Recent progress on blowup phenomena in nonlinear Schrodinger equations

J. Colliander

University of Toronto

Georgia Tech PDE Seminar
1. **Nonlinear Schrödinger Initial Value Problem**

2. **Critical Regimes & Low Regularity GWP?**

3. **$H^{1/2}$ Critical Case**

4. **Energy Critical Case**

5. **Energy Supercritical Case**

6. **Critical Norm Explosion for $H^{1/2}$ Critical Case**
Nonlinear Schrödinger Initial Value Problem

Consider the initial value problem \( NLS \):

\[
\begin{align*}
&i \frac{\partial}{\partial t} u + \Delta u = \pm |u|^{p-1} u \\
&(u(0, x)) = u_0(x).
\end{align*}
\]

We seek \( u: (-T^*, T^*) \times \mathbb{R}^d \rightarrow \mathbb{C} \).

Time Invariant Quantities:

- Mass: \( \| u(t) \|_{L^2_x} \)
- Hamiltonian: \( \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx \pm \frac{2}{p+1} \int_{\mathbb{R}^d} |u(t)|^{p+1} dx \)
Consider the initial value problem $NLS_p^\pm(\mathbb{R}^d)$:

\[
\begin{aligned}
    &i\partial_t u + \Delta u = \pm |u|^{p-1} u \\
    &u(0, x) = u_0(x).
\end{aligned}
\]
Consider the initial value problem $NLS_p^\pm(\mathbb{R}^d)$:

\[
\begin{aligned}
    i\partial_t u + \Delta u &= \pm |u|^{p-1} u \\
    u(0, x) &= u_0(x).
\end{aligned}
\]

We seek $u : (-T^*, T^*) \times \mathbb{R}^d \to \mathbb{C}$.

(+ focusing, − defocusing)
Consider the initial value problem $\text{NLS}_p^\pm(\mathbb{R}^d)$:

\[
\begin{cases}
  i\partial_t u + \Delta u = \pm |u|^{p-1}u \\
  u(0, x) = u_0(x).
\end{cases}
\]

We seek $u : (-T_*, T^*) \times \mathbb{R}^d \rightarrow \mathbb{C}$.

(+ focusing, $-$ defocusing)

**Time Invariant Quantities**

Mass $= \|u(t)\|_{L_x^2}$

Hamiltonian $= \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx + \frac{2}{p+1} |u(t)|^{p+1} dx$
Dilation Invariance

If $u$ solves $\text{NLS}^{\pm} (\mathbb{R}^d)$ on $(-T^*, T^*) \times \mathbb{R}^2$ then

$$u_{\lambda}(\tau, y) := \lambda^{-2} u(\tau \lambda^{-2}, y \lambda^{-1})$$

solves $\text{NLS}^{\pm} (\mathbb{R}^d)$ on $(-\lambda^2 T^*, \lambda^2 T^*) \times \mathbb{R}^2$.

Dilation invariant norms play decisive role in the theory of $\text{NLS}^{\pm} (\mathbb{R}^d)$:

$$\| D_{\sigma y} u_{\lambda} \|_{L^q (\mathbb{R}^d)} = \left( \frac{1}{\lambda} \right)^{2p-1} \sigma - d \right) \| D_{\sigma x} u \|_{L^q (\mathbb{R}^d)}.$$

$\dot{W}_{\sigma, q}$ is critical if

$$2p-1 + \sigma - d q = 0.$$  

$\text{NLS}^{\pm} (\mathbb{R}^d)$ is $\dot{H}^{s_c}$-critical for $s_c := d^2 - 2p - 1$.

$L^2$ and $\dot{H}^1$ critical cases distinguished by conservation laws.
Dilation Invariance

If \( u \) solves \( NLS_{p}^{\pm}(\mathbb{R}^d) \) on \((-T_*, T*) \times \mathbb{R}^2\) then

\[
    u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})
\]

solves \( NLS_{p}^{\pm}(\mathbb{R}^d) \) on \((-\lambda^2 T_*, \lambda^2 T*) \times \mathbb{R}^2\).
If \( u \) solves \( NLS_p^\pm(\mathbb{R}^d) \) on \( (-T^*, T^*) \times \mathbb{R}^2 \) then

\[
u_\lambda(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})\]

solves \( NLS_p^\pm(\mathbb{R}^d) \) on \( (-\lambda^2 T^*, \lambda^2 T^*) \times \mathbb{R}^2 \).

Dilation invariant norms play decisive role in the theory of \( NLS_p^\pm(\mathbb{R}^d) \):

- \( \dot{W}_{\sigma, q} \) is critical if
  \[
  2p - 1 + \sigma - d q = 0.
  \]

\( \dot{H}^{s_c} \) is critical if
  \[
  s_c = \frac{d}{2} - \frac{2}{1-p}.
  \]

\( \dot{H}^1 \) critical cases distinguished by conservation laws.
Dilation Invariance

- If $u$ solves $NLS_{p}^{\pm}(\mathbb{R}^{d})$ on $(-T_{*}, T^{*}) \times \mathbb{R}^{2}$ then

$$u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})$$

solves $NLS_{p}^{\pm}(\mathbb{R}^{d})$ on $(-\lambda^{2} T_{*}, \lambda^{2} T^{*}) \times \mathbb{R}^{2}$. 

- Dilation invariant norms play decisive role in the theory of $NLS_{p}^{\pm}(\mathbb{R}^{d})$:
If $u$ solves $NLS^\pm_p(\mathbb{R}^d)$ on $(-T_*, T^*)\times \mathbb{R}^2$ then

$$u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})$$

solves $NLS^\pm_p(\mathbb{R}^d)$ on $(-\lambda^2 T_*, \lambda^2 T^*)\times \mathbb{R}^2$.

Dilation invariant norms play decisive role in the theory of $NLS^\pm_p(\mathbb{R}^d)$:

$$\|D_y^\sigma u_{\lambda}\|_{L^q(\mathbb{R}_y^d)} = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1} + \sigma - \frac{d}{q}} \|D_x^\sigma u\|_{L^q(\mathbb{R}_x^d)}.$$
If \( u \) solves \( NLS_p^\pm(\mathbb{R}^d) \) on \( (-T_*, T^*) \times \mathbb{R}^2 \) then

\[
  u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})
\]

solves \( NLS_p^\pm(\mathbb{R}^d) \) on \( (-\lambda^2 T_*, \lambda^2 T^*) \times \mathbb{R}^2 \).

Dilation invariant norms play decisive role in the theory of \( NLS_p^\pm(\mathbb{R}^d) \):

\[
  \| D_y^\sigma u_{\lambda} \|_{L^q(\mathbb{R}^d_y)} = \left( \frac{1}{\lambda} \right)^{\frac{2}{p-1} + \sigma - \frac{d}{q}} \| D_x^\sigma u \|_{L^q(\mathbb{R}^d_x)}.
\]

\( \dot{W}^{\sigma, q} \) is critical if \( \frac{2}{p-1} + \sigma - \frac{d}{q} = 0 \).
If \( u \) solves \( NLS_{p}^{\pm}(\mathbb{R}^{d}) \) on \((-T_{*}, T^{*}) \times \mathbb{R}^{2}\) then

\[
u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})\]

solves \( NLS_{p}^{\pm}(\mathbb{R}^{d}) \) on \((-\lambda^{2} T_{*}, \lambda^{2} T^{*}) \times \mathbb{R}^{2}\).

Dilation invariant norms play decisive role in the theory of \( NLS_{p}^{\pm}(\mathbb{R}^{d}) \):

\[
\| D_{y}^{\sigma} u_{\lambda} \|_{L^{q}(\mathbb{R}^{d}_{y})} = \left( \frac{1}{\lambda} \right)^{\frac{2}{p-1} + \sigma - \frac{d}{q}} \| D_{x}^{\sigma} u \|_{L^{q}(\mathbb{R}^{d}_{x})}.\]

\( \dot{W}_{\sigma, q} \) is critical if \( \frac{2}{p-1} + \sigma - \frac{d}{q} = 0 \).

\( NLS_{p}^{\pm}(\mathbb{R}^{d}) \) is \( \dot{H}^{s_{c}} \)-critical for \( s_{c} := \frac{d}{2} - \frac{2}{p-1} \).
If \( u \) solves \( NLS^\pm_p(\mathbb{R}^d) \) on \( (-T_*, T^*) \times \mathbb{R}^2 \) then

\[
u_{\lambda}(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})
\]

solves \( NLS^\pm_p(\mathbb{R}^d) \) on \( (-\lambda^2 T_*, \lambda^2 T^*) \times \mathbb{R}^2 \).

Dilation invariant norms play decisive role in the theory of \( NLS^\pm_p(\mathbb{R}^d) \):

\[
\| D^\sigma_y u_{\lambda} \|_{L^q(\mathbb{R}^d_y)} = \left( \frac{1}{\lambda} \right)^{\frac{2}{p-1} + \sigma - \frac{d}{q}} \| D^\sigma_x u \|_{L^q(\mathbb{R}^d_x)}.
\]

\( \dot{W}^{\sigma, q} \) is critical if \( \frac{2}{p-1} + \sigma - \frac{d}{q} = 0 \).

\( NLS^\pm_p(\mathbb{R}^d) \) is \( \dot{H}^{sc} \)-critical for \( sc := \frac{d}{2} - \frac{2}{p-1} \).

\( L^2 \) and \( \dot{H}^1 \) critical cases distinguished by conservation laws.
Critical Regimes

NLS theory for $\pm^p(\mathbb{R}^d)$ is qualitatively similar in regimes:

- Mass subcritical ($s_c < 0$)
- Mass critical ($s_c = 0$)
- Mass supercritical/Energy subcritical ($0 < s_c < 1$)
- Energy critical ($s_c = 1$)
- Energy supercritical ($s_c > 1$).

Optimal local-in-time well-posedness (LWP) for NLS:

$$\forall s \geq \max(0, s_c) \exists \text{unique continuous data-to-solution map } H_s \ni u_0 \mapsto u \in C([0, T_{lwp}] ; H_s) \cap L^{q_t} L^p_x$$

with $T_{lwp} = T_{lwp}(\|u_0\|_{H_s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)
  - Energy critical ($s_c = 1$)
  - Energy supercritical ($s_c > 1$).

Optimal local-in-time well-posedness ($LWP$) for $NLS_p^\pm(\mathbb{R}^d)$:

$\forall s \geq \max(0, s_c)$ \exists unique continuous data-to-solution map $H_s \ni u_0 \mapsto u \in C([0, T_{lwp}); H_s) \cap L^q_t L^p_x$ with $T_{lwp} = T_{lwp}(\|u_0\|_{H_s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

Optimal maximal-in-time well-posedness ($GWP$) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)

Optimal local-in-time well-posedness ($LWP$) for $NLS_p^\pm(\mathbb{R}^d)$:

$\forall s \geq \max(0, s_c) \exists$ unique continuous data-to-solution map $H_s \ni u_0 \mapsto u \in C([0, T_{lwp}]; H_s) \cap L^q_t L^p_x$ with $T_{lwp} = T_{lwp}(\|u_0\|_{H_s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

Optimal maximal-in-time well-posedness ($GWP$) is known only in the defocusing energy critical case.

What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
Critical Regimes

- Theory for $NLS_p^{\pm}(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)

Optimal local-in-time well-posedness ($LWP$) for $NLS_p^{\pm}(\mathbb{R}^d)$:

$\forall s \geq \max(0, s_c) \exists$ unique continuous data-to-solution map $H^s \ni u_0 \mapsto u \in C([0, T_{lwp}); H^s) \cap L^q_t L^p_x$ with $T_{lwp} = T_{lwp}(\|u_0\|_{H^s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

Optimal maximal-in-time well-posedness ($GWP$) is known only in the defocusing energy critical case.

What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)
  - Energy critical ($s_c = 1$)
Theory for $NLS_p^{\pm}(\mathbb{R}^d)$ is qualitatively similar in regimes:

- Mass subcritical ($s_c < 0$)
- Mass critical ($s_c = 0$)
- Mass supercritical/Energy subcritical ($0 < s_c < 1$)
- Energy critical ($s_c = 1$)
- Energy supercritical ($s_c > 1$).
Critical Regimes

- Theory for $NLS_{p}^{\pm}(\mathbb{R}^{d})$ is qualitatively similar in regimes:
  - Mass subcritical ($s_{c} < 0$)
  - Mass critical ($s_{c} = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_{c} < 1$)
  - Energy critical ($s_{c} = 1$)
  - Energy supercritical ($s_{c} > 1$).

- Optimal local-in-time well-posedness (LWP) for $NLS_{p}^{\pm}(\mathbb{R}^{d})$: for all $s \geq \max(0, s_{c})$, there exists a unique continuous data-to-solution map $u_{0} \mapsto u \in C([0, T_{\text{LWP}}]; H^{s}) \cap L^{q}_{t}L^{p}_{x}$ with $T_{\text{LWP}} = T_{\text{LWP}}(\|u_{0}\|_{H^{s}})$ if $s > s_{c}$ and $T_{\text{LWP}} = T(u_{0})$ if $s = s_{c}$.

- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $\text{NLS}_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)
  - Energy critical ($s_c = 1$)
  - Energy supercritical ($s_c > 1$).

- Optimal local-in-time well-posedness (LWP) for $\text{NLS}_p^\pm(\mathbb{R}^d)$:

  $\forall s \geq \max(0, s_c) \exists$ unique continuous data-to-solution map $H_s \ni u_0 \mapsto u \in C([0, T_{lwp}); H_s) \cap L^q_t L^p_x$ with $T_{lwp} = T_{lwp}(\|u_0\|_{H_s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case.

What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)
  - Energy critical ($s_c = 1$)
  - Energy supercritical ($s_c > 1$).

- Optimal local-in-time well-posedness (LWP) for $NLS_p^\pm(\mathbb{R}^d)$: 
  $\forall \ s \geq \max(0, s_c) \ \exists \ \text{unique continuous data-to-solution map } H^s \ni u_0 \mapsto u \in C([0, T_{lwp}]; H^s) \cap L_t^q L_x^p$

  with $T_{lwp} = T_{lwp}(\|u_0\|_{H^s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$. 

- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. 

What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:
  - Mass subcritical ($s_c < 0$)
  - Mass critical ($s_c = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_c < 1$)
  - Energy critical ($s_c = 1$)
  - Energy supercritical ($s_c > 1$).

- Optimal local-in-time well-posedness (LWP) for $NLS_p^\pm(\mathbb{R}^d)$:
  \[ \forall s \geq \max(0, s_c) \exists \text{unique continuous data-to-solution map} \]
  \[ H^s \ni u_0 \mapsto u \in C([0, T_{lwp}]; H^s) \cap L^q_t L^p_x \]
  with \( T_{lwp} = T_{lwp}(\|u_0\|_{H^s}) \) if \( s > s_c \) and \( T_{lwp} = T(u_0) \) if \( s = s_c \).

- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?
Critical Case: LWP Theory

Restrict attention to \( NLS \pm \delta \). Typical \( L^2 \) critical case?

[Cazenave-Weissler] For all \( u_0 \in L^2 \), there exists a \( \text{LWP} (u_0) \) determined by

\[
\| e^{it \Delta} u_0 \|_{L^4_{tx}([0, T_{\text{LWP}}] \times \mathbb{R}^2)} < 100.
\]

Existence of a unique solution \( u \in C([0, T_{\text{LWP}}]; L^2) \cap L^4_{tx}([0, T_{\text{LWP}}] \times \mathbb{R}^2) \).

Define the maximal forward existence time \( T^* (u_0) \) by

\[
\| u \|_{L^4_{tx}([0, T^* - \delta] \times \mathbb{R}^2]} < \infty \text{ for all } \delta > 0 \text{ but diverges to } \infty \text{ as } \delta \downarrow 0.
\]

\( \exists \) small data scattering threshold \( \mu_0 > 0 \) where

\[
\| u_0 \|_{L^2} < \mu_0 \Rightarrow \| u \|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.
\]
Critical Case: LWP Theory

Restrict attention to $\text{NLS}_3^{\pm}(\mathbb{R}^2)$. Typical $L^2$ critical case?
Critical Case: LWP Theory

Restrict attention to $NLS_3^\pm (\mathbb{R}^2)$. Typical $L^2$ critical case?

[Cazenave-Weissler]
Critical Case: LWP Theory

Restrict attention to $\text{NLS}_3^\pm(\mathbb{R}^2)$. Typical $L^2$ critical case?

[Cazenave-Weissler]

\[ \forall u_0 \in L^2 \text{ there exists } T_{lwp}(u_0) \text{ determined by} \]

\[ \| e^{it\Delta} u_0 \|_{L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}. \]

\[ \exists \text{ unique solution } u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2). \]
Critical Case: LWP Theory

Restrict attention to $NLS^\pm_3(\mathbb{R}^2)$. Typical $L^2$ critical case?

[Cazenave-Weissler]

$\forall u_0 \in L^2$ there exists $T_{lwp}(u_0)$ determined by

$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}.$$  

$\exists$ unique solution $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$.

Define the maximal forward existence time $T^*(u_0)$ by

$$\|u\|_{L^4_{tx}([0, T^* - \delta] \times \mathbb{R}^2)} < \infty$$

for all $\delta > 0$ but diverges to $\infty$ as $\delta \downarrow 0$. 

$\exists$ small data scattering threshold $\mu_0 > 0$

$\|u_0\|_{L^2} < \mu_0 \Rightarrow \|u\|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0$. 

$L^2$ Critical Case: LWP Theory

Restrict attention to $NLS_3^\pm(\mathbb{R}^2)$. Typical $L^2$ critical case?

[Cazenave-Weissler]

- $\forall u_0 \in L^2$ there exists $T_{lwp}(u_0)$ determined by

$$\|e^{it\Delta}u_0\|_{L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}.$$

- $\exists$ unique solution $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$.

- Define the maximal forward existence time $T^*(u_0)$ by

$$\|u\|_{L^4_{tx}([0, T^* - \delta] \times \mathbb{R}^2)} < \infty$$

for all $\delta > 0$ but diverges to $\infty$ as $\delta \downarrow 0$.

- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.$$
$L^2$ Critical Case: GWP Theory

\[ H_1 \rightarrow \text{GWP mass threshold} \parallel Q \parallel_{L^2} \text{ for } NLS^{−3}(R^2) : \]
\[ \|u_0\|_{L^2} < \|Q\|_{L^2} \Rightarrow H_1 \ni u_0 \mapsto u, T^* = \infty. \]

[Weinstein] Here $Q$ is the ground state solution to $−Q + \Delta Q = Q^3$. $e^{itQ(x)}$ is the ground state soliton solution to $NLS^{−3}(R^2)$.

'I Method' yields $H_s$-GWP for $s > 47/3$ (soon).

[Grillakis-Fang], [CKSTT] $NLS^{+5}(R^1)$ is similarly $H_s$-GWP for $s > 49/3$.

[Tzirakis] $NLS^{+4d+1}(R^d)$ is $H_s$-GWP for $s > d + 8d + 10$. [Visan-Zhang]
$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $\text{NLS}_3^+ (\mathbb{R}^2)$. 

$\parallel u_0 \parallel_{L^2} < \parallel Q \parallel_{L^2} \Rightarrow H^1 \ni u_0 \mapsto -T^* = \infty$. 

[Weinstein] Here $Q$ is the ground state solution to $-Q + \Delta Q = Q$. 

$e^{itQ(x)}$ is the ground state soliton solution to $\text{NLS}^+ (\mathbb{R}^2)$. 

'I Method' yields $H^s$-GWP for $s > 4/7$ ($s > 1/2$). 

[Grillakis-Fang], [CKSTT] $\text{NLS}^+ (\mathbb{R}^2)$ is similarly $H^s$-GWP for $s > 4/9$. 

[Tzirakis] $\text{NLS}^+ (\mathbb{R}^d)$ is $H^s$-GWP for $s > d + 8d + 10$. 

[Visan-Zhang]
$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $NLS_3^+ (\mathbb{R}^2)$.
- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $NLS_3^- (\mathbb{R}^2)$.

[Weinstein] Here $Q$ is the ground state solution to $-Q + \Delta Q = Q$. $e^{itQ(x)}$ is the ground state soliton solution to $\text{NLS}^-_{3} (\mathbb{R}^2)$.

[I Method] yields $H^s$-GWP for $s > \frac{47}{2}$ soon.

[Grillakis-Fang], [CKSTT] $\text{NLS}^+_{4} (\mathbb{R})$ is similarly $H^s$-GWP for $s > \frac{49}{2}$.

[Tzirakis] $\text{NLS}^+_{d+1} (\mathbb{R}^d)$ is $H^s$-GWP for $s > d+10$.
$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $NLS_3^+(\mathbb{R}^2)$.
- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $NLS_3^-(\mathbb{R}^2)$:
\textbf{L}^2 \textbf{C}ritical \textbf{C}ase: \textbf{GWP} \textbf{Theory}

- \( H^1 \)-GWP for \( NLS_3^+ (\mathbb{R}^2) \).

- \( H^1 \)-GWP mass threshold \( \| Q \|_{L^2} \) for \( NLS_3^- (\mathbb{R}^2) \):

  \[ \| u_0 \|_{L^2} < \| Q \|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty. \]

[Weinstein]
$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $\text{NLS}_3^+(\mathbb{R}^2)$.
- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $\text{NLS}_3^-(\mathbb{R}^2)$:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty.$$ 

[Weinstein]

Here $Q$ is the ground state solution to $-Q + \Delta Q = Q^3$. $e^{it}Q(x)$ is the ground state soliton solution to $\text{NLS}_3^-(\mathbb{R}^2)$. 

[Weinstein]
**$L^2$ Critical Case: GWP Theory**

- $H^1$-GWP for $NLS_3^+(\mathbb{R}^2)$.
- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $NLS_3^-(\mathbb{R}^2)$:

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty.$$  

[Weinstein]

Here $Q$ is the *ground state* solution to $-Q + \Delta Q = Q^3$. $e^{it}Q(x)$ is the *ground state soliton* solution to $NLS_3^-(\mathbb{R}^2)$.

- 'I Method' yields $H^s$-GWP for $s > \frac{4}{7}$ ($s > \frac{1}{2}$ soon).  

[Grillakis-Fang], [CKSTT]
\(L^2\) CRITICAL CASE: GWP THEORY

- \(H^1\)-GWP for \(NLS_3^+ (\mathbb{R}^2)\).
- \(H^1\)-GWP mass threshold \(\|Q\|_{L^2}\) for \(NLS_3^- (\mathbb{R}^2)\):

\[
\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty.
\]

[Weinstein]

Here \(Q\) is the ground state solution to \(-Q + \Delta Q = Q^3\). \(e^{it}Q(x)\) is the ground state soliton solution to \(NLS_3^- (\mathbb{R}^2)\).

- ‘I Method’ yields \(H^s\)-GWP for \(s > \frac{4}{7}\) (\(s > \frac{1}{2}\) soon).
[Grillakis-Fang], [CKSTT]
**$L^2$ Critical Case: GWP Theory**

- $H^1$-GWP for $NLS_3^+(\mathbb{R}^2)$.

- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $NLS_3^-(\mathbb{R}^2)$:

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u, T^* = \infty.$$ 

[Weinstein]

Here $Q$ is the ground state solution to $-Q + \Delta Q = Q^3$. $e^{it}Q(x)$ is the ground state soliton solution to $NLS_3^-(\mathbb{R}^2)$.

- 'I Method' yields $H^s$-GWP for $s > \frac{4}{7}$ (soon).

[Grillakis-Fang], [CKSTT] $NLS_5^+(\mathbb{R}^1)$ is similarly $H^s$-GWP for $s > \frac{4}{9}$.

[Tzirakis] $NLS_{\frac{d}{d+1}}^+(\mathbb{R}^d)$ is $H^s$-GWP for $s > \frac{d+8}{d+10}$.

[Visan-Zhang]
Explicit Blowup Solutions arise as pseudoconformal image of \( e^{i|Q(x) + 1|/4t} \).

\[ S(t, x) = \frac{1}{t^2} Q(x/t) e^{-\left|\frac{x}{2}\right|^2} + it. \]

\( S \) has minimal mass:

\[ \| S(\cdot^{-1}) \|_{L^2} = \| Q \|_{L^2}. \]

All mass in \( S \) is conically concentrated into a point.

Minimal mass \( \mathcal{H}_1 \) blowup solution characterization:

\[ u_0 \in \mathcal{H}_1, \quad \| u_0 \|_{L^2} = \| Q \|_{L^2}, \quad T^* (u_0) < \infty \implies u = S \text{ up to an explicit solution symmetry.} \]
Explicit Blowup Solutions

Arise as pseudoconformal image of $Q(x)$:

$$S(t,x) = 1/t Q(x_t) e^{-i|x|^2/4t + i t}$$. $S$ has minimal mass:

$$\|S(-1)\|_{L^2} = \|Q\|_{L^2}$$. All mass in $S$ is conically concentrated into a point. Minimal mass $H^1$ blowup solution characterization:

$u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $T^* (u_0) < \infty$ implies that $u$ up to an explicit solution symmetry. [Merle]
Explicit Blowup Solutions

- Arise as \textit{pseudoconformal} image of $e^{it} Q(x)$:

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + \frac{i}{t}}.$$
Explicit Blowup Solutions

- Arise as pseudoconformal image of $e^{it} Q(x)$:

$$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + \frac{i}{t}}.$$ 

- $S$ has minimal mass:

$$\| S(-1) \|_{L^2_x} = \| Q \|_{L^2}.$$ 

All mass in $S$ is conically concentrated into a point.
Explicit Blowup Solutions

- Arise as pseudoconformal image of $e^{it} Q(x)$:

$$ S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + i \frac{t}{t}}. $$

- $S$ has minimal mass:

$$ \| S(-1) \|_{L^2_x} = \| Q \|_{L^2}. $$

All mass in $S$ is conically concentrated into a point.

- Minimal mass $H^1$ blowup solution characterization:

$u_0 \in H^1, \| u_0 \|_{L^2} = \| Q \|_{L^2}, \ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]
Virial Identity \implies \exists \ Many \ Blowup \ Solutions
Critical Case: Blowup Solution Properties

Virial Identity $\implies$ Many Blowup Solutions

Integration by parts and the equation yields

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 |u(t, x)|^2 dx = 8H[u_0].$$
Virial Identity $\implies$ Many Blowup Solutions

- Integration by parts and the equation yields

$$\partial_t^2 \int_{\mathbb{R}^2_x} |x|^2 |u(t, x)|^2 \, dx = 8H[u_0].$$

- $H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 \, dx < \infty$ blows up.
Virial Identity $\implies$ Many Blowup Solutions

- Integration by parts and the equation yields

$$\partial_t^2 \int_{\mathbb{R}^2_x} |x|^2 |u(t, x)|^2 \, dx = 8H[u_0].$$

- $H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 \, dx < \infty$ blows up.

- How do these solutions blow up?
$L^2$ Critical Case: Mass Concentration

Merle-Tsutsumi $H^1$ blowups parabolically concentrate at least the ground state mass. Explicit blowups concentrate mass much faster. Fantastic recent progress on the $H^1$ blowup theory.
$L^2$ Critical Case: Mass Concentration

$H^1$ Theory of Mass Concentration
$L^2$ Critical Case: Mass Concentration

$H^1 \cap \{\text{radial}\} \ni u_0 \rightarrow u$, $T^* < \infty$ implies

$$\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{1/2}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.$$ 

[Merle-Tsutsumi]
$H^1$ Theory of Mass Concentration

- $H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u, \ T^* < \infty$ implies

$$\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{1/2}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.$$ 

[Merle-Tsutsumi]

- $H^1$ blowups parabolically concentrate at least the ground state mass. Explicit blowups $S$ concentrate mass much faster.
Critical Case: Mass Concentration

\[ H^1 \cap \{ \text{radial} \} \ni u_0 \mapsto u, \quad T^* < \infty \implies \lim_{t \uparrow T^*} \int_{|x| < (T^*-t)^{1/2}} |u(t,x)|^2 \, dx \geq \|Q\|_{L^2}^2. \]

[Merle-Tsutsumi]

\( H^1 \) blowups \textbf{parabolically} concentrate at least the ground state mass. Explicit blowups \( S \) concentrate mass much faster.

Fantastic recent progress on the \( H^1 \) blowup theory. [Merle-Raphaël]
$L^2$ Critical Case: Mass Concentration

Theory of Mass Concentration

$\ni \quad u_0 \mapsto u, T^* < \infty$ implies $\limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side } (I)} c \leq (T^* - t) \frac{1}{2} \int_I |u(t, x)|^2 \, dx \geq \|u_0\| - M_{L^2}$.

[Bourgain] $L^2$ blowups parabolically concentrate some mass. For large $L^2$ data, do there exist tiny concentrations? Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
$L^2$ Critical Case: Mass Concentration

$L^2$ Theory of Mass Concentration

\[ u_0 \xrightarrow{\to} -u, T^* < \infty \implies \limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side } (I)} c \leq (T^* - t)^{1/2} \int_I |u(t, x)|^2 dx \geq \|u_0\| - M \]

[Bourgain] $L^2$ blowups parabolically concentrate some mass. For large $L^2$ data, do there exist tiny concentrations? Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
$L^2$ Critical Case: Mass Concentration

$L^2$ Theory of Mass Concentration

$\exists \ u_0 \mapsto u, \ T^* < \infty \implies \limsup_{t \uparrow T^*} \sup_{\text{cubes } l, \text{side}(l) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 \, dx \geq \|u_0\|_{L^2}^{-M}.$

[Bourgain]

$L^2$ blowups parabolically concentrate some mass.
$L^2$ Critical Case: Mass Concentration

$\mathbf{L^2}$ Theory of Mass Concentration

- $L^2 \ni u_0 \mapsto u, \quad T^* < \infty$ implies

$$\limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 \, dx \geq \|u_0\|_{L^2}^{-M}.$$  

[Bourgain]

$L^2$ blowups parabolically concentrate some mass.

- For large $L^2$ data, do there exist tiny concentrations?

[Extensions in \cite{Merle-Vega}, \cite{Carles-Keraani}, \cite{Bégout-Vargas}.
\( L^2 \) Critical Case: Mass Concentration

\( L^2 \) Theory of Mass Concentration

- \( L^2 \ni u_0 \mapsto u, T^* < \infty \) implies

\[
\limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 \, dx \geq \|u_0\|_{L^2}^{-M}.
\]

[Bourgain]

\( L^2 \) blowups parabolically concentrate some mass.

- For large \( L^2 \) data, do there exist tiny concentrations?

- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].
Typical blowups leave an $L^2$ stain at time $T^*$

[Merle-Raphaël]:

$$H^1 \cap \{ \|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \} \ni u_0 \mapsto u$$ solving

$NLS_3^{-}(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.

$\exists \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi\mathbb{Z})$ and $u^*$ such that

$$u(t) - \lambda(t)^{-1}Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\theta(t)} \to u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!
Consider focusing $NLS^{-3}(\mathbb{R}^2)$:

$\|u_0\|_{L^2} < \|Q\|_{L^2} \Rightarrow ?? u_0 \mapsto \to \to u$ with $\|u\|_{L^4_{tx}} < \infty$.

(Also, $L^2$ solutions of $NLS^3(\mathbb{R}^2)$ satisfy ?? $\|u\|_{L^4_{tx}} < \infty$.)

Minimal Mass Blowup Characterization

$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \mapsto \to \to u, T^* < \infty \Rightarrow ?? u = S$, modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$.

Concentrated mass amounts are quantized The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles. Are there any general upper bounds?
Consider focusing $\text{NLS}_3^-(\mathbb{R}^2)$:
Consider focusing $NLS_3^{-}(\mathbb{R}^2)$:

**Scattering Below the Ground State Mass**

$$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies \text{???} \quad u_0 \mapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$ 

(Also, $L^2$ solutions of $NLS_3^{+}(\mathbb{R}^2)$ satisfy $\text{???} \quad \|u\|_{L^4_{tx}} < \infty$.)
Consider focusing $\text{NLS}^{-3}(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass**
  \[ \|u_0\|_{L^2} < \|Q\|_{L^2} \implies \text{??? } u_0 \mapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty. \]
  (Also, $L^2$ solutions of $\text{NLS}^+\mathbb{R}^2$ satisfy ??? $\|u\|_{L^4_{tx}} < \infty$.)

- **Minimal Mass Blowup Characterization**
  \[ \|u_0\|_{L^2} = \|Q\|_{L^2}, \quad u_0 \mapsto u, \quad T^* < \infty \implies \text{??? } u = S, \]
  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$. 

**Concentrated mass amounts are quantized**

The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.

Are there any general upper bounds?
$L^2$ Critical Case: Conjectures/Questions

Consider focusing $\text{NLS}_3^-(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass**
  \[ \|u_0\|_{L^2} < \|Q\|_{L^2} \implies ??? \quad u_0 \mapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty. \]
  (Also, $L^2$ solutions of $\text{NLS}_3^+(\mathbb{R}^2)$ satisfy $??? \quad \|u\|_{L^4_{tx}} < \infty.$)

- **Minimal Mass Blowup Characterization**
  \[ \|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \mapsto u, T^* < \infty \implies ??? \quad u = S, \]
  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$.

- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.
Consider focusing $\text{NLS}_3^-(\mathbb{R}^2)$:

- **Scattering Below the Ground State Mass**
  \[ \|u_0\|_{L^2} < \|Q\|_{L^2} \implies ??? u_0 \mapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty. \]
  
  (Also, $L^2$ solutions of $\text{NLS}_3^+(\mathbb{R}^2)$ satisfy ??? $\|u\|_{L^4_{tx}} < \infty$.)

- **Minimal Mass Blowup Characterization**
  \[ \|u_0\|_{L^2} = \|Q\|_{L^2}, \quad u_0 \mapsto u, \quad T^* < \infty \implies ??? u = S, \]
  
  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$.

- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.

- **Are there any general upper bounds?**
$L^2$ Critical Case: Partial Results

For $0.86 \sim 1.5 (1 + \sqrt{11}) < s < 1$, $H_s \cap \{\text{radial}\} \ni u_0 \mapsto u$, $T^* < \infty = \Rightarrow \limsup_{t \uparrow T^*} \int |x| < (T^* - t) s/2 - |u(t, x)|^2 dx \geq \|Q\|_2^2$. $H_s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

$\|u_0\|_{L^2} = \|Q\|_{L^2}$, $u_0 \in H_s \sim 0.86 < s < 1$, $T^* < \infty = \Rightarrow \exists t_n \uparrow T^*$ s.t. $u(t_n) \rightarrow Q$ in $H_{\tilde{s}}(s)$ (mod symmetry sequence).

For $H_s$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}$, $u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani]

This is an $H_s$ analog of an $H_1$ result of [Weinstein] which preceded the minimal $H_1$ blowup solution characterization. Same results for NLS $-^{4d+1}(R^d)$ in $H_s$, $s > d+8$. [Visan-Zhang]


\[ L^2 \text{ Critical Case: Partial Results} \]

For \( 0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1 \), \( H^s \cap \{ \text{radial} \} \ni u_0 \mapsto u, T^* < \infty \implies \]

\[
\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2-}} |u(t, x)|^2 \, dx \geq \| Q \|_{L^2}^2.
\]

\( H^s \)-blowup solutions concentrate ground state mass.

[With Raynor, Sulem and Wright]
$L^2$ Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1$, $H^s \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty \implies$

  \[
  \limsup_{t \uparrow T^*} \int_{|x| < (T^*-t)^{s/2-}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.
  \]

$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies$

  $\exists \ t_n \uparrow T^* \text{ s.t. } u(t_n) \to Q \text{ in } H^\tilde{s}(s) \ (\text{mod symmetry sequence})$. 

[Visan-Zhang]
For $0.86 \sim \frac{1}{5} (1 + \sqrt{11}) < s < 1$, $H^s \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty \implies$

$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2-}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.$$ 

$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

$\|u_0\|_{L^2} = \|Q\|_{L^2}$, $u_0 \in H^s$, $\sim 0.86 < s < 1$, $T^* < \infty \implies$

$\exists \, t_n \uparrow T^*$ s.t. $u(t_n) \rightharpoonup Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence).
\( L^2 \) Critical Case: Partial Results

- For \( 0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1, H^s \cap \{ \text{radial} \} \ni u_0 \to u, T^* < \infty \implies \exists t_n \uparrow T^* \) s.t. \( u(t_n) \to Q \) in \( H^\tilde{s}(s) \) (mod symmetry sequence).

\[
\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.
\]

\( H^s \)-blowup solutions concentrate ground state mass.

[With Raynor, Sulem and Wright]

- \( \|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies \exists t_n \uparrow T^* \) s.t. \( u(t_n) \to Q \) in \( H^\tilde{s}(s) \) (mod symmetry sequence).

For \( H^s \) blowups with \( \|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \) in \( H^1 \) (mod symmetry sequence). [Hmidi-Keraani]
$L^2$ Critical Case: Partial Results

- For $0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \mapsto u, T^* < \infty \implies$

$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2-}} |u(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$  

$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies$

  $\exists \ t_n \uparrow T^*$ s.t. $u(t_n) \rightharpoonup Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence).

For $H^s$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an $H^s$ analog of an $H^1$ result of [Weinstein] which preceded the minimal $H^1$ blowup solution characterization.
Critical Case: Partial Results

For $0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1$, $H^s \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty \implies$

$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2-}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.$$  

$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

$\|u_0\|_{L^2} = \|Q\|_{L^2}$, $u_0 \in H^s$, $\sim 0.86 < s < 1$, $T^* < \infty \implies \exists t_n \uparrow T^*$ s.t. $u(t_n) \rightharpoonup Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence).

For $H^s$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}$, $u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an $H^s$ analog of an $H^1$ result of [Weinstein] which preceded the minimal $H^1$ blowup solution characterization.

Same results for $NLS_{\frac{4}{d}+1}^\sim(\mathbb{R}^d)$ in $H^s$, $s > \frac{d+8}{d+10}$. [Visan-Zhang]
\section*{$L^2$ Critical Case: Partial Results}

The \textit{Spacetime norm divergence rate}

\[ \|u\|_{L^4_t L^4_x}(\mathbb{R}^2) \gtrsim (T^* - t)^{-\beta} \]

is linked with mass concentration rate

\[ \limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side } (I)} c(t, x) \leq (T^* - t)^{-\frac{1}{2}} + \beta \int_I |u(t,x)|^2 \, dx \geq \|u_0\|_2 - M_L^2. \]
*Critical Case: Partial Results*

\[ \|u\|_{L^4_{tx}([0,t] \times \mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta} \]

is linked with mass concentration rate

\[
\limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 \, dx \geq \|u_0\|_{L^2}^{-\beta}^M.
\]

[Work in progress with Roudenko]
\[ H^{1/2} \text{ Critical Case} \]
Consider $NLS^{-}_3(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?
Consider $NLS_3^- (\mathbb{R}^3)$. Also $L^3_{x^*}$-Critical. Typical Case?

- LWP theory similar to $NLS_3^{\pm} (\mathbb{R}^2)$:

$$L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3)$$

$$L^4_{tx} \hookrightarrow L^5_{tx}.$$
Consider $NLS_3^-(\mathbb{R}^3)$. Also $L^3_{\times}$-Critical. Typical Case?

- LWP theory similar to $NLS_3^{\pm}(\mathbb{R}^2)$:

$$L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3)$$
$$L^4_{tx} \hookrightarrow L^5_{tx}.$$

- There cannot be an $H^1$-GWP mass threshold.
Consider $NLS_3^-(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?

- LWP theory similar to $NLS_3^{\pm}(\mathbb{R}^2)$:

\[ L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3) \]
\[ L^4_{tx} \hookrightarrow L^5_{tx}. \]

- There cannot be an $H^1$-GWP mass threshold.
- No explicit blowup solutions are known.
Consider $NLS_3^-(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?

- LWP theory similar to $NLS_3^{\pm}(\mathbb{R}^2)$:
  \[
  L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3)
  
  L^4_{tx} \hookrightarrow L^5_{tx}.
  \]

- There cannot be an $H^1$-GWP mass threshold.
- No explicit blowup solutions are known.
- Virial identity $\Rightarrow \exists$ many blowup solutions.
Consider $NLS_3^{-}(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?

- LWP theory similar to $NLS_3^{\pm}(\mathbb{R}^2)$:

$$L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3)$$

$$L^4_{tx} \hookrightarrow L^5_{tx}.$$

- There cannot be an $H^1$-GWP mass threshold.
- No explicit blowup solutions are known.

$H^1 \cap \{\text{radial}\} \ni u_0 \hookrightarrow u, T^* < \infty$ then for any $a > 0$

$$\|\nabla u(t)\|_{L^2_{|x| < a}} \uparrow \infty \text{ as } t \uparrow T^*.$$ 

Thus, radial solutions must explode at the origin.
**Proof.**

By Hamiltonian conservation,

\[ \|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4}^4 \mid_x < a + \frac{1}{2} \|u(t)\|_{L^4}^4 \mid_x > a. \]

Inner contribution estimated using Gagliardo-Nirenberg by

\[ C(Mass, a) \|\nabla u(t)\|_{L^2}^3 \mid_x < a. \]

Exterior region estimated by pulling out two factors in \( L^\infty \times \) then using radial Sobolev to get control by

\[ \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}. \]

Absorb the exterior kinetic energy to left side

\[ \|\nabla u(t)\|_{L^2} \lesssim C(a, Mass[H[u_0]], H[u_0]) + C(a, Mass[u_0]) \|\nabla u(t)\|_{L^2}^3 \mid_x < a. \]
**Critical Case: Radial $NLS_3^{-}(\mathbb{R}^3)$**

**Proof.**

By Hamiltonian conservation,

$$\|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4}^4_{|x|<a} + \frac{1}{2} \|u(t)\|_{L^4}^4_{|x|>a}.$$
$H^{1/2}$ Critical Case: Radial $NLS^\sigma_3(\mathbb{R}^3)$

**Proof.**

By Hamiltonian conservation,

$$\|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4_{|x|<a}}^4 + \frac{1}{2} \|u(t)\|_{L^4_{|x|>a}}^4.$$  

Inner contribution estimated using Gagliardo-Nirenberg by $C(Mass, a)\|\nabla u(t)\|_{L^2_{|x|<a}}^3$.  

Exterior region estimated by pulling out two factors in $L^\infty_x$ then using radial Sobolev to get control by $\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}$.  

Absorb the exterior kinetic energy to left side

$$\|\nabla u(t)\|_{L^2}^2 \lesssim C(a, Mass [u_0], H[u_0]) + C(a, Mass [u_0])\|\nabla u(t)\|_{L^2_{|x|<a}}^3.$$
$H^{1/2}$ Critical Case: Radial $NLS^{-}_3(\mathbb{R}^3)$

**Proof.**

By Hamiltonian conservation,

$$\|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4_{|x|<a}}^4 + \frac{1}{2} \|u(t)\|_{L^4_{|x|>a}}^4.$$

Inner contribution estimated using Gagliardo-Nirenberg by $C(Mass, a)\|\nabla u(t)\|_{L^2_{|x|<a}}^3$. Exterior region estimated by pulling out two factors in $L^\infty_x$ then using radial Sobolev to get control by $\|u(t)\|_{L^2}^3 \|\nabla u(t)\|_{L^2}$. 

Absorb the exterior kinetic energy to left side

$$\|\nabla u(t)\|_{L^2}^2 \lesssim C(a, Mass[u_0], H[u_0]) + C(a, Mass[u_0]) \|u(t)\|_{L^2}^3 \|\nabla u(t)\|_{L^2}.$$
**Proof.**

By Hamiltonian conservation,

\[ \| \nabla u(t) \|_{L^2}^2 = H[u_0] + \frac{1}{2} \| u(t) \|_{L^4_{|x|<a}}^4 + \frac{1}{2} \| u(t) \|_{L^4_{|x|>a}}^4. \]

Inner contribution estimated using Gagliardo-Nirenberg by
\[ C(Mass, a) \| \nabla u(t) \|_{L^2_{|x|<a}}^3. \]
Exterior region estimated by pulling out two factors in \( L^\infty_x \) then using radial Sobolev to get control by
\[ \| u(t) \|_{L^2_{|x|>a}}^3 \| \nabla u(t) \|_{L^2}. \]
Absorb the exterior kinetic energy to left side

\[ \| \nabla u(t) \|_{L^2}^2 \lesssim C(a, Mass[u_0], H[u_0]) + C(a, Mass[u_0]) \| \nabla u(t) \|_{L^2_{|x|<a}}^3. \]
Radial blowup solutions of energy subcritical NLS $(\mathbb{R}^d)$ with $p < 5$ must explode at the origin. For $H^{1/2}$-critical NLS $-(\mathbb{R}^2)$, there exists $H^{1/2} \cap \{\text{radial}\} \ni v_0 \mapsto -\rightarrow v_0$, $T^* (v_0) < \infty$ which blows up precisely on a circle! [Raphaël]

Numerics/heuristics suggest: Finite time blowup solutions of NLS $(\mathbb{R}^3)$ satisfy $\|u(t)\|_{L^3} \uparrow \infty$ as $t \uparrow T^*$. [Recently proved for $H^{1/2} \cap \{\text{radial}\}$ data by Merle-Raphaël [Work in progress with Raynor, Sulem, Wright, different proof]

(Analogous to [Escauriaza-Seregin-ˇSverˇak] on Navier-Stokes)

$H^{1/2}$-blowups parabolically concentrate in $L^3$ and $H^{1/2}$?

[Work in progress with Roudenko]
Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.
Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.

For $H^{1/2}$-critical $NLS_5^-(\mathbb{R}^2)$, there exists $H^1 \cap \{\text{radial}\} \ni v_0 \mapsto v$, $T^*(v_0) < \infty$ which blows up precisely on a circle! [Raphaël]
Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.

For $H^{1/2}$-critical $NLS_5^-(\mathbb{R}^2)$, there exists $H^1 \cap \{\text{radial}\} \ni v_0 \mapsto v$, $T^*(v_0) < \infty$ which blows up precisely on a circle! [Raphaël]

Numerics/heuristics suggest: Finite time blowup solutions of $NLS_3(\mathbb{R}^3)$ satisfy $\|u(t)\|_{L^3_x} \uparrow \infty$ as $t \uparrow T^*$.

[Recently proved for $H^1 \cap \{\text{radial}\}$ data by Merle-Raphaël]
[Work in progress with Raynor, Sulem, Wright, different proof]
(Analogous to [Escauriaza-Seregin-Šverák] on Navier-Stokes)
$H^{1/2}$ Critical Case: Remarks

- Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.

- For $H^{1/2}$-critical $NLS_5^{-}(\mathbb{R}^2)$, there exists $H^1 \cap \{\text{radial}\} \ni v_0 \mapsto v, \ T^*(v_0) < \infty$ which blows up precisely on a circle! [Raphaël]

- Numerics/heuristics suggest: Finite time blowup solutions of $NLS_3(\mathbb{R}^3)$ satisfy $\|u(t)\|_{L_x^3} \uparrow \infty$ as $t \uparrow T^*$.
  [Recently proved for $H^1 \cap \{\text{radial}\}$ data by Merle-Raphaël]
  [Work in progress with Raynor, Sulem, Wright, different proof]
  (Analogous to [Escauriaza-Seregin-Šverák] on Navier-Stokes)

- $H^{1/2}$-blowups parabolically concentrate in $L^3$ and $H^{1/2}$?
  [Work in progress with Roudenko]
$H^1(\mathbb{R}^d)$, $d \geq 3$ Critical Case
Defocusing energy critical $\text{NLS}^{+}_{1+4/(d-2)}(\mathbb{R}^d)$, $d \geq 3$ is globally well-posed and scatters in $H^1$.
Defocusing energy critical $NLS_{1+4/(d-2)}^+(\mathbb{R}^d)$, $d \geq 3$ is globally well-posed and scatters in $H^1$:

[Bourgain], [Grillakis]: Radial Case for $d = 3$
[CKSTT]: $d = 3$
[Tao]: Radial Case for $d = 4$
[Ryckman-Visan], [Visan], [Tao-Visan]: $d \geq 4$
$H^1(\mathbb{R}^d)$, $d \geq 3$ Critical Case

- Defocusing energy critical $NLS_{1+4/(d-2)}^+(\mathbb{R}^d)$, $d \geq 3$ is globally well-posed and scatters in $H^1$:
  - [Bourgain], [Grillakis]: Radial Case for $d = 3$
  - [CKSTT]: $d = 3$
  - [Tao]: Radial Case for $d = 4$
  - [Ryckman-Visan], [Visan], [Tao-Visan]: $d \geq 4$

Induction on Energy; Interaction Morawetz; Mass Freezing
$H^1(\mathbb{R}^d), \; d \geq 3 \; \text{Critical Case}$

- Defocusing energy critical $NLS_{1+4/(d-2)}^+ (\mathbb{R}^d), \; d \geq 3$ is globally well-posed and scatters in $H^1$:
  - [Bourgain], [Grillakis]: Radial Case for $d = 3$
  - [CKSTT]: $d = 3$
  - [Tao]: Radial Case for $d = 4$
  - [Ryckman-Visan], [Visan], [Tao-Visan]: $d \geq 4$

  **Induction on Energy; Interaction Morawetz; Mass Freezing**

- Focusing energy critical case?
  - [Kenig-Merle]: $E[u_0] < E[Q]$ and $\| \nabla u_0 \|_{L^2} < \| \nabla Q \|_{L^2} \Rightarrow$ global-in-time and scatters.
$H^1(\mathbb{R}^2)$ ”CRITICAL” CASE

NLS is energy subcritical for all $p$. Is there an “energy critical” NLS equation on $\mathbb{R}^2$?

Consider the defocusing initial value problem

\[
i \frac{\partial}{\partial t} u + \Delta u = u (|u|^2 - 1) u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R}^2)
\]

with Hamiltonian $H[u(t)] := \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 + \int_{\mathbb{R}^2} e^{4\pi |u(t, x)|^2} - 1 4\pi dx$. 

$H^1(\mathbb{R}^2)$ ”Critical” Case

- $NLS_p(\mathbb{R}^2)$ is energy subcritical for all $p$. Is there an ”energy critical” $NLS$ equation on $\mathbb{R}^2$?
$H^1(\mathbb{R}^2)$ "Critical" Case

- $\text{NLS}_p(\mathbb{R}^2)$ is energy subcritical for all $p$. Is there an "energy critical" $\text{NLS}$ equation on $\mathbb{R}^2$?

- Consider the defocusing initial value problem $\text{NLS}_{exp}(\mathbb{R}^2)$

\[
\begin{cases}
  i\partial_t u + \Delta u = u(e^{4\pi|u|^2} - 1) \\
  u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R}^2)
\end{cases}
\]
$H^1(\mathbb{R}^2)$ ”Critical” Case

- $NLS_p(\mathbb{R}^2)$ is energy subcritical for all $p$. Is there an “energy critical” $NLS$ equation on $\mathbb{R}^2$?

- Consider the defocusing initial value problem $NLS_{\exp}(\mathbb{R}^2)$

$$\begin{cases} i\partial_t u + \Delta u = u(e^{4\pi|u|^2} - 1) \\ u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R}^2) \end{cases}$$

with Hamiltonian

$$H[u(t)] := \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 + \int_{\mathbb{R}^2} \frac{e^{4\pi|u(t,x)|^2} - 1}{4\pi} \, dx.$$
$H^1(\mathbb{R}^2)$ "Critical" Case

If $H[u_0] - M[u_0] \leq 1$ then $\text{NLS} \exp(R^2)$ is globally well-posed.

Uniform continuity of data-to-solution map fails to hold for data satisfying $H[u_0] - M[u_0] > 1$.

[Work in progress with Ibrahim, Majdoub, Masmoudi]

Well-posedness result relies upon Strichartz estimates, Moser-Trudinger inequality, and a log-Sobolev inequality. (Largely based on similar result for $\text{NLKG}$ by [Ibrahim-Majdoub-Masmoudi])

Ill-posedness result relies upon optimizing sequence for Moser-Trudinger and small dispersion approximation following [Christ-C-Tao].

Scattering?
$H^1(\mathbb{R}^2)$ “Critical” Case

- If $H[u_0] - M[u_0] \leq 1$ then $NLS_{exp}(\mathbb{R}^2)$ is globally well-posed. Uniform continuity of data-to-solution map fails to hold for data satisfying $H[u_0] - M[u_0] > 1$.

[Work in progress with Ibrahim, Majdoub, Masmoudi]
$H^1(\mathbb{R}^2)$ "Critical" Case

- If $H[u_0] - M[u_0] \leq 1$ then $NLS_{\exp}(\mathbb{R}^2)$ is globally well-posed. Uniform continuity of data-to-solution map fails to hold for data satisfying $H[u_0] - M[u_0] > 1$.
  [Work in progress with Ibrahim, Majdoub, Masmoudi]

- Well-posedness result relies upon Strichartz estimates, Moser-Trudinger inequality, and a log-Sobolev inequality.
  (Largely based on similar result for $NLKG$ by [Ibrahim-Majdoub-Masmoudi])
If \( H[u_0] - M[u_0] \leq 1 \) then \( NLS_{\exp}(\mathbb{R}^2) \) is globally well-posed.

Uniform continuity of data-to-solution map fails to hold for data satisfying \( H[u_0] - M[u_0] > 1 \).

[Work in progress with Ibrahim, Majdoub, Masmoudi]

Well-posedness result relies upon Strichartz estimates, Moser-Trudinger inequality, and a log-Sobolev inequality. (Largely based on similar result for \( NLKG \) by [Ibrahim-Majdoub-Masmoudi])

Ill-posedness result relies upon optimizing sequence for Moser-Trudinger and small dispersion approximation following [Christ-C-Tao].

\( H^1(\mathbb{R}^2) \) "CRITICAL" CASE
$H^1(\mathbb{R}^2)$ ’’Critical’’ Case

- If $H[u_0] - M[u_0] \leq 1$ then $NLS_{exp}(\mathbb{R}^2)$ is globally well-posed. Uniform continuity of data-to-solution map fails to hold for data satisfying $H[u_0] - M[u_0] > 1$. [Work in progress with Ibrahim, Majdoub, Masmoudi]

- Well-posedness result relies upon Strichartz estimates, Moser-Trudinger inequality, and a log-Sobolev inequality. (Largely based on similar result for $NLKG$ by [Ibrahim-Majdoub-Masmoudi])

- Ill-posedness result relies upon optimizing sequence for Moser-Trudinger and small dispersion approximation following [Christ-C-Tao].

- Scattering?
Energy Supercritical Case

Consider $\text{NLS}^7(\mathbb{R}^3)$. Typical case?

Numerical experiments by [Blue-Sulem] and also for corresponding NLKG [Strauss-Vazquez] suggest GWP and scattering.

Conjecture: $\text{NLS}^7(\mathbb{R}^3)$ is GWP and scatters in $\mathcal{H}^{7/6}(\mathbb{R}^3)$.

[See discussion by Bourgain, GAFA Special Volume, 2000]
Consider $NLS_7^+ (\mathbb{R}^3)$. Typical case?

Numerical experiments by [Blue-Sulem] and also for corresponding NLKG [Strauss-Vazquez] suggest GWP and scattering.

Conjecture: $NLS_7^+ (\mathbb{R}^3)$ is GWP and scatters in $H^{7/6} (\mathbb{R}^3)$.

[See discussion by Bourgain, GAFA Special Volume, 2000]
Consider $NLS_7^+(\mathbb{R}^3)$. Typical case?

- Numerical experiments by [Blue-Sulem] and also for corresponding $NLKG$ [Strauss-Vazquez] suggest GWP and scattering.

Conjecture: $NLS_7^+(\mathbb{R}^3)$ is GWP and scatters in $H^{7/6}(\mathbb{R}^3)$.

[See discussion by Bourgain, GAFA Special Volume, 2000]
Consider $NLS_7^+(\mathbb{R}^3)$. Typical case?

- Numerical experiments by [Blue-Sulem] and also for corresponding $NLKG$ [Strauss-Vazquez] suggest GWP and scattering.

**Conjecture:** $NLS_7^+(\mathbb{R}^3)$ is GWP and scatters in $H^{7/6}(\mathbb{R}^3)$. [See discussion by Bourgain, GAFA Special Volume, 2000]
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright....details remain.]
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright....details remain.]

Question: Qualitative properties mass supercritical NLS blowup?
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright…details remain.]

Question: Qualitative properties mass supercritical NLS blowup? Restrict attention to $H^{1/2}$-critical $NLS_3^-$ ($\mathbb{R}^3$).
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright....details remain.]

**Question:** Qualitative properties mass supercritical NLS blowup?

Restrict attention to $H^{1/2}$-critical $NLS_3^-$ ($\mathbb{R}^3$).

- $T^*$ defined via divergence of $\|u\|_{L^{5}_{tx}}$ or $\|D^{1/2}u\|_{L^{10/3}_{tx}}$.
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright….details remain.]

Question: Qualitative properties mass supercritical NLS blowup? Restrict attention to $H^{1/2}$-critical $NLS_3^-$ ($\mathbb{R}^3$).

- $T^*$ defined via divergence of $\|u\|_{L^5_{tx}}$ or $\|D^{1/2}u\|_{L^{10/3}_{tx}}$.
- Finite energy radial blowups explode at spatial origin.
[Work in progress with Raynor, Sulem, Wright....details remain.]

**Question:** Qualitative properties mass supercritical NLS blowup? Restrict attention to $H^{1/2}$-critical $\text{NLS}_3^-$ ($\mathbb{R}^3$).

- $T^*$ defined via divergence of $\|u\|_{L^5_{tx}}$ or $\|D^{1/2}u\|_{L^{10/3}_{tx}}$.
- Finite energy radial blowups explode at spatial origin.
- Heuristics and numerics suggest asymptotic profile $Q$ which decays near spatial infinity like $|y|^{-1} \implies Q \notin L^3(\mathbb{R}^3)$. Sobolev embedding $H^{1/2} \hookrightarrow L^3$ suggests as $t \uparrow T^*$

$$\|u(t)\|_{H^{1/2}} \sim |\log(T^* - t)| \to \infty.$$
Critical Norm Explosion?

[Work in progress with Raynor, Sulem, Wright....details remain.]

**Question:** Qualitative properties mass supercritical NLS blowup? Restrict attention to $H^{1/2}$-critical $NLS_3^{-}\left(\mathbb{R}^3\right)$.

- $T^*$ defined via divergence of $\|u\|_{L^5_{tx}}$ or $\|D^{1/2} u\|_{L^{10/3}_{tx}}$.
- Finite energy radial blowups explode at spatial origin.
- Heuristics and numerics suggest asymptotic profile $Q$ which decays near spatial infinity like $|y|^{-1} \implies Q \notin L^3(\mathbb{R}^3)$.
  Sobolev embedding $H^{1/2} \hookrightarrow L^3$ suggests as $t \uparrow T^*$

$$\|u(t)\|_{H^{1/2}} \sim |\log(T^* - t)| \to \infty.$$

- Frequency heuristic: Bounded $H^{1/2}$ blowup inconsistent with mass conservation.
Contradiction Strategy

Contradiction Hypothesis (CH):
Assume \( \exists \Lambda < \infty \) such that \( \|u\|_{L^\infty_t H^1/2} < \Lambda \).

Concentration Property:
If \( H^1 \cap \{\text{radial}\} \ni u_0 \mapsto -u \) solves NLS\(^-3(R^3) \), \( T^* < \infty \) and we assume (CH) then
\[
\liminf_{t \uparrow T^*} \|u(t)\|_{L^3} |x| < (T^* - t)^{1/2} - c^* = \sqrt{2/3} \approx 0.827.
\]

The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for \( L^2 \) conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]
Contradiction Hypothesis (CH): Assume $\exists \Lambda < \infty$ such that $\|u\|_{L^\infty_t H^{1/2}_x([0, T^*) \times \mathbb{R}^3)} < \Lambda$. 

The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for $L^2$ conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]
**Contradiction Strategy**

- **Contradiction Hypothesis (CH):** Assume $\exists \Lambda < \infty$ such that
  $$\|u\|_{L^\infty_t H^{1/2}_x([0, T^*) \times \mathbb{R}^3)} < \Lambda.$$  

- **Concentration Property:** If $H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u$ solves $NLS_3^-(\mathbb{R}^3)$, $T^* < \infty$ and we assume (CH) then
  $$\liminf_{t \uparrow T^*} \|u(t)\|_{L^3_{|x|<(T^*-t)^{1/2}}} \geq \frac{\sqrt{2}}{\pi^{2/3}} = c^*.$$
Contradiction Hypothesis (CH): Assume $\exists \Lambda < \infty$ such that
\[
\|u\|_{L^\infty_t H^{1/2}_x([0,T^*) \times \mathbb{R}^3)} < \Lambda.
\]

Concentration Property: If $H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u$ solves $NLS_3^-(\mathbb{R}^3)$, $T^* < \infty$ and we assume (CH) then
\[
\liminf_{t \uparrow T^*} \|u(t)\|_{L^3_{|x|<(T^*-t)^{1/2}-}} \geq \frac{\sqrt{2}}{\pi^{2/3}} = c^*.
\]
Contradiction Hypothesis (CH): Assume $\exists \Lambda < \infty$ such that

$$\|u\|_{L^\infty_t H^{1/2}_x([0, T^*) \times \mathbb{R}^3)} < \Lambda.$$ 

Concentration Property: If $H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u$ solves $\text{NLS}_3^-(\mathbb{R}^3)$, $T^* < \infty$ and we assume (CH) then

$$\liminf_{t \uparrow T^*} \|u(t)\|_{L^3_{|x|<(T^*-t)^{1/2}}} \geq \frac{\sqrt{2}}{\pi^{2/3}} = c^*.$$ 

The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for $L^2$ conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]
Contradiction Strategy

Frequency level Sets:

\[ R_{\mu}(t) := \sup \left\{ R \mid \| P_{\xi} \| > R_{\mu}(t) \right\} \]

Concentration \(=\)

\[ \forall = R_{c^*}(t) \geq \left( T^* - t \right)^{-1/2}. \]

\[ M := \sup \left\{ \mu \mid R_{\mu}(t) = O\left( R_{c^*}(t) \right) \right\} \text{ as } t \uparrow T^*. \]

By design

\[ R_{M^*} + \gamma_0(t) = o\left( R_{M^*}(t) \right) \text{ for all } \gamma_0 > 0 \text{ as } t \uparrow T^*. \]

There exists \( \mu_0 > 0 \) such that

\[ R_{M^*}(t) \sim R_{M^*} - \mu_0(t) \text{ as } t \uparrow T^*. \]

Fix a number \( K \) by the condition

\[ K^{1/2} \mu_0 = 3\Lambda. \]
Contradiction Strategy

Frequency level Sets:

\[ R_{\mu}(t) := \sup \{ R : \| P_{|\xi|>R} u(t) \|_{H_x^{1/2}} > \mu \} \]
Contradiction Strategy

- Frequency level Sets:

\[ R_\mu(t) := \sup \{ R : \| P_{|\xi|>R} u(t) \|_{\dot{H}^{1/2}_x} > \mu \} \]
Contradiction Strategy

- Frequency level Sets:

\[ R_\mu(t) := \sup \{ R : \| P_{|\xi|>R} u(t) \|_{H_{x}^{1/2}} > \mu \} \]

Concentration \( \implies R_{c^{*}}(t) \geq (T^{*} - t)^{-1/2} \).
Frequency level Sets:

\[ R_\mu(t) := \sup\{ R : \| P_{|\xi|>R} u(t) \|_{H^1_x} > \mu \} \]

Concentration \[ \Rightarrow R_{c^*}(t) \geq (T^* - t)^{-1/2}. \]

\[ M := \sup\{ \mu : R_\mu(t) = O(R_{c^*}(t)) \text{ as } t \uparrow T^* \} \]
Contradiction Strategy

- Frequency level Sets:

\[ R_\mu(t) := \sup\{ R : \| P|\xi|>RU(t)\|_{H^{1/2}_x} > \mu \} \]

Concentration \implies R_{c^*}(t) \geq (T^* - t)^{-1/2}.

\[ M := \sup\{ \mu : R_\mu(t) = O(R_{c^*}(t)) \text{ as } t \uparrow T^* \} \]

By design \( R_{M+\gamma_0}(t) = o(R_M(t)) \) for all \( \gamma_0 > 0 \) as \( t \uparrow T^* \).

There exists \( \mu_0 > 0 \) such that \( R_M(t) \sim R_{M-\mu_0}(t) \) as \( t \uparrow T^* \).
**Contradiction Strategy**

- **Frequency level Sets:**

  \[ R_\mu(t) := \sup\{ R : \| P_{|\xi| > R} u(t) \|_{H_x^{1/2}} > \mu \} \]

  **Concentration** \[ \implies R_{c^*}(t) \geq (T^*-t)^{-1/2}. \]

  \[ M := \sup\{ \mu : R_\mu(t) = O(R_{c^*}(t)) \text{ as } t \uparrow T^* \} \]

  By design \( R_{M+\gamma_0}(t) = o(R_M(t)) \) for all \( \gamma_0 > 0 \) as \( t \uparrow T^* \).

  There exists \( \mu_0 > 0 \) such that \( R_M(t) \sim R_{M-\mu_0}(t) \) as \( t \uparrow T^* \).

- **Fix a number** \( K \) by the condition

  \[ K^{1/2} \mu_0 = 3\Lambda. \]
Contradiction Strategy

Solution Decomposition:

At a time \( t_0 < T^* \), decompose \( u(t_0) = u_{\text{low}}(t_0) + u_{\text{gap}}(t_0) + u_{\text{hi}}(t_0) \) with respect to frequency regions

\( |\xi| < R_M + \gamma_0(t_0) \),

\( R_M + \gamma_0(t_0) < |\xi| < R_M(t_0) \),

\( R_M(t_0) < |\xi| \).

Evolve \( u_{\text{low}} \) and \( u_{\text{gap}} \) forward on \([t_0, T^*)\) using \( NLS^-_3(R_3) \). Evolve \( u_{\text{hi}} \) according to \( \tilde{NLS} \) so that

\( u(t) = u_{\text{low}}(t) + u_{\text{gap}}(t) + u_{\text{hi}}(t) \).

K\text{th Doubling Time after } t_0:\n
\( t_1 := \inf \{ t \in (t_0, T^*) : R_M(t_1) > KR_M(t_0) - \mu_0(t_0) \} \).
Solution Decomposition: At a time $t_0 < T^*$, decompose

$$u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0)$$
Contradiction Strategy

- Solution Decomposition: At a time $t_0 < T^*$, decompose

$$u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0)$$
Solution Decomposition: At a time $t_0 < T^*$, decompose

$$ u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0) $$

with respect to frequency regions

$$ |\xi| < R_{M+\gamma_0}(t_0) $$

$$ R_{M+\gamma_0}(t_0) < |\xi| < R_M(t_0) $$

$$ R_M(t_0) < |\xi|. $$
**Contradiction Strategy**

- **Solution Decomposition:** At a time $t_0 < T^*$, decompose
  
  $$u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0)$$

  with respect to frequency regions

  $$|\xi| < R_{M+\gamma_0}(t_0)$$

  $$R_{M+\gamma_0}(t_0) < |\xi| < R_M(t_0)$$

  $$R_M(t_0) < |\xi|.$$  

  Evolve $u^l$ and $u^g$ forward on $[t_0, T^*)$ using $\text{NLS}_3^-(\mathbb{R}^3)$. Evolve $u^h$ according to $\text{NLS}$ so that

  $$u(t) = u^l(t) + u^g(t) + u^h(t).$$
**Contradiction Strategy**

- **Solution Decomposition:** At a time $t_0 < T^*$, decompose
  
  $$u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0)$$

  with respect to frequency regions

  $$|\xi| < R_{M+\gamma_0}(t_0)$$

  $$R_{M+\gamma_0}(t_0) < |\xi| < R_{M}(t_0)$$

  $$R_{M}(t_0) < |\xi|. $$

  Evolve $u^l$ and $u^g$ forward on $[t_0, T^*)$ using $NLS_3^-(\mathbb{R}^3)$. Evolve $u^h$ according to $\tilde{NLS}$ so that

  $$u(t) = u^l(t) + u^g(t) + u^h(t).$$

- **$K$th Doubling Time after $t_0$:**

  $$t_1 := \inf\{ t \in (t_0, T^*) : R_M(t_1) > KR_{M-\mu_0}(t_0) \}.$$
Contradiction Strategy

Contradiction Strategy

Suppose we show the high frequency mass freezing property $\|P|_\xi|_R^M(t_0) u(t_1) \|_{L^2} \geq \frac{1}{2} \|P|_\xi|_R^M(t_0) u(t_0) \|_{L^2} \gg \mu_0 R^{-1/2} M^{-\mu_0}(t_0)$.

For small $\gamma_0$, we cannot place this mass inside the gap $R^M(t_0) < |\xi| < R^M(t_1)$ so we have to put it in the high frequency box $|\xi| > R^M(t_1)$.

Since, at time $t_1$, $R^M(t_1) \geq KR^M - \mu_0(t_0)$, we conclude $\|u(t_1)\|_{H^1/2} \geq 3\Lambda$, a contradiction.
Contradiction Strategy

- **High Frequency Mass Freezing Contradiction:** Suppose we show the high frequency mass freezing property

\[
\| P_{|\xi| > R_M(t_0)} u(t_1) \|_{L^2} \geq \frac{1}{2} \| P_{|\xi| > R_M(t_0)} u(t_0) \|_{L^2} \\
\geq \mu_0 R_M^{-1/2}(t_0).
\]

For small \( \gamma_0 \), we cannot park this mass inside the gap \( R_M(t_0) < |\xi| < R_M(t_1) \) so we have to put it in the high frequency box \( |\xi| > R_M(t_1) \).

Since, at time \( t_1 \), \( R_M(t_1) \geq K R_M - \mu_0(t_0) \), we conclude \( \| u(t_1) \|_{H^{1/2}} \geq 3 \Lambda \), a contradiction.
Contradiction Strategy

- **High Frequency Mass Freezing Contradiction**: Suppose we show the high frequency mass freezing property

\[
\|P_{|\xi|>R_M(t_0)} u(t_1)\|_{L^2} \geq \frac{1}{2} \|P_{|\xi|>R_M(t_0)} u(t_0)\|_{L^2} \geq \mu_0 R_M^{-1/2} R_M(t_0).
\]

For small \( \gamma_0 \), we can not park this mass inside the gap \( R_M(t_0) < |\xi| < R_M(t_1) \) so we have to put it in the high frequency boondox \( |\xi| > R_M(t_1) \).
Contradiction Strategy

- High Frequency Mass Freezing Contradiction: Suppose we show the high frequency mass freezing property

\[
\|P_{|\xi|>R_M(t_0)}u(t_1)\|_{L^2} \geq \frac{1}{2}\|P_{|\xi|>R_M(t_0)}u(t_0)\|_{L^2} \\
\gtrsim \mu_0 R_M^{-1/2}(t_0).
\]

- For small $\gamma_0$, we cannot park this mass inside the gap $R_M(t_0) < |\xi| < R_M(t_1)$ so we have to put it in the high frequency boondox $|\xi| > R_M(t_1)$.

- Since, at time $t_1$, $R_M(t_1) \geq KR_{M-\mu_0}(t_0)$, we conclude

\[
\|u(t_1)\|_{H^{1/2}} \geq 3\Lambda,
\]

a contradiction.
Main issue is to control the $L^2$ mass increment of $P|\xi| > R_M(t_0)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$.

We must control 4-linear spacetime integrals like

$$\int_{t_0}^{t_1} \int_{R^3} P > R_M(t_0) (u_{\text{long}}) P > R_M(t_0) u_{\text{long}} \, dx \, dt.$$ 

Since $L^4$ is $H^{1/4}$-critical and we have $H^{1/2}$ control on $u$, we can control such integrals with some gain:

$$\lesssim \left( \frac{t_1 - t_0}{5} \right)^{1/5} \left\| u \right\|_{L^5(t_0, t_1) \times R^3}^{5/2} \Lambda^{3/2}.$$

Assuming that $\left\| u \right\|_{L^5(t_0, t_1) \times R^3} \lesssim (T^* - t)^{-1/5}$ and the Concentration Property, we contradict (CH) proving critical norm explosion.
Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t_0)} u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$. Since $L^4$ is $H_1/4$-critical and we have $H_1/2$ control on $u$ we can control such integrals with some gain: $\lesssim \{ (t_1-t_0)^{1/5} \| u \|_{L^5(t_0,t_1;\mathbb{R}^3)} \}^{5/2} \Lambda^{3/2}$.
Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t_0)} u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$.

We must control 4-linear spacetime integrals like

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} P_{>R_M(t_0)}(u^I u^g u^h) P_{>R_M(t_0)} u^h \, dx dt.$$
Main issue is to control the $L^2$ mass increment of $\mathcal{P}_{|\xi|>R_M(t_0)} u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$.

We must control 4-linear spacetime integrals like

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \mathcal{P}_{>R_M(t_0)}(u^I u^g u^h) \mathcal{P}_{>R_M(t_0)} u^h \, dx \, dt.$$

Since $L^4_{tx}$ is $H^{1/4}$-critical and we have $H^{1/2}$ control on $u$ we can control such integrals with some gain:

$$\lesssim \left\{ (t_1 - t_0)^{1/5} \| u \|_{L^5_{t,x}([t_0, t_1] \times \mathbb{R}^3)} \right\}^{5/2} \Lambda^{3/2}.$$
Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t_0)}u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$.

We must control 4-linear spacetime integrals like

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} P_{>R_M(t_0)}(u^l u^g u^h) P_{>R_M(t_0)} u^h \, dx \, dt.$$ 

Since $L^4_{tx}$ is $H^{1/4}$-critical and we have $H^{1/2}$ control on $u$ we can control such integrals with some gain:

$$\lesssim \left\{ (t_1 - t_0)^{1/5} \|u\|_{L^5_{t,x}([t_0,t_1] \times \mathbb{R}^3)} \right\}^{5/2} \Lambda^{3/2}.$$ 

Assuming that $\|u\|_{L^5_{tx}([0,t] \times \mathbb{R}^3)} \lesssim (T^* - t)^{-1/5+}$ and the Concentration Property we contradict (CH) proving critical norm explosion.
Remarks

Spacetime $L^5$ upper bound is consistent with heuristics.

Concentration Property following [Merle-Tsutsumi] proof assumed $H^1 \cap \{\text{radial}\}$ data. The rest of the argument is at the critical level.

Under $(CH)$ bound, Bourgain's $L^2$ critical concentration result extends to the NLS$-3$ ($\mathbb{R}^3$) case to prove $L^3$ and $H^1/2$ concentration. [with Roudenko] This relaxes the $H^1 \cap \{\text{radial}\}$ assumptions to $H^1/2$.

Extends to the general mass supercritical case?
Remarks

- Spacetime $L^5_{tx}$ upper bound is consistent with heuristics.
Spacetime $L^5_{tx}$ upper bound is consistent with heuristics.

Concentration Property following [Merle-Tsutsumi] proof assumed $H^1 \cap \{\text{radial}\}$ data. The rest of the argument is at the critical level.
Remarks

- Spacetime $L^5_{tx}$ upper bound is consistent with heuristics.
- **Concentration Property** following [Merle-Tsutsumi] proof assumed $H^1 \cap \{\text{radial}\}$ data. The rest of the argument is at the critical level.
- Under (CH) bound, Bourgain’s $L^2$ critical concentration result extends to the $NLS^-_3(\mathbb{R}^3)$ case to prove $L^3$ and $H^{1/2}$ concentration. [with Roudenko] This relaxes the $H^1 \cap \{\text{radial}\}$ assumptions to $H^{1/2}$.
Remarks

- Spacetime $L^5_{tx}$ upper bound is consistent with heuristics.
- **Concentration Property** following [Merle-Tsutsumi] proof assumed $H^1 \cap \{\text{radial}\}$ data. The rest of the argument is at the critical level.
- Under (CH) bound, Bourgain’s $L^2$ critical concentration result extends to the $NLS_3^{-} (\mathbb{R}^3)$ case to prove $L^3$ and $H^{1/2}$ concentration. [with Roudenko]
  This relaxes the $H^1 \cap \{\text{radial}\}$ assumptions to $H^{1/2}$.
- Extends to the general mass supercritical case?