Weak Turbulence for NLS

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1 Weak Turbulence for NLS

2 Overview of proof
- Resonant finite dimensional truncations approximate NLS
- Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...
- ...we get a low $\rightarrow$ high frequency travelling wave across $\Lambda$
- Combinatorial construction of $\Lambda \subset \mathbb{Z}^2$ such that...

3 Remarks
Defocusing cubic Nonlinear Schrödinger on $\mathbb{T}^2$

Consider the initial value problem:

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\begin{align*}
&i \partial_t u + \Delta u = |u|^2 u \\
&u(0, x) = u_0(x), \quad x \in \mathbb{T}^2.
\end{align*}
\] (NLS)

Local-in-time well-posedness (LWP) is known for $u_0 \in H^s$, $s > 0$. [Bourgain 1993]

Time Invariant Quantities:

- Mass $\|u(t)\|_{L^2}$
- Hamiltonian $\int_{\mathbb{T}^2} |\nabla u(t)|^2 \, dx + \frac{1}{2} |u(t)|^4 \, dx$

Global-in-time well-posedness is known for $u_0 \in H^s$, $s > 2 \frac{3}{2}$. [De Silva, Pavlovic, Staffilani, Tzirakis 2006]
Defocusing cubic Nonlinear Schrödinger on \( \mathbb{T}^2 \)

Consider the initial value problem:

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- Global-in-time well-posedness is known for \( u_0 \in H^s, s > \frac{2}{3} \). [De Silva, Pavlovic, Staffilani, Tzirakis 2006]
What happens to smooth solutions?

Suppose \( u_0 \in H^s(T^2) \) for \( s > 1 \). What happens to \( \|u(t)\|_{H^s} \)?

Recall that

\[
\|f\|_{H^s_x} = \left\| (1 + |\xi|)^s \hat{f}(\xi) \right\|_{L^2_\xi}.
\]

LWP \( \Rightarrow \|u(t)\|_{H^s} \lesssim e^{Ct} \).

LWP + Dispersive Smoothing \( \Rightarrow \|u(t)\|_{H^s} \lesssim (1 + |t|)^{\alpha(s)} \).

[Bourgain 1996, Staffilani 1998]

Weak Turbulence Conjecture:

Exist \( s \) solutions with \( \|u(t)\|_{H^s} \to \infty \) as \( t \to \infty \)?

Does the conserved mass stay put in frequency space or does it cascade up to high frequencies?
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(Very) weak turbulence result

Theorem (C-Keel-Staffilini-Takaoka-Tao)

Given \( s > 1, \epsilon \ll 1, K \gg 1 \), there exists a smooth solution \( u(t) \) of NLS and a time \( T \) such that

\[
\| u(0) \|_{H^s} \leq \epsilon, \quad \| u(T) \|_{H^s} \geq K.
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Overview of proof:
Preliminary reductions

Gauge Freedom:

If $u$ solves \textit{NLS}, then $v(t, x) = e^{-i \frac{G}{2} t} u(t, x)$ solves \textit{NLS}.

Fourier Ansatz:

Recast the dynamics in Fourier coefficients,

$$v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i (n \cdot x + |n|^2 t)}.$$}

$$i \partial_t a_n = 2 G a_n + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2: n_1 - n_2 + n_3 = n} a_{n_1} a_{n_2} a_{n_3} e^{i \omega_4 t} a_n(0) = \hat{u}_0(n), n \in \mathbb{Z}^2.$$}

$$\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2.$$
Preliminary reductions

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\[
\begin{align*}
    i\frac{\partial}{\partial t} a_n &= 2Ga_n + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = n [a_{n_1}a_{n_2}a_{n_3}] e^{i\omega_4 t} \\
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- **Gauge Freedom:**
  If $u$ solves NLS then $v(t, x) = e^{-i2Gt} u(t, x)$ solves
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  \begin{cases}
  i\partial_t v + \Delta v = (2G + |v|^2)v \\
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  n_1 - n_2 + n_3 = n \\
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\( \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 \).
PRELIMINARY REDUCTIONS
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- Diagonal decomposition of sum:

\[
\sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = \sum_{n_1, n_3 \in \mathbb{Z}^2} n_1 + \sum_{n_1, n_2, n_3 \in \mathbb{Z}^2} n_1 - n_2 + n_3 = \sum_{n_1, n_3 \in \mathbb{Z}^2} n_1 + \sum_{n_1, n_3 \in \mathbb{Z}^2} n_3
\]

Choice of $G$:

\[
G = -\|u_0\|_2 L_2
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Preliminary reductions

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\[
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- Choice of $G$:

\[
G = -\|u_0\|_{L^2}^2.
\]
Resonant truncation

\[-i \frac{\partial}{\partial t} a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} a_{n_2} a_{n_3} e^{i \omega_4 t}.\]

\[\Gamma_{\text{res}}(n) = \{ n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0 \}.\]

Resonant truncation of \( F_{\text{NLS}} \) is

\[-i \frac{\partial}{\partial t} b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n)} b_{n_1} b_{n_2} b_{n_3}.\]

\[\left( R F_{\text{NLS}} \right)\]
Resonant truncation

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\[-i \partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i \omega_4 t}. \quad (\mathcal{F}NLS)\]
Resonant truncation

- \textit{NLS} dynamic is recast as

\[-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\bar{a}_{n_2}a_{n_3}e^{i\omega_4 t}. \quad (\mathcal{F NLS})\]

where

\[\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n\}.\]
Resonant truncation

- $NLS$ dynamic is recast as

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- $\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0\}.$
Resonant truncation

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- \[ \Gamma_{res}(n) = \{ n_1, n_2, n_3 \in \Gamma(n) : \omega_4 = 0 \}. \]

\[ = \{ \text{Triples } (n_1, n_2, n_3) : (n_1, n_2, n_3, n_4) \text{ is a rectangle} \} \]
Resonant truncation

- NLS dynamic is recast as

\[ -i \partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a}_{n_2} a_{n_3} e^{i\omega_4 t}. \quad (\mathcal{F}NLS) \]

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- The resonant truncation of \( \mathcal{F}NLS \) is

\[ -i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b}_{n_2} b_{n_3}. \quad (R\mathcal{F}NLS) \]
Finite dimensional resonant truncation

A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if $n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda \Rightarrow n \in \Lambda$.

A finite dimensional resonant truncation of $\mathcal{F}_{\text{NLS}}$ is\
$$-i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda} b_{n_1} b_{n_2} b_{n_3}.$$

($R_{\mathcal{F}_{\text{NLS}}\Lambda}$)\

$\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$, $R_{\mathcal{F}_{\text{NLS}}\Lambda}$ is an ODE.

If $\text{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}_{\text{NLS}}$-evolution $a_n(0) \mapsto -\to b_n(t)$ is nicely approximated by $R_{\mathcal{F}_{\text{NLS}}\Lambda}$-ODE $a_n(0) \mapsto -\to b_n(t)$.

Given $\epsilon, s, K$, build $\Lambda$ so that $R_{\mathcal{F}_{\text{NLS}}\Lambda}$ has weak turbulence.
A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$
 finite dimensional resonant truncation

A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

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A finite dimensional resonant truncation of $\mathcal{F}NLS$ is

$$-i \partial_t b_n = -b_n |b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} \overline{b}_{n_2} b_{n_3}. \quad (R \mathcal{F}NLS_{\Lambda})$$
A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

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$\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $R\mathcal{F}NLS_{\Lambda}$ is an ODE.
Finite dimensional resonant truncation

A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

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A finite dimensional resonant truncation of $\mathcal{F}\text{NLS}$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} \bar{b}_{n_2} b_{n_3}. \ (R\mathcal{F}\text{NLS}_\Lambda)$$

$\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2 \ R\mathcal{F}\text{NLS}_\Lambda$ is an ODE.

If spt($a_n(0)$) $\subset \Lambda$ then $\mathcal{F}\text{NLS}$-evolution $a_n(0) \mapsto a_n(t)$ is nicely approximated by $R\mathcal{F}\text{NLS}_\Lambda$-ODE $a_n(0) \mapsto b_n(t)$. 
A set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Lambda \implies n \in \Lambda.$$

A finite dimensional resonant truncation of $\mathcal{F}NLS$ is

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n) \cap \Lambda^3} b_{n_1} b_{n_2} b_{n_3}. \quad (RFNLS_{\Lambda})$$

All resonant-closed finite $\Lambda \subset \mathbb{Z}^2$ $RFNLS_{\Lambda}$ is an ODE.

If $\text{spt}(a_n(0)) \subset \Lambda$ then $\mathcal{F}NLS$-evolution $a_n(0) \mapsto a_n(t)$ is nicely approximated by $RFNLS_{\Lambda}$-ODE $a_n(0) \mapsto b_n(t)$.

Given $\epsilon, s, K$, build $\Lambda$ so that $RFNLS_{\Lambda}$ has weak turbulence.
Resonant finite dimensional truncations approximate \textit{NLS}
Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...

Define a nuclear family to be a rectangle $(n_1, n_2, n_3, n_4)$ where the frequencies $n_1, n_3$ (the 'parents') live in generation $\Lambda_j$ and $n_2, n_4$ (‘children’) live in generation $\Lambda_j + 1$.

$\forall 1 \leq j < M$ and $\forall n_1 \in \Lambda_j \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_j + 1$ are children.

$\forall 1 \leq j < M$ and $\forall n_2 \in \Lambda_j + 1 \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_j + 1$ are children and $n_1, n_3 \in \Lambda_j$ are parents.

The sibling of a frequency is never its spouse.

Besides nuclear families, $\Lambda$ contains no other rectangles.

The function $n \mapsto -a_n(0)$ is constant on each generation $\Lambda_j$. 
Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...

Imagine a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ with properties.
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Imagine we build a resonant $\Lambda \subset \mathbb{Z}^2$ such that...

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- $\forall \ 1 \leq j < M$ and $\forall \ n_1 \in \Lambda_j \ \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.

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- The sibling of a frequency is never its spouse.

Besides nuclear families, $\Lambda$ contains no other rectangles.

The function $n \mapsto \overrightarrow{a}_n(0)$ is constant on each generation $\Lambda_j$. 

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- $\forall 1 \leq j < M$ and $\forall n_2 \in \Lambda_{j+1}$ there exists a unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.

- The sibling of a frequency is never its spouse.
- Besides nuclear families, $\Lambda$ contains no other rectangles.
- The function $n \mapsto a_n(0)$ is constant on each generation $\Lambda_j$. 
The toy model ODE

Assume we can construct such a $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$. The properties imply $RFNLS_{\Lambda}$ simplifies to the toy model ODE:

$$i \partial_t b_j(t) = |b_j(t)|^2 b_j(t) - 2b_j(t) - 2b_{j-1}(t) - 2b_{j+1}(t).$$

$L_2 \sim \sum_j |b_j(t)|^2 = \sum_j |b_j(0)|^2 S$. We also want $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_M$ to satisfy

$$\sum_{n \in \Lambda_M} |n|^2 S \gg \sum_{n \in \Lambda_1} |n|^2 S.$$
The toy model ODE

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$$i\partial_t b_j(t) = |b_j(t)|^2 b_j(t) - 2b_{j-1}(t)^2\overline{b_j}(t) - 2b_{j+1}(t)^2\overline{b_j}(t).$$
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$$H^s \sim \sum_j |b_j(t)|^2 \left( \sum_{n \in \Lambda_j} |n|^{2s} \right).$$
The toy model ODE

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$$\sum_{n\in\Lambda_M} |n|^{2s} \gg \sum_{n\in\Lambda_1} |n|^{2s}.$$
Toy model travelling wave solution

Using dynamical systems methods, we construct a Toy Model ODE evolution \( b_j(0) \mapsto b_j(t) \) such that:

\[
(b_1(0), b_2(0), \ldots, b_M(0)) \sim (1, 0, \ldots, 0) \quad \text{and} \quad (b_1(t_2), b_2(t_2), \ldots, b_M(t_2)) \sim (0, 1, \ldots, 0) \quad \text{for } t_2 = \ldots.
\]

Bulk of conserved mass is transferred from \( \Lambda_1 \) to \( \Lambda_M \).

Weak turbulence follows, provided we can construct such a \( \Lambda \).
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\[
\vdots
\]
\[
\vdots
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$$(b_1(t_2), b_2(t_2), \ldots, b_M(t_2)) \sim (0, 1, \ldots, 0)$$

$$\ldots$$

$$(b_1(t_M), b_2(t_M), \ldots, b_M(t_M)) \sim (0, 0, \ldots, 1)$$
Using dynamical systems methods, we construct a Toy Model ODE evolution $b_j(0) \mapsto b_j(t)$ such that:

\[
(b_1(0), b_2(0), \ldots, b_M(0)) \sim (1, 0, \ldots, 0) \\
(b_1(t_2), b_2(t_2), \ldots, b_M(t_2)) \sim (0, 1, \ldots, 0) \\
\vdots \\
(b_1(t_M), b_2(t_M), \ldots, b_M(t_M)) \sim (0, 0, \ldots, 1)
\]

Bulk of conserved mass is transferred from $\Lambda_1$ to $\Lambda_M$. Weak turbulence follows, provided we can construct such a $\Lambda$. 
Combinatorial construction of \( \Lambda \subset \mathbb{Z}^2 \)
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