Blowup properties for critical and supercritical NLS at low regularity

J. Colliander

Toronto and M.S.R.I.
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1. **Nonlinear Schrödinger Initial Value Problem**

2. **Critical Regimes & Low Regularity GWP?**

3. \( H^{1/2} \) Critical Case

4. **Energy Critical Case**

5. **Energy Supercritical Case**

6. **Critical Norm Explosion for \( H^{1/2} \) Critical Case**
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Critical Regimes & Low Regularity

GWP?

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Energy Supercritical Case
Consider the initial value problem $NLS_p^{\pm}(\mathbb{R}^d)$:

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\begin{cases}
i \partial_t u + \Delta u = \pm |u|^{p-1} u \\
u(0, x) = u_0(x)
\end{cases}
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We seek $u : (-T_*, T^*) \times \mathbb{R}^d \rightarrow \mathbb{C}$. (+ focusing, − defocusing)
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(+ focusing, $-$ defocusing)

**Time Invariant Quantities**

**Mass**

$$\text{Mass} = \|u(t)\|_{L^2}^2$$

**Hamiltonian**

$$\text{Hamiltonian} = \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx \mp \frac{2}{p+1} |u(t)|^{p+1} dx$$
Dilation Invariance

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**Dilation Invariance**

- If $u$ solves $\text{NLS}_p^\pm(\mathbb{R}^d)$ on $(-T_*, T*) \times \mathbb{R}^2$ then

  $$u_\lambda(\tau, y) := \lambda^{\frac{2}{1-p}} u(\tau \lambda^{-2}, y \lambda^{-1})$$

  solves $\text{NLS}_p^\pm(\mathbb{R}^d)$ on $(-\lambda^2 T_*, \lambda^2 T*) \times \mathbb{R}^2$. 
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- Dilation invariant norms play decisive role in the theory of $\text{NLS}_p^\pm(\mathbb{R}^d)$.
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\[\hat{W}_\sigma, q \text{ is critical if } 2 \frac{1}{p} - 1 + \sigma - d q = 0.\]
Dilation Invariance

- If $u$ solves $NLS_{\pm}^p(\mathbb{R}^d)$ on $(-T_*, T*) \times \mathbb{R}^2$ then
  
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- Dilation invariant norms play decisive role in the theory of $NLS_{\pm}^p(\mathbb{R}^d)$:

  $$\|D_\sigma y u_\lambda\|_{L^q(\mathbb{R}^d_y)} = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1} + \sigma - \frac{d}{q}} \|D_\sigma x u\|_{L^q(\mathbb{R}^d_x)}.$$
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- $L^2$ and $\dot{H}^1$ critical cases distinguished by conservation laws.
Critical Regimes

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Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

Energy Critical Case

Energy Super-critical Case
Critical Regimes

- Theory for $NLS_p^\pm(\mathbb{R}^d)$ is qualitatively similar in regimes:

  - Mass subcritical ($s_0 < 0$)
  - Mass critical ($s_0 = 0$)
  - Mass supercritical/Energy subcritical ($0 < s_0 < 1$)
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- Optimal local-in-time well-posedness (LWP) for $NLS^\pm_p(\mathbb{R}^d)$:
Critical Regimes

- Theory for $NLS_{p}^{±}(\mathbb{R}^{d})$ is qualitatively similar in regimes:
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- Optimal local-in-time well-posedness (LWP) for $NLS_{p}^{±}(\mathbb{R}^{d})$: 

  $\forall s \geq \max(0, s_{c}) \exists$ unique continuous data-to-solution map $H^{s} \ni u_{0} \mapsto u \in C([0, T_{lwp}); H^{s}) \cap L^{q}_{t}L^{p}_{x}$ with $T_{lwp} = T_{lwp}(\|u_{0}\|_{H^{s}})$ if $s > s_{c}$ and $T_{lwp} = T(u_{0})$ if $s = s_{c}$.

Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case.

What is the fate of local-in-time solutions with critical initial regularity?
Critical Regimes

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- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?
LOW REGULARITY POLEMICS

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**Low Regularity Polemics**

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Some answers:

- $\exists$ global-in-time classical solutions of energy critical NLS was unknown until finite energy solutions were shown to be global.
- Typical, slightly bigger than necessary, finite time blowup solutions of mass critical NLS with nice initial data are "$L^2$-typical" after removing the blowup.
- Invariant measures live on low regularity phase space.
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The talk will first discuss ideas and questions in the $L^2$ mass critical case, the $H^{1/2}$ critical case, and the $H^1$ energy critical case. Then, I’ll discuss a qualitative property of blowup in an $H^{1/2}$ critical case.
Typical blowups leave an $L^2$ stain at time $T^*$

[Merle-Raphaël]:

$$H^1 \cap \{ \| Q \|_{L^2} < \| u_0 \|_{L^2} < \| Q \|_{L^2} + \alpha^* \} \ni u_0 \mapsto u$$ solving

$NLS^\sim_3(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.

$\exists \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi\mathbb{Z})$ and $u^*$ such that

$$u(t) - \lambda(t)^{-1}Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\theta(t)} \rightarrow u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!
**$L^2$ Critical Case: LWP Theory**

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Restrict attention to $NLS_3^\pm(\mathbb{R}^2)$. Typical $L^2$ critical case?
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[Cazenave-Weissler]
**$L^2$ Critical Case: LWP Theory**

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[Cazenave-Weissler]

$\forall u_0 \in L^2$ there exists $T_{lwp}(u_0)$ determined by

$$\| e^{it\Delta} u_0 \|_{L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}.$$

$\exists$ unique solution $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$. 

\[\begin{aligned}
\text{Blowup properties for critical and super-critical NLS at low regularity} \\
\text{J. Colliander} \\
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  $\exists$ unique solution $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$.

- Define the maximal forward existence time $T^*(u_0)$ by
  \[ \| u \|_{L^4_{tx}([0, T^*-\delta] \times \mathbb{R}^2)} < \infty \]
  for all $\delta > 0$ but diverges to $\infty$ as $\delta \downarrow 0$. 
\[ L^2 \text{ Critical Case: LWP Theory} \]

Restrict attention to \( NLS_3^\pm(\mathbb{R}^2) \). Typical \( L^2 \) critical case?

[Cazenave-Weissler]

- \( \forall u_0 \in L^2 \) there exists \( T_{lwp}(u_0) \) determined by
  \[
  \| e^{it\Delta} u_0 \|_{L^4_t([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100}.
  \]

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  for all \( \delta > 0 \) but diverges to \( \infty \) as \( \delta \downarrow 0 \).

- \( \exists \) small data scattering threshold \( \mu_0 > 0 \)
  \[
  \| u_0 \|_{L^2} < \mu_0 \implies \| u \|_{L^4_{tx}(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0.
  \]
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- $H^1$-GWP for $NLS_3^+(\mathbb{R}^2)$. 

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$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $NLS_3^+(\mathbb{R}^2)$.
- $H^1$-GWP mass threshold $\|Q\|_{L^2}$ for $NLS_3^-(\mathbb{R}^2)$.
\(L^2\) Critical Case: GWP Theory

- \(H^1\)-GWP for \(NLS^+_3(\mathbb{R}^2)\).
- \(H^1\)-GWP mass threshold \(\|Q\|_{L^2}\) for \(NLS^-_3(\mathbb{R}^2)\):

\[
\text{Weinstein} \\
\text{Here } Q \text{ is the ground state solution to } -Q + \Delta Q = Q^3. \\
e^{itQ}(x) \text{ is the ground state soliton solution to } NLS^-_3(R^2). \\
\text{I Method} \text{ yields } H^s\text{-GWP for } s > \frac{47}{2}. \\
\text{CKSTT} \text{ NLS}^+ + 5(R^1) \text{ is similarly } H^s\text{-GWP for } s > \frac{49}{2}. \\
\text{Tzirakis} \text{ NLS} + d + 1(R^d) \text{ is } H^s\text{-GWP for } s > d + 8d + 10. \\
\text{Visan-Zhang} \text{ NLS} + d + 1(R^d) \text{ is } H^s\text{-GWP for } s > d + 8d + 10.
$L^2$ Critical Case: GWP Theory

- $H^1$-GWP for $NLS^+_3(\mathbb{R}^2)$.
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[Weinstein]
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- 'I Method' yields $H^s$-GWP for $s > \frac{4}{7}$ ($s > \frac{1}{2}$?). [CKSTT]
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- 'I Method' yields $H^s$-GWP for $s > \frac{4}{7}$ ($s > \frac{1}{2}$?).

  [CKSTT]
  $NLS_5^+(\mathbb{R}^1)$ is similarly $H^s$-GWP for $s > \frac{4}{9}$.

  [Tzirakis]
  $NLS_{4d+1}^+(\mathbb{R}^d)$ is $H^s$-GWP for $s > \frac{d+8}{d+10}$.

  [Visan-Zhang]
**$L^2$ Critical Case: Blowup Solution Properties**

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Explicit Blowup Solutions
**$L^2$ Critical Case: Blowup Solution Properties**

**Explicit Blowup Solutions**

- Arise as *pseudoconformal* image of $e^{it}Q(x)$:

  $$S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + \frac{i}{t}}.$$
**$L^2$ Critical Case: Blowup Solution Properties**

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- $S$ has minimal mass:

  $$\|S(-1)\|_{L^2_x} = \|Q\|_{L^2}.$$

  All mass in $S$ is *conically* concentrated into a point.
Explicit Blowup Solutions

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  \[ S(t, x) = \frac{1}{t} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + i} . \]

- $S$ has minimal mass:
  \[ \| S(-1) \|_{L^2_x} = \| Q \|_{L^2} . \]

All mass in $S$ is *conically* concentrated into a point.

- Minimal mass $H^1$ blowup solution characterization:
  $u_0 \in H^1$, $\| u_0 \|_{L^2} = \| Q \|_{L^2}$, $T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]
$L^2$ CRITICAL CASE: BLOWUP SOLUTION PROPERTIES

**Blowup properties for critical and super-critical NLS at low regularity**

J. Colliander

**Nonlinear Schrödinger Initial Value Problem**

Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

Energy Critical Case

Energy Supercritical Case

**Virial Identity** $\implies \exists$ Many Blowup Solutions

Integration by parts and the equation yields

$$\frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^2} |u(t, x)|^2 \, dx = 8 H[u_0].$$

$H[u_0] < 0$, $\int |x|^2 |u_0(x)|^2 \, dx < \infty$ blows up.

How do these solutions blow up?
$L^2$ Critical Case: Blowup Solution Properties

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$H^1$ blowups parabolically concentrate at least the ground state mass. Explicit blowups $S$ concentrate mass much faster. Fantastic recent progress on the $H^1$ blowup theory.

[Merle-Tsutsumi]

[Merle-Raphaël]
$L^2$ Critical Case: Mass Concentration

$H^1$ Theory of Mass Concentration
\( \mathbb{L}^2 \textbf{ Critical Case: Mass Concentration} \)

**H^1 Theory of Mass Concentration**

- \( H^1 \cap \{ \text{radial} \} \ni u_0 \mapsto u, \ T^* < \infty \) implies

\[
\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{1/2}} |u(t, x)|^2 \, dx \geq \|Q\|^2_{L^2}.
\]

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  \textbf{[Merle-Tsutsumi]}
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[Bourgain] $L^2$ blowups parabolically concentrate some mass. For large $L^2$ data, do there exist tiny concentrations? Extensions in [Merle-Vega], [Carles-Keraani], [Begout-Vargas].
\textbf{L}^2 \textbf{Critical Case: Mass Concentration}

\textit{L}^2 \textit{Theory of Mass Concentration}
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- \( L^2 \ni u_0 \longrightarrow u, T^* < \infty \) implies

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\limsup_{t \uparrow T^*} \sup_{\text{cubes } l, \text{side}(l) \lesssim (T^* - t)^{1/2}} \int_I |u(t, x)|^2 dx \geq \| u_0 \|_{L^2}^{-M}.
\]

[Bourgain]

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\[ L^2 \text{ Critical Case: Mass Concentration} \]

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$L^2$ CRITICAL CASE: CONJECTURES/QUESTIONS
$L^2$ Critical Case: Conjectures/Questions

Consider focusing $NLS_3^-(\mathbb{R}^2)$:
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- **Scattering Below the Ground State Mass**

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \implies ??? \quad u_0 \mapsto u \text{ with } \|u\|_{L^4_{tx}} < \infty.$$  

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  modulo a solution symmetry. An intermediate step would extend characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$. 
Consider focusing $NLS_3^-(\mathbb{R}^2)$:

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- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.
**L^2 Critical Case:**
**Conjectures/Questions**

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- **Concentrated mass amounts are quantized**
  The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles.

- **Are there any general upper bounds?**
\textbf{L}^2 \textbf{C}RITICAL \textbf{C}ASE: \textbf{P}ARTIAL \textbf{R}ESULTS

\textbf{Blowup properties for critical and super-critical NLS at low regularity}

J. Colliander

\textbf{Nonlinear Schrödinger Initial Value Problem}

\textbf{Critical Regimes & Low Regularity GWP?}

\textbf{H}^{1/2} \textbf{Critical Case}

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$L^2$ Critical Case: Partial Results

For $0.86 \sim \frac{1}{5} (1 + \sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \mapsto u, T^* < \infty \implies$

$$\limsup_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2-}} |u(t, x)|^2 dx \geq \|Q\|_{L^2}^2.$$ 

$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]
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- \( \| u_0 \|_{L^2} = \| Q \|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, \ T^* < \infty \implies \exists \ t_n \uparrow T^* \text{ s.t. } u(t_n) \rightarrow Q \text{ in } H^{\tilde{s}(s)} \text{ (mod symmetry sequence).} \)
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  $\exists \ t_n \uparrow T^* \ s.t. \ u(t_n) \to Q \ in \ H^{\tilde{s}(s)} \ (\text{mod symmetry sequence}).$ For $H^s$ blowups with
  $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \to V \in H^1 \ (\text{mod symmetry sequence}).$ [Hmidi-Keraani]
\textbf{L}^2 \textbf{C}ritical \textbf{C}ase: \textbf{P}artial \textbf{R}esults

- For $0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1$, $H^s \cap \{\text{radial}\} \ni u_0 \mapsto u, T^* < \infty \implies$

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$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $u_0 \in H^s$, $\sim 0.86 < s < 1$, $T^* < \infty \implies \exists t_n \uparrow T^*$ s.t. $u(t_n) \to Q$ in $H^{\tilde{s}(s)}$ (mod symmetry sequence). For $H^s$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}$, $u(t_n) \rightharpoonup V \in H^1$ (mod symmetry sequence). [Hmidi-Keraani] This is an $H^s$ analog of an $H^1$ result of [Weinstein] which preceded the minimal $H^1$ blowup solution characterization.
L² CRITICAL CASE: PARTIAL RESULTS

- For $0.86 \sim \frac{1}{5}(1 + \sqrt{11}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \mapsto u, T^* < \infty \implies$

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$H^s$-blowup solutions concentrate ground state mass. [With Raynor, Sulem and Wright]

- $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \implies$

  $\exists \ t_n \uparrow T^* \text{ s.t. } u(t_n) \rightarrow Q \text{ in } H^\tilde{s}(s) \text{ (mod symmetry sequence)}. \text{ For } H^s \text{ blowups with}$

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- Same results for $NLS_{4^{-}}_{d+1}^{d} (\mathbb{R}^d)$ in $H^s$, $s > \frac{d+8}{d+10}$. [Visan-Zhang]
$L^2$ Critical Case: Partial Results
\[ L^2 \text{ Critical Case: Partial Results} \]

- Spacetime norm divergence rate
  \[ \|u\|_{L^4_t([0,t] \times \mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta} \]
  is linked with mass concentration rate
  \[ \limsup_{t \uparrow T^*} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{\frac{1}{2} + \frac{\beta}{2}}} \int_I |u(t, x)|^2 dx \geq \|u_0\|_{L^2}^{-M}. \]

[Work in progress with Roudenko]
$H^{1/2}$ Critical Case

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$H^{1/2}$ Critical Case

Consider $NLS^-_3(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?
Consider $\text{NLS}_3^-(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?

- LWP theory similar to $\text{NLS}_3^{\pm}(\mathbb{R}^2)$:

\[
L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3) \hookrightarrow L^4_{tx} \hookrightarrow L^5_{tx}.
\]
H^{1/2} Critical Case

Consider NLS_{3}^{-}(\mathbb{R}^{3}). Also L_{x}^{3}-Critical. Typical Case?

- LWP theory similar to NLS_{3}^{\pm}(\mathbb{R}^{2}):
  \[ L^{2}(\mathbb{R}^{2}) \leftrightarrow H^{1/2}(\mathbb{R}^{3}) \]
  \[ L_{t,x}^{4} \leftrightarrow L_{t,x}^{5}. \]

- There cannot be an H^{1}-GWP mass threshold.
**$H^{1/2}$ Critical Case**

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$H^{1/2}$ Critical Case

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$H^{1/2}$ Critical Case

Consider $NLS^+_3(\mathbb{R}^3)$. Also $L^3_x$-Critical. Typical Case?

- LWP theory similar to $NLS^\pm_3(\mathbb{R}^2)$:
  \[ L^2(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^3) \]
  \[ L^4_{tx} \hookrightarrow L^5_{tx}. \]

- There cannot be an $H^1$-GWP mass threshold.
- No explicit blowup solutions are known.
- Virial identity $\Rightarrow \exists$ many blowup solutions.
- $H^1 \cap \{\text{radial}\} \ni u_0 \hookrightarrow u, T^* < \infty$ then for any $a > 0$
  \[ \|\nabla u(t)\|_{L^2_{|x|<a}} \uparrow \infty \text{ as } t \uparrow T^*. \]

Thus, radial solutions must explode at the origin.
**Proof.**
Proof.

By Hamiltonian conservation,

\[ \| \nabla u(t) \|_{L^2}^2 = H[u_0] + \frac{1}{2} \| u(t) \|_{L^4}^4 \left| x < a \right| + \frac{1}{2} \| u(t) \|_{L^4}^4 \left| x > a \right|. \]
**$H^{1/2}$ Critical Case: Radial $\text{NLS}_3^-(\mathbb{R}^3)$**

**Proof.**

By Hamiltonian conservation,

$$\|\nabla u(t)\|^2_{L^2} = H[u_0] + \frac{1}{2}\|u(t)\|^4_{L^4_{|x|<a}} + \frac{1}{2}\|u(t)\|^4_{L^4_{|x|>a}}.$$

Inner contribution estimated using Gagliardo-Nirenberg by $C(\text{Mass}, a)\|\nabla u(t)\|^3_{L^2_{|x|<a}}$. 
**Proof.**

By Hamiltonian conservation,

\[ \|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4}^4 + \frac{1}{2} \|u(t)\|_{L^4}^4. \]

Inner contribution estimated using Gagliardo-Nirenberg by 
\[ C(\text{Mass}, a)\|\nabla u(t)\|_{L^2}^3. \]

Exterior region estimated by pulling out two factors in \( L^\infty \) then using radial Sobolev to get control by 
\[ \|u(t)\|_{L^2}^3 \|\nabla u(t)\|_{L^2}. \]
**Proof.**

By Hamiltonian conservation,

\[ \|\nabla u(t)\|_{L^2}^2 = H[u_0] + \frac{1}{2} \|u(t)\|_{L^4_{|x|<a}}^4 + \frac{1}{2} \|u(t)\|_{L^4_{|x|>a}}^4. \]

Inner contribution estimated using Gagliardo-Nirenberg by \( C(\text{Mass}, a)\|\nabla u(t)\|_{L^2_{|x|<a}}^3 \). Exterior region estimated by pulling out two factors in \( L^\infty_x \) then using radial Sobolev to get control by \( \|u(t)\|_{L^2_{|x|<a}}^3 \|\nabla u(t)\|_{L^2} \). Absorb the exterior kinetic energy to left side

\[ \|\nabla u(t)\|_{L^2}^2 \lesssim C(a, \text{Mass}[u_0], H[u_0]) + C(a, \text{Mass}[u_0]) \|\nabla u(t)\|_{L^2_{|x|<a}}^3. \]
$H^{1/2}$ Critical Case: Remarks
Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.
$H^{1/2}$ Critical Case: Remarks

- Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.
- For $H^{1/2}$-critical $NLS_5^-(\mathbb{R}^2)$, there exists $H^1 \cap \{\text{radial}\} \ni \nu_0 \mapsto \nu, \ T^*(\nu_0) < \infty$ which blows up precisely on a circle! [Raphaël]
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- Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.

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- Numerics/heuristics suggest: Finite time blowup solutions of $NLS_3(\mathbb{R}^3)$ satisfy $\|u(t)\|_{L^3_x} \uparrow \infty$ as $t \uparrow T^*$. [Work in progress with Raynor, Sulem, Wright] (Analogous to [Escauriaza-Seregin-Šverák] on Navier-Stokes)
Radial blowup solutions of energy subcritical $NLS_p(\mathbb{R}^d)$ with $p < 5$ must explode at the origin.

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$H^{1/2}$-blowups parabolically concentrate in $L^3$ and $H^{1/2}$.
[Work in progress with Roudenko]
$H^1(\mathbb{R}^d), d \geq 3$ Critical Case

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Energy Critical Case

Energy Sub-Critical Case
$H^1(\mathbb{R}^d), d \geq 3$ CRITICAL CASE

- Defocusing energy critical $NLS^{+}_{1+4/(d-2)}(\mathbb{R}^d), d \geq 3$ is globally well-posed and scatters in $H^1$: 

\[ H^{1/2} \]

Critical Case

Energy Critical Case

Energy Supercritical Case
$H^1(\mathbb{R}^d), d \geq 3$ Critical Case

Defocusing energy critical $NLS_{1+4/(d-2)}^+(\mathbb{R}^d), d \geq 3$ is globally well-posed and scatters in $H^1$:

[Bourgain], [Grillakis]: Radial Case for $d = 3$
[CKSTT]: $d = 3$
[Tao]: Radial Case for $d = 4$
[Ryckman-Visan], [Visan], [Tao-Visan]: $d \geq 4$
$H^1(\mathbb{R}^d)$, $d \geq 3$ Critical Case

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Induction on Energy; Interaction Morawetz; Mass Freezing
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Induction on Energy; Interaction Morawetz; Mass Freezing

- Focusing energy critical case?
$H^1(\mathbb{R}^2)$ "Critical" Case
\( H^1(\mathbb{R}^2) \) "Critical" Case

- \( NLS_p(\mathbb{R}^2) \) is energy subcritical for all \( p \). Is there an "energy critical" \( NLS \) equation on \( \mathbb{R}^2 \)?
\[ H^1(\mathbb{R}^2) \] ”Critical” Case

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\begin{cases}
  i\partial_t u + \Delta u = u(e^{4\pi|u|^2} - 1) \\
  u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R}^2)
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with Hamiltonian

\[
H[u(t)] := \int_{\mathbb{R}^{2}} |\nabla u(t, x)|^2 + \int_{\mathbb{R}^{2}} \frac{e^{4\pi|u(t,x)|^2} - 1}{4\pi} \, dx.
\]
$H^1(\mathbb{R}^2)$ "Critical" Case

Blowup properties for critical and super-critical NLS at low regularity

J. Colliander

Nonlinear Schrödinger Initial Value Problem

Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

Energy Critical Case

Energy Super-critical Case
$H^1(\mathbb{R}^2)$ "Critical" Case

- If $H[u_0] - M[u_0] \leq 1$ then $NLS_{exp}(\mathbb{R}^2)$ is globally well-posed. Uniform continuity of data-to-solution map fails to hold for data satisfying $H[u_0] - M[u_0] > 1$.

[Work in progress with Ibrahim, Majdoub, Masmoudi]
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- Well-posedness result relies upon Strichartz estimates, Moser-Trudinger inequality, and a log-Sobolev inequality. (Largely based on similar result for NLKG by [Ibrahim-Majdoub-Masmoudi])

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- Scattering?
energY Supercritical Case

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$H^{1/2}$ Critical Case

Energy Critical Case

Energy Supercritical Case

Consider NLS $+7(R^3)$. Typical case?

Numerical experiments by [Blue-Sulem] and also for corresponding NLKG [Strauss-Vazquez] suggest GWP and scattering. Conjecture: NLS $+7(R^3)$ is GWP and scatters in $H^{7/6}(R^3)$.

[See discussion by Bourgain, GAFA Special Volume, 2000]
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Consider $NLS_7^+(\mathbb{R}^3)$. Typical case?

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Critically Norm Explosion?

[Work in progress with Raynor, Sulem, Wright....details remain.]
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**Question:** Qualitative properties mass supercritical NLS blowup?
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CRITICAL NORM EXPLOSION?

[Work in progress with Raynor, Sulem, Wright....details remain.]

**Question:** Qualitative properties mass supercritical NLS blowup? Restrict attention to $H^{1/2}$-critical $NLS_3^-(\mathbb{R}^3)$.

- $T^*$ defined via divergence of $\|u\|_{L^5_{tx}}$ or $\|D^{1/2}u\|_{L^{10/3}_{tx}}$. 

Frequency heuristic: Bounded $H^{1/2}$ blowup inconsistent with mass conservation.
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**Blowup properties for critical and super-critical NLS at low regularity**

J. Colliander

**Nonlinear Schrödinger Initial Value Problem**

Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

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Energy Super-critical Case
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[Work in progress with Raynor, Sulem, Wright....details remain.]

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- $T^*$ defined via divergence of $\|u\|_{L^5_{tx}}$ or $\|D^{1/2}u\|_{L^{10/3}_{tx}}$.
- Finite energy radial blowups explode at spatial origin.
- Heuristics and numerics suggest asymptotic profile $Q$ which decays near spatial infinity like $|y|^{-1} \implies Q \notin L^3(\mathbb{R}^3)$. Sobolev embedding $H^{1/2} \hookrightarrow L^3$ suggests as $t \uparrow T^*$

$$\|u(t)\|_{H^{1/2}} \sim |\log(T^* - t)| \to \infty.$$
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CONTRADICTION STRATEGY
**Contradiction Strategy**

- **Contradiction Hypothesis (CH):** Assume \( \exists \, \Lambda < \infty \) such that

\[
\| u \|_{L_t^\infty L_x^{1/2}([0, T^*) \times \mathbb{R}^3)} < \Lambda.
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- Concentration Property: If \( H^1 \cap \{\text{radial}\} \ni u_0 \mapsto u \) solves \( NLS^-_{3}(\mathbb{R}^3) \), \( T^* < \infty \) and we assume (CH) then
  \[
  \liminf_{t \uparrow T^*} \|u(t)\|_{L^3_{|x|<(T^*-t)^{1/2}} \geq \frac{\sqrt{2}}{\pi^{2/3}} = c^*.
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**Contradiction Strategy**

- **Contradiction Hypothesis (CH):** Assume $\exists \Lambda < \infty$ such that
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The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for $L^2$ conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]
**Contradiction Strategy**

**Blowup properties for critical and super-critical NLS at low regularity**

J. Colliander

**Nonlinear Schrödinger Initial Value Problem**

Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

Energy Critical Case

Energy Super-critical Case
**Contradiction Strategy**

- **Frequency level Sets:**

  \[ R_\mu(t) := \sup \left\{ R : \| P_{|\xi| > R} u(t) \|_{H_x^{1/2}} > \mu \right\} \]
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M := \sup\{\mu : R_\mu(t) = O(R_{c^*}(t)) \text{ as } t \uparrow T^* \}
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  By design \(R_{M+\gamma_0}(t) = o(R_M(t))\) for all \(\gamma_0 > 0\) as \(t \uparrow T^*.\)

  There exists \(\mu_0 > 0\) such that \(R_M(t) \sim R_{M-\mu_0}(t)\) as \(t \uparrow T^*.\)
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- **Frequency level Sets:**
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- **Fix a number \( K \) by the condition**
  \[ K^{1/2} \mu_0 = 3 \Lambda. \]
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**Contradiction Strategy**

- **Solution Decomposition:** At a time $t_0 < T^*$, decompose

$$u(t_0) = u^{\text{low}}(t_0) + u^{\text{gap}}(t_0) + u^{\text{hi}}(t_0)$$
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  with respect to frequency regions

  $$|\xi| < R_{M+\gamma_0}(t_0)$$

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Evolve $u^l$ and $u^g$ forward on $[t_0, T^*)$ using $\tilde{NLS}_3^{-}(\mathbb{R}^3)$. Evolve $u^h$ according to $\tilde{NLS}$ so that

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  Evolve $u^l$ and $u^g$ forward on $[t_0, T^*)$ using $\overline{\text{NLS}_3}(\mathbb{R}^3)$.
  Evolve $u^h$ according to $\overline{\text{NLS}}$ so that
  \[ u(t) = u^l(t) + u^g(t) + u^h(t). \]

- **$K$th Doubling Time after $t_0$:**
  \[ t_1 := \inf\{ t \in (t_0, T^*) : R_M(t_1) > KR_{M-\mu_0}(t_0) \} \]
Contradiction Strategy

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**Contradiction Strategy**

- **High Frequency Mass Freezing Contradiction**: Suppose we show the high frequency mass freezing property

\[
\| P_{|\xi|>R_M(t_0)} u(t_1) \|_{L^2} \geq \frac{1}{2} \| P_{|\xi|>R_M(t_0)} u(t_0) \|_{L^2} \\
\geq \mu_0 R_M^{-1/2}(t_0).
\]

For small $\gamma_0$, we cannot park this mass inside the gap $R_M(t_0) < |\xi| < R_M(t_1)$ so we have to put it in the high frequency boondox $|\xi| > R_M(t_1)$.
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- Since, at time \( t_1 \), \( R_M(t_1) \geq KR_M-\mu_0(t_0) \), we conclude

\[
\| u(t_1) \|_{H^{1/2}} \geq 3 \Lambda,
\]

a contradiction.
High Frequency Mass Freezing Property
Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t_0)}u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$. 
HIGH FREQUENCY MASS FREEZING PROPERTY

- Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t)}u^h(\cdot)$ under the $\tilde{NLS}$ evolution from $t_0$ to $t_1$.

- We must control 4-linear spacetime integrals like

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^3} P_{>R_M(t_0)}(u^l \overline{ug} u^h) P_{>R_M(t_0)} \overline{u^h} \, dx dt.$$
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Since $L^4_{tx}$ is $H^{1/4}$-critical and we have $H^{1/2}$ control on $u$ we can control such integrals with some gain:

$$\lesssim \left\{ (t_1 - t_0)^{1/5} \| u \|_{L^5_{t,x}([t_0,t_1] \times \mathbb{R}^3)} \right\}^{5/2} \Lambda^{3/2}.$$
High Frequency Mass Freezing Property

- Main issue is to control the $L^2$ mass increment of $P_{|\xi|>R_M(t_0)} u^h(\cdot)$ under the $\tilde{\text{NLS}}$ evolution from $t_0$ to $t_1$.
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- Since $L^4_{tx}$ is $H^{1/4}$-critical and we have $H^{1/2}$ control on $u$ we can control such integrals with some gain:
  $$\lesssim \left\{ (t_1 - t_0)^{1/5} \| u \|_{L^5_{t,x}([t_0,t_1] \times \mathbb{R}^3)} \right\}^{5/2} \Lambda^{3/2}.$$
- Assuming that $\| u \|_{L^5_{tx}([0,t] \times \mathbb{R}^3)} \lesssim (T^* - t)^{-1/5+}$ and the Concentration Property we contradict (CH) proving critical norm explosion.
Remarks

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Remarks

- Spacetime $L^5_{tx}$ upper bound is consistent with heuristics.
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- Under (CH) bound, Bourgain's $L^2$ critical concentration result extends to the $NLS_3^{-}(\mathbb{R}^3)$ case to prove $L^3$ and $H^{1/2}$ concentration. [with Roudenko] This relaxes the $H^1 \cap \{\text{radial}\}$ assumptions to $H^{1/2}$. 
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- Extends to the general mass supercritical case?