model ODE

207 The system (12) defines a vector flow  $t \mapsto \mathbf{b}(t) = \{b_1(t), \dots, b_N(t)\} \in \mathbb{C}^N$ .  
 208 A calculation shows that  $|\mathbf{b}(t)|^2 = \sum_{j=1}^N |b_j(t)|^2 = |\mathbf{b}(0)|^2$ . In particular, the unit  
 209 sphere  $\mathbb{S} = \{\mathbf{x} \in \mathbb{C}^N : |\mathbf{x}| = 1\}$  in  $\mathbb{C}^N$  is invariant under the Toy Model flow.

211 Inside the sphere, we have the *coordinate circles*  $\mathbb{T}_1, \dots, \mathbb{T}_N$  defined by  $\mathbb{T}_j =$   
 212  $\{(b_1, \dots, b_N) \in \mathbb{S} : |b_j| = 1, b_k = 0 \forall k \neq j\}$ . For each  $j \in \{1, \dots, N\}$ , the vector  
 213 function  $b_j(t) = e^{-i(t+\theta)}$ ,  $b_k(t) = 0 \forall k \neq j$  is an *explicit oscillator solution* of  
 214 (12) that traverses  $\mathbb{T}_j$ . Here  $\theta \in \mathbb{R}$  is an arbitrary phase parameter.

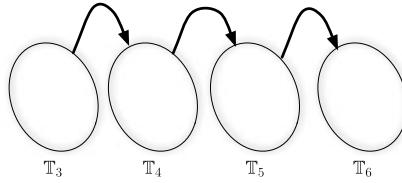
215 Between  $\mathbb{T}_1$  and  $\mathbb{T}_2$ , we also have an explicit *slider solution* of (12):

$$216 b_1(t) = \frac{e^{-it}\omega}{\sqrt{1 + e^{2\sqrt{3}t}}}; \quad b_2(t) = \frac{e^{-it}\omega^2}{\sqrt{1 + e^{-2\sqrt{3}t}}}; \quad b_k(t) = 0 \quad \forall k \neq 1, 2,$$

217 where  $\omega = e^{\frac{2\pi i}{3}}$  is a cube root of unity. This solution approaches the coordinate  
 218 circle  $\mathbb{T}_1$  exponentially fast as  $t \rightarrow -\infty$  and approaches the coordinate circle  $\mathbb{T}_2$   
 219 as  $t \rightarrow +\infty$ . There are also slider solutions between  $\mathbb{T}_j$  and  $\mathbb{T}_{j+1}$  for each  $j \in$   
 220  $\{1, \dots, N-1\}$ .

221 Using delicate dynamical systems arguments, we jiggle the sliders to construct  
 222 a solution of (1) which starts near  $\mathbb{T}_3$  and, in a finite time, travels very close to  $\mathbb{T}_4$ .  
 223 After another finite time, it departs from near  $\mathbb{T}_4$  and moves very close to  $\mathbb{T}_5$ . The  
 224 solution continues this pattern of riding across jiggle sliders between  $\mathbb{T}_j$  and  $\mathbb{T}_{j+1}$   
 225 until it arrives very close to  $\mathbb{T}_{N-2}$ . Let  $S(t)$  denote the flow map for (12), so  $\mathbf{b}(t) =$   
 226  $S(t)\mathbf{b}(0)$  solves (12).

231  
232  
233  
234  
235  
236

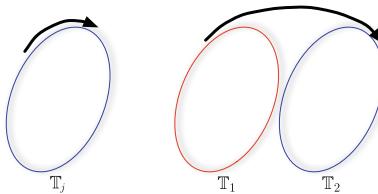


237 **Fig. 1** Explicit oscillator solution around  $\mathbb{T}_j$  and the slider solution from  $\mathbb{T}_1$  to  $\mathbb{T}_2$

238  
239

240 **Theorem 2 (Arnold Diffusion for (12)).** Let  $N \geq 6$ . Given any  $\epsilon > 0$ , there exists  
241 a point  $\mathbf{x}_3 \in \mathbb{C}^N$  within  $\epsilon$  of  $\mathbb{T}_3$  (using the usual metric on  $\mathbb{S}$ ), a point  $\mathbf{x}_{N-2} \in \mathbb{C}^N$   
242 within  $\epsilon$  of  $\mathbb{T}_{N-2}$ , and a time  $t \geq 0$  such that  $S(t)\mathbf{x}_3 = \mathbf{x}_{N-2}$ .

243  
244



251 **Fig. 2** The idea behind the proof of Theorem 2 is to concatenate jiggled slider solutions to flow  
252 from nearby the coordinate circle  $\mathbb{T}_3$  to arrive nearby  $\mathbb{T}_{N-2}$

253  
254

255 We briefly explain how Theorem 2 implies Theorem 1. We can inflate the Toy  
256 Model solution of Theorem 2 into a solution of  $R\mathcal{F}NLS$  (11) by recalling that  
257  $b_j(t) = r_n(t)$ ,  $\forall n \in \Lambda_j$ . Thus, this solution of  $R\mathcal{F}NLS$  initially starts mostly  
258 supported on  $\Lambda_3$  but evolves to a time when it is mostly supported on  $\Lambda_{N-2}$ . The  
259 Norm Explosion condition (8) then implies this solution satisfies the Sobolev norm  
260 claims in Theorem 1. Finally, an approximation result, which shows this  $R\mathcal{F}NLS$   
261 evolution is appropriately close to the  $\mathcal{F}NLS$  (8) evolution emerging from the same  
262 data, completes the proof of Theorem 1.

263

264

265

## 266 5 Construction of the Resonant Set $\Lambda$

267

268 We construct a subset  $\Lambda$  of the lattice  $\mathbb{Z}^2$  satisfying the eight properties listed in  
269 Sect. 3 in two stages. First, we build an abstract *combinatorial model*  $\Sigma = \Sigma_1 \cup$   
270  $\dots \cup \Sigma_N$  which will turn out to be a subset of  $\mathbb{C}^{N-1}$ . Next, we define a *placement*  
271 function  $f : \Sigma \rightarrow \mathbb{C}$  so that  $f(\Sigma) = \Lambda \subset \mathbb{Z}^2$ .

272

273 We describe the construction of the combinatorial model  $\Sigma$ . We define the *standard unit square*  
274  $S \subset \mathbb{C}$  to be the four element set  $S = \{0, 1, 1+i, i\}$ . We decompose  
275 this set  $S = S_1 \cup S_2$  where  $S_1 = \{1, i\}$  and  $S_2 = \{0, 1+i\}$ . We define  $\Sigma_j \subset \mathbb{C}^{N-1}$   
276 to be the set of all  $(N-1)$ -tuples  $(z_1, \dots, z_{N-1})$  such that  $z_1, \dots, z_{j-1} \in S_2$  and

277  $z_j, \dots, z_{N-1} \in S_1$ . Thus,  $\Sigma_j = S_2^{j-1} \times S_1^{N-j}$ . We define  $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_N$ . The  
 278 set  $\Sigma_j$  is the  $j^{\text{th}}$  generation of  $\Sigma$ .

279 Consider the four element set  $F \subset \Sigma_j \cup \Sigma_{j+1}$  defined by  
 280

$$281 \quad F = \{(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_N) : w \in S\}$$

283 where  $z_1, \dots, z_{j-1} \in S_2$  and  $z_{j+1}, \dots, z_N \in S_1$  are all fixed and  $w$  varies among  
 284 the four points of the standard unit square. The two elements of  $F$  corresponding to  
 285  $w \in S_1$ , which we denote  $F_1, F_i$ , are in generation  $\Sigma_j$  (the parents) while the two  
 286 elements corresponding to  $w \in S_2$ , which we denote  $F_0, F_{1+i}$ , are in generation  
 287  $\Sigma_{j+1}$  (the children). We call the four element set  $F$  a *combinatorial nuclear family*  
 288 *connecting generations*  $\Sigma_j, \Sigma_{j+1}$ . For each  $j$ , there exists  $2^{N-2}$  combinatorial  
 289 nuclear families connecting generations  $\Sigma_j, \Sigma_{j+1}$ . The existence and uniqueness  
 290 conditions (3), (4) and the nondegeneracy condition (5) can now be checked to hold  
 291 true for  $\Sigma$ .

292 Next, we motivate aspects of the construction of the placement function which  
 293 embeds  $\Sigma$  into a subset of the frequency lattice  $\mathbb{Z}^2$ . We identify  $\mathbb{Z}^2$  with the Gaussian  
 294 integers  $\mathbb{Z}[i]$  in the discussion below. Suppose  $f_1 : \Sigma_1 \rightarrow \mathbb{C}$  is defined. This means  
 295 that the frequencies in the first generation have been placed on the plane. We want to  
 296 define  $f_2 : \Sigma_2 \rightarrow \mathbb{C}$ , that is we want to place down the next generation of frequen-  
 297 cies, in such a way that the images of combinatorial nuclear families connecting  
 298 generations  $\Sigma_1, \Sigma_2$  form rectangles in the frequency lattice. We want the combi-  
 299 natorial nuclear families to map to nuclear families linking generations  $\Lambda_j, \Lambda_{j+1}$ .  
 300 The diagonal of the rectangle going from two parent frequencies in  $\Lambda_1$  is deter-  
 301 mined by  $f_1$ . The constraint that the image of the combinatorial nuclear families  
 302 form rectangles in the plane does not determine the placement of the child frequen-  
 303 cies. Indeed, there is the freedom to choose the angle between the diagonals of the  
 304 rectangle. Therefore, for each  $j \in \{1, \dots, N-1\}$  and for each combinatorial nuclear  
 305 family  $F$  connecting generations  $\Sigma_j, \Sigma_{j+1}$ , we define an angle  $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$ .

306 The placement function is then defined recursively with respect to the generation  
 307 index  $j$  using the angles associated to the nuclear families. Suppose that we have  
 308 defined the placement function components  $f_j : \Sigma_j \rightarrow \mathbb{C}$  for all  $j \in \{1, \dots, k\}$  for  
 309 some  $k < N - 1$ . We need to define  $f_{k+1} : \Sigma_{k+1} \rightarrow \mathbb{C}$  to set up the recursion. By  
 310 the combinatorial construction of  $\Sigma$ , each element of  $\Sigma_{k+1}$  is a child of a unique (up  
 311 to trivial permutations) combinatorial nuclear family linking  $\Sigma_k, \Sigma_{k+1}$ . We define  
 312  $f_{k+1} : \Sigma_{k+1} \rightarrow \mathbb{C}$  by requiring

$$313 \quad f_{k+1}(F_{1+i}) = \frac{1 + e^{i\theta(F)}}{2} f_k(F_1) + \frac{1 - e^{i\theta(F)}}{2} f_k(F_i)$$

$$314 \quad f_{k+1}(F_0) = \frac{1 + e^{i\theta(F)}}{2} f_k(F_1) - \frac{1 - e^{i\theta(F)}}{2} f_k(F_i)$$

318 for all combinatorial nuclear families  $F$  connecting  $\Sigma_k, \Sigma_{k+1}$ .  
 319

Using some measure theory, the density of the complex rationals  $\mathbb{Q}[i]$  in  $\mathbb{C}$ , and the fact that the angles of Pythagorean<sup>2</sup> triangles are dense in  $\mathbb{R}/2\pi\mathbb{Z}$ , we show that there are choices of the initial placement function  $f_1$  and the angles  $\theta(F)$  which define a lattice subset  $\Lambda$  satisfying the required properties:

**Theorem 3 (Construction of a good placement function).** *Let  $N \geq 2$ ,  $s > 1$ , and let  $\mathcal{R}$  be a sufficiently large integer (depending on  $N$ ). Then there exists an initial placement function  $f_1 : \Sigma_1 \rightarrow \mathbb{C}$  and choices of angles  $\theta(F)$  for each nuclear family  $F$  (and thus an associated complete placement function  $f : \Sigma \rightarrow \mathbb{C}$ ) with the following properties:*

- (Nondegeneracy) *The function  $f$  is injective.*
- (Integrality) *We have  $f(\Sigma) \subset \mathbb{Z}[i]$ .*
- (Magnitude) *We have  $C(N)^{-1}\mathcal{R} \leq |f(x)| \leq C(N)\mathcal{R}$  for all  $x \in \Sigma$ .*
- (Closure and Faithfulness) *If  $x_1, x_2, x_3 \in \Sigma$  are distinct elements of  $\Sigma$  are such that  $f(x_1), f(x_2), f(x_3)$  form the three corners of a right-angled triangle, then  $x_1, x_2, x_3$  belong to a combinatorial nuclear family.*
- (Norm explosion) *We have*

$$\sum_{n \in f(\Sigma_{N-2})} |n|^{2s} > \frac{1}{2} 2^{(s-1)(N-5)} \sum_{n \in f(\Sigma_3)} |n|^{2s}.$$

The reader is invited to consult [3] for further details.

## References

1. J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* **3**(2), 107–156 (1993)
2. J. Bourgain, Problems in Hamiltonian PDE’s. *Geom. Funct. Anal.* (Special Volume, Part I), 32–56 (2000)
3. J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Weakly turbulent solutions for the cubic defocussing nonlinear Schrödinger equation. Preprint (2008), pp. 1–54. <http://arxiv.org/abs/0808.1742v1>
4. S. Dyachenko, A.C. Newell, A.C.A. Pushkarev, and V.E. Zakharov, Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schrödinger equation. *Physica D* **57**(1–2), 96–160 (1992)

---

<sup>2</sup> Pythagorean triangles are right triangles with integer sides.