Recent progress on blowup phenomena in nonlinear Schrodinger equations

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Georgia Tech PDE Seminar

1. Nonlinear Schrödinger Initial Value Problem
2. Critical Regimes & Low Regularity GWP?
3. $H^{1/2}$ Critical Case
4. Energy Critical Case
5. Energy Supercritical Case
6. Critical Norm Explosion for $H^{1/2}$ Critical Case

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<u>Time Invariant Quantities</u>

$$\text{Mass} = \|u(t)\|_{L^2_x}$$

$$\text{Hamiltonian} = \int_{R^d} |\nabla u(t)|^2 dx \mp \frac{2}{p+1} |u(t)|^{p+1} dx$$

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- If u solves $NLS_p^{\pm}(\mathbb{R}^d)$ on $(-T_*, T^*) \times \mathbb{R}^2$ then

$$u_\lambda(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau\lambda^{-2}, y\lambda^{-1})$$

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$$\|D_y^\sigma u_\lambda\|_{L^q(\mathbb{R}_y^d)} = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1}+\sigma-\frac{d}{q}} \|D_x^\sigma u\|_{L^q(\mathbb{R}_x^d)}.$$

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- L^2 and $\dot{H}^1$ critical cases distinguished by conservation laws.

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- Optimal local-in-time well-posedness (LWP) for $NLS_p^{\pm}(\mathbb{R}^d)$: $\forall\ s \geq \max(0, s_c)$ $\exists$ unique continuous data-to-solution map

$$H^s \ni u_0 \longmapsto u \in C([0, T_{lwp}]; H^s) \cap L_t^q L_x^p$$

with $T_{lwp} = T_{lwp}(\|u_0\|_{H^s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

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- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?

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  for all $\delta > 0$ but diverges to ∞ as $\delta \downarrow 0$.

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- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

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  [Grillakis-Fang], [CKSTT]
  $NLS_5^+(\mathbb{R}^1)$ is similarly H^s-GWP for $s > \frac{4}{9}$.
  [Tzirakis]
  $NLS^+_{\frac{4}{d}+1}(\mathbb{R}^d)$ is H^s-GWP for $s > \frac{d+8}{d+10}$.
  [Visan-Zhang]

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$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}.$$

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All mass in S is conically concentrated into a point.

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- Minimal mass H^1 blowup solution characterization:
  $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

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Virial Identity $\implies$ $\exists$ **Many Blowup Solutions**

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- How do these solutions blow up?

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$$\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{1/2-}} |u(t,x)|^2 dx \geq \|Q\|_{L^2}^2.$$

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- H^1 blowups parabolically concentrate at least the ground state mass. Explicit blowups S concentrate mass much faster.
- Fantastic recent progress on the H^1 blowup theory. [Merle-Raphaël]

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[Bourgain]

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- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].

TYPICAL BLOWUPS LEAVE AN L^2 STAIN AT TIME T^*

[Merle-Raphaël]:

$H^1 \cap \{\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*\} \ni u_0 \longmapsto u$ solving $NLS_3^-(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.
$\exists\ \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi\mathbb{Z})$ and u^* such that

$$u(t) - \lambda(t)^{-1} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\theta(t)} \to u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!

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(Also, L^2 solutions of $NLS_3^+(\mathbb{R}^2)$ satisfy$^{???}$ $\|u\|_{L^4_{tx}} < \infty$.)

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modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in H^s for $s < 1$.

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- **Concentrated mass amounts are quantized**
The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.

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- **Scattering Below the Ground State Mass**

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies^{???} u_0 \longmapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$

  (Also, L^2 solutions of $NLS_3^+(\mathbb{R}^2)$ satisfy$^{???}$ $\|u\|_{L^4_{tx}} < \infty$.)
- **Minimal Mass Blowup Characterization**

  $$\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \longmapsto u, T^* < \infty \implies^{???} u = S,$$

  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in H^s for $s < 1$.
- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.
- **Are there any general upper bounds?**

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- Same results for $NLS^-_{\frac{4}{d}+1}(\mathbb{R}^d)$ in H^s, $s > \frac{d+8}{d+10}$. [Visan-Zhang]

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- Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

$$\limsup_{t\uparrow T^*} \sup_{cubes\ \ I, side(I)\le (T^*-t)^{\frac{1}{2}+\frac{\beta}{2}}} \int_I |u(t,x)|^2 dx \ge \|u_0\|_{L^2}^{-M}.$$

[Work in progress with Roudenko]

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- Virial identity $\implies$ $\exists$ many blowup solutions.
- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ then for any $a > 0$

$$\|\nabla u(t)\|_{L^2_{|x|<a}} \uparrow \infty \text{ as } t \uparrow T^*.$$

Thus, radial solutions must explode at the origin.

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$$\|\nabla u(t)\|_{L^2}^2 \lesssim C(a, Mass[u_0], H[u_0]) + C(a, Mass[u_0])\|\nabla u(t)\|_{L^2_{|x|<a}}^3.$$

$\square$

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- $H^{1/2}$-blowups parabolically concentrate in L^3 and $H^{1/2}$?
  [Work in progress with Roudenko]

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- Focusing energy critical case?
  [Kenig-Merle]: $E[u_0] < E[Q]$ and $\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2} \implies$ global-in-time and scatters.

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with Hamiltonian

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- **Conjecture:** $NLS_7^+(\mathbb{R}^3)$ is GWP and scatters in $H^{7/6}(\mathbb{R}^3)$. [See discussion by Bourgain, GAFA Special Volume, 2000]

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- Frequency heuristic: Bounded $H^{1/2}$ blowup inconsistent with mass conservation.

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The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for L^2 conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]

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Contradiction Strategy

- Frequency level Sets:

$$R_\mu(t) := \sup\{R : \|P_{|\xi|>R} u(t)\|_{\dot{H}^{1/2}_x} > \mu\}$$

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- Fix a number K by the condition

$$K^{1/2}\mu_0 = 3\Lambda.$$

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- **Kth Doubling Time after t_0:**

$$t_1 := \inf\{t \in (t_0, T^*) : R_M(t_1) > KR_{M-\mu_0}(t_0)\}$$

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- Since, at time $t_1, R_M(t_1) \geq KR_{M-\mu_0}(t_0)$, we conclude

$$\|u(t_1)\|_{H^{1/2}} \geq 3\Lambda,$$

a **contradiction**.

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- Since L^4_{tx} is $H^{1/4}$-critical and we have $H^{1/2}$ control on u we can control such integrals with some gain:

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- Assuming that $\|u\|_{L^5_{tx}([0,t]\times\mathbb{R}^3)}\lesssim (T^*-t)^{-1/5+}$ and the Concentration Property we **contradict** (CH) proving critical norm explosion.

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- Extends to the general mass supercritical case?