Weak Turbulence for *NLS*

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1 Weak Turbulence for NLS

2 Overview of proof

- Resonant finite dimensional truncations approximate *NLS*
- Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...
- ...we get a low $\longrightarrow$ high frequency travelling wave across Λ
- Combinatorial construction of $\Lambda \subset \mathbb{Z}^2$ such that...

3 Remarks

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- Global-in-time well-posedness is known for $u_0 \in H^s, s > \frac{2}{3}$. [De Silva, Pavlovic, Staffilani, Tzirakis 2006]

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Does the conserved mass stay put in frequency space or does it cascade up to high frequencies?

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$$\begin{cases} i\partial_t a_n = 2Ga_n + \displaystyle\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1}\overline{a}_{n_2}a_{n_3}e^{i\omega_4 t} \\ \qquad a_n(0) = \widehat{u_0}(n), \qquad n \in \mathbb{Z}^2. \end{cases} \qquad (\mathcal{F}NLS_G)$$

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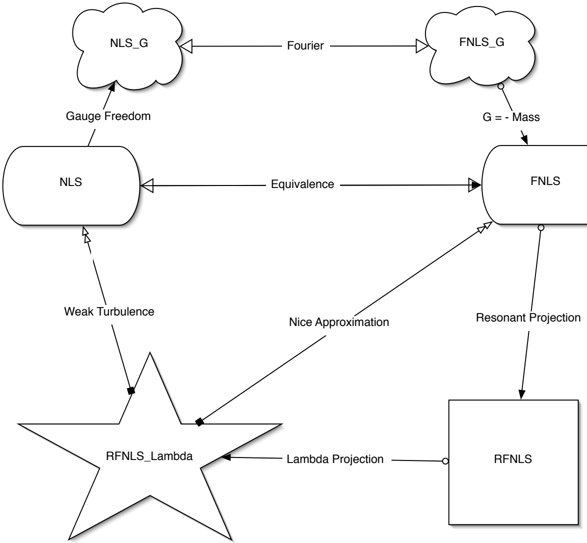
- Given ϵ, s, K, build Λ so that $R\mathcal{F}NLS_\Lambda$ has weak turbulence.

Resonant finite dimensional truncations approximate *NLS*



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- Besides nuclear families, Λ contains no other rectangles.
- The function $n \longmapsto a_n(0)$ is constant on each generation Λ_j.

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$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

THE TOY MODEL ODE

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$$i\partial_t b_j(t) = |b_j(t)|^2 b_j(t) - 2b_{j-1}(t)^2 \overline{b}_j(t) - 2b_{j+1}(t)^2 \overline{b}_j(t).$$

$$L^2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2$$

$$H^s \sim \sum_j |b_j(t)|^2 \Big(\sum_{n\in\Lambda_j} |n|^{2s}\Big).$$

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We also want $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ to satisfy

$$\sum_{n\in\Lambda_M} |n|^{2s} \gg \sum_{n\in\Lambda_1} |n|^{2s}.$$

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&\vdots
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&\cdot \\
&\cdot \\
&\cdot \\
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Bulk of conserved mass is transferred from Λ_1 to Λ_M. Weak turbulence follows, provided we can construct such a Λ.

Combinatorial construction of $\Lambda \subset \mathbb{Z}^2$

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Remarks