Mapping/-in-time issues for semilinear NLS

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J. Colliander

Rochester Talk ?

$NLS_p^{\pm}(\mathbb{R}^d)$.

Conservation — Energy or Hamiltonian, momentum, mass } linked

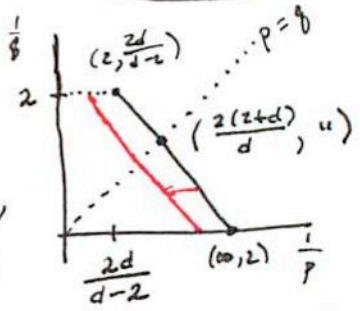
local versions $\partial_t |u|^2 = \nabla \cdot 2 I_m (u \nabla \bar{u})$

Dilation Symmetry

$$u_\lambda : [0, \lambda^2 T]_\tau \times \mathbb{R}^d_y \longmapsto \mathbb{C} \quad \Longleftrightarrow \quad u : [0,T]_t \times \mathbb{R}^d_x \longmapsto \mathbb{C}$$

$$u(\tau, y) = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1}} u\left(\frac{\tau}{\lambda^2}, \frac{x}{\lambda}\right)$$

$$\| D_x^{s_c} u_\lambda \|_{L^p} \sim \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1} + s - \frac{d}{p}} \quad : \quad L^2\text{-based } \dot{H}^{s_c} : \quad s_c = \frac{d}{2} - \frac{2}{p-1}$$



Strichartz Estimates

$$\|u\|_{S^0} = \sup_{(q,r)} \|u\|_{L_{t\in I}^q L_x^r} \leq \|u(t_0)\|_{L_x^2} + \|(i\partial_t + \Delta) u\|_{L_{t\in I}^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad ; \quad (q,r) \neq (\tilde{q}, \tilde{r}) \quad ; \quad \text{/ Hölder dual.}$$

$(2, \infty)$

LWP Theory

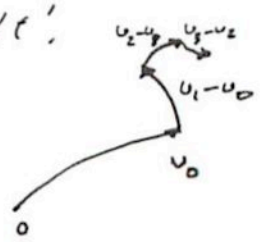
Duhamel's Formula : $u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} \left(|u|^{p-1} u \right)(t')\, dt'$.

$$v^{(-1)} \equiv 0 \longrightarrow -i \int_0^t e^{i(t-t')} \left(|v^{(-1)}|^{p-1} v^{(-1)} \right)(t')\, dt' \longrightarrow \cdots$$

$$v^0 = e^{it\Delta} u_0.$$

$$v^{j+1} = e^{it\Delta} u_0 + i \int_0^t e^{i(t-t')\Delta} \left(|v^j|^{p-1} v^j \right)(t')\, dt'.$$

$\{ v^j \}_{j=-1}^{\infty}$



$$\| D_x^{s_c} (\underbrace{v^{j+1}}_{v} - \underbrace{v^j}_{w}) \|_{S^0} \leq \| D_x^{s_c} (v^p - w^p) \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$$\leq \| (v^{p-1} + w^{p-1}) \underbrace{D_x^{s_c} (v - w)}_{S^0} \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

Symmetrize over Hölder

$$\leq \left(\|v\|_{L_t^{q_1} L_x^{r_1}}^{p-1} + \|w\|_{L_t^{q_1} L_x^{r_1}}^{p-1} \right) \| D_x^{s} (v^{j+1} - v^j) \|_{S^0}.$$

$$\frac{2}{q_1} + \frac{d}{r_1} = \frac{d}{2} - s_c$$

The condition $\| e^{it\Delta} u_0 \|_{L^{q_1}_t L^{r_1}_x} < \frac{1}{100}$ propagates to imply

$$\forall j \qquad \| u^{(j)} \|_{L^{q_1}_t L^{r_1}_x} < \frac{1}{50}.$$

$$\Longrightarrow \qquad \| u \|_{L^{q_1}_t L^{r_1}_x} < \frac{1}{50} \quad \text{provided} \quad \| e^{it\Delta} u_0 \|_{L^{q_1}_t L^{r_1}_x} < \frac{1}{100}.$$

$\in I$ $\qquad$ Critical LWP $\qquad$ $\in I$

small data scattering $\qquad$ T^c_{lwp}.

focusing or defocusing

Maximality Criterion.

$$\| u \|_{L^{q_1}_{t\in[0,T^*)} L^{r_1}_x} = +\infty$$

$$T^* - \delta < +\infty \quad \forall \ \delta > 0$$

blowup: $T^* < \infty$

$T^* = \infty$ → other? ∞. → bounded scatter

Q: $NLS^{\pm}_p(\mathbb{R}^d)$: which case occurs? Describe solution.

$$\frac{2}{q(s)} + \frac{d}{r_1} = \frac{d}{2} - s.$$

Subcritical LWP

Hölder in time

$$\| e^{it\Delta} u_0 \|_{L^{q_1}_{t\in I} L^{r_1}_x} \le |I|^{\beta(s)} \| e^{it\Delta} u_0 \|_{L^{q(s)}_{t\in I} L^{r_1}_x}$$

$$\le |I|^{\beta(s)} \| u_0 \|_{H^s}$$

Scaling lower bound / Subcritical LWP lifetime

t near T^*. $\quad v(\tau, y) = \lambda^{\frac{2}{p-1}} u(t + \tau\lambda^2, \lambda x)$.

$\| v|_{\tau=0} \|_{H^s} = \lambda^{s-s_c} \| u(t) \|_{H^s}$. Choose λ s.t. $\| v|_{\tau=0} \|_{H^s} = 1$.

$v|_{\tau=0} \longmapsto v$ valid for $\tau \in \mathcal{O}(1)$.

$$\lambda \sim \left(\frac{1}{\| u(t) \|_{H^s}} \right)^{\frac{1}{s-s_c}}$$

$\Downarrow$

$\lambda^2 < (T^* - t)$ $\qquad$ LWP size is λ^2

$$\boxed{\begin{aligned} |[0, T_{H^s\text{-lwp}}]| &\sim \left(\frac{1}{\| u(t) \|_{H^s}} \right)^{\frac{2}{s-s_c}} \\ \| u(t) \|_{H^s} &> \left\{ \frac{1}{(T^* - t)} \right\}^{\frac{s-s_c}{2}} \end{aligned}}$$

Subcritical Norms Explode

Critical Norm Behavior unclear from LWP Theory.

Maximal-in-time knowledge $\Leftarrow$
- small data scattering
- global conservation laws / subcritical regularity eg. [W]
- spatially localized conservation laws virial, Morawetz
- Almost conservation laws / Freq. localized conserv. B99, CKSTT

Critical techniques [B99], [CKSTT]
- induction on energy; minimal blowup...
- profile decomposition [MV], [KM], ...
- perturbation theory / super LWP.

Restrict attention to $s_c \geq 0$ case.

Defocusing Scattering Conjecture $NLS_p^+(\mathbb{R}^d)$.

$H^{s_c} \ni u_0 \longmapsto u$, GWP + $\|D^{s_c} u\|_{S^0} < \infty$.

small data theory

Evidence: energy subcritical setting

$0 \leq s_c < 1$; $u_0 \in H^1 \Longrightarrow u_0 \longmapsto u$ GWP.

H^1 control not necessary.

$0 \leq s_c < 1$: $\exists\ s_{GWP} \in (s_c, 1)$ s.t. $\forall\ s > s_{GWP}$

$u_0 \in H^s \Longrightarrow u_0 \longmapsto u$ GWP.

} Energy subcritical methods. Bilow, I-method, Colliander/Planchon.

$0 \leq s_c < 1$: $d=1$, $L^8_{t,x \in \mathbb{R}}$ Morawetz $\Longrightarrow$ $\|D^s u\|_{S^0} < \infty$ for $s \in (s_{Moraw}, 1)$

$d=3$, $L^4_{t,x \in \mathbb{R}}$ Morawetz

$s_c = 1$ energy critical setting

	Radial	general
$NLS_5^+(\mathbb{R}^3)$	[B99] [KM]	[CKSTT 03]
$NLS_3^+(\mathbb{R}^4)$	[B99] [KM]	[RV]
$NLS_{\frac{7}{3}}^+(\mathbb{R}^5)$	[KM]	
$NLS_{\frac{4}{d-2}}^+(\mathbb{R}^d)$	[Tao 04] [Vz]	[V], [TV]

Hartree w. Bluc, Tzirakis

Minimal Mass Blowup characterization in H^1 (F. Merle)

$u_0 \in H^1$, $u_0 \longmapsto u$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $T^* < \infty$.

$\Longrightarrow \quad u(t) = S(t - T^*)$ (up to H^1 symmetries of equation)

Conjecture: Minimal mass blowup characterization in L^2

$L^2 \ni u_0 \longmapsto u$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $T^* < \infty$

$\Longrightarrow \quad u(t) = S(t - T^*)$ (mod symmetries).

[Richards]

H^s ?

Virial Identity

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \, dx = 16 \, H[u_0]$$

Negative Energy initial data have variance $\searrow 0$ in finite time.

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = |(u,u)| = |(u, \tfrac{1}{d}(\nabla \cdot x) u)| \le \frac{1}{d} \underbrace{\|\nabla u(t)\|_{L^2}}_{\nearrow \infty} \underbrace{\|\, |x| u \|_{L^2}}_{\searrow 0}$$

$s_c = 0$

Mass Critical Setting / pseudoconformal NLS

Radial | General

$NLS_5^+(\mathbb{R})$

$NLS_3^+(\mathbb{R}^2)$

$(d \geq 3)$ $NLS_{\frac{4}{d}}^+(\mathbb{R}^d)$

? ? New Morawetz ? [FG] [CKSTT].

[TVZ]

$d \geq 3$ Morawetz

(Blue-c]
L^2-GWP $\Leftrightarrow$ L^2-Scattering)

$s_c = \frac{1}{2}$

$H^{1/2}$-critical setting (typical of $0 < s_c < 1$?)

Radial | General

$NLS_5^+(\mathbb{R}^2)$

$NLS_3^+(\mathbb{R}^3)$

$NLS_{\frac{4}{d-1}+1}^+(\mathbb{R}^d)$

CRSW

Profile decomposition / Bubbles
Lin-Strauss Morawetz
Almost Conservation vs. Critical explosion

Crit Norm Conjecture:
$s_c > 0$: critical norm divergence if $T^* < \infty$

$s_c > 1$

Energy supercritical setting

Radial | General

$NLS_7^+(\mathbb{R}^3)$

Numerically ✓
[SVZ], [BJSoulem].

? ?

$d \geq 3$; $NLS_p^+(\mathbb{R}^d)$?

Focusing Case $\quad NLS_p^- (\mathbb{R}^d)$.

$s_c < 0$ $\quad L^2$ - subcritical case.

LWP: $L^2 \ni u_0 \longmapsto u$ GWP in L^2.

Dynamics ?

$s_c = 0$ $\quad L^2$ - critical case / pseudoconformal NLS: $NLS^-_{1+\frac{4}{d}}(\mathbb{R}^d)$

Weinstein Inequality:

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \underbrace{\frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \, dx}_{\text{optimal G-N}} \quad ; \quad p = \frac{4}{d} + 1 .$$

$$E[u] \geq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx \left\{ 1 - \left(\frac{\|u\|_{L^2_x}}{\|Q\|_{L^2_x}} \right)^{\frac{4}{d}} \right\}.$$

Q is the unique (up to translations) positive solution of

$$-Q + \Delta Q = -Q^{1+\frac{4}{d}} .$$ The "ground state."

$\hookrightarrow e^{it} Q(x)$ "ground state soliton" nondispersive.

$\Longrightarrow \quad u_0 \in H^1, \ \|u_0\|_{L^2} < \|Q\|_{L^2} \rightsquigarrow u_0 \longmapsto u$ GWP.

H^s (subcritical mass + $H^1 \Rightarrow$ GWP)

L^2 - Scattering below ground state mass conjecture

$u_0 \in L^2, \ \|u_0\|_{L^2} < \|Q\|_{L^2} \Longrightarrow u_0 \longmapsto u$, GWP + $\|u\|_{S^0} < \infty$.

Minimal Mass Blowup Solution

$$pc\left[e^{it} Q(x)\right] = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} := S(t,x).$$

$S(-1) \longmapsto S(t)$ explicit blowup at $t = 0$.

$\|S(-1)\|_{L^2} = \|Q\|_{L^2}$.

$\Longrightarrow$ Weinstein's smallness condition is optimal.