Nonlinear Schrödinger Initial Value Problem

Critical Regimes & Low Regularity GWP?

$H^{1/2}$ Critical Case

Energy Critical Case

Energy Su-

Blowup properties for critical and supercritical NLS at low regularity

J. Colliander

Toronto and M.S.R.I.

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4 Energy Critical Case

5 Energy Supercritical Case

6 Critical Norm Explosion for $H^{1/2}$ Critical Case

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Time Invariant Quantities

$$\text{Mass} = \|u(t)\|_{L^2_x}$$

$$\text{Hamiltonian} = \int_{R^d} |\nabla u(t)|^2 dx \mp \frac{2}{p+1} |u(t)|^{p+1} dx$$

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- If u solves $NLS_p^{\pm}(\mathbb{R}^d)$ on $(-T_*, T^*) \times \mathbb{R}^2$ then

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- L^2 and $\dot{H}^1$ critical cases distinguished by conservation laws.

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- Optimal local-in-time well-posedness (LWP) for $NLS_p^{\pm}(\mathbb{R}^d)$: $\forall\ s \geq \max(0, s_c)\ \exists$ unique continuous data-to-solution map

$$H^s \ni u_0 \longmapsto u \in C([0, T_{lwp}]; H^s) \cap L_t^q L_x^p$$

with $T_{lwp} = T_{lwp}(\|u_0\|_{H^s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.

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  with $T_{lwp} = T_{lwp}(\|u_0\|_{H^s})$ if $s > s_c$ and $T_{lwp} = T(u_0)$ if $s = s_c$.
- Optimal maximal-in-time well-posedness (GWP) is known only in the defocusing energy critical case. What is the fate of local-in-time solutions with critical initial regularity?

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The talk will first discuss ideas and questions in the L^2 mass critical case, the $H^{1/2}$ critical case, and the H^1 energy critical case. Then, I'll discuss a qualitative property of blowup in an $H^{1/2}$ critical case.

Typical blowups leave an L^2 stain at time T^*

[Merle-Raphaël]:

$H^1 \cap \{\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*\} \ni u_0 \longmapsto u$ solving $NLS_3^-(\mathbb{R}^2)$ on $[0, T^*)$ (maximal) with $T^* < \infty$.
$\exists\ \lambda(t), x(t), \theta(t) \in \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}/(2\pi\mathbb{Z})$ and u^* such that

$$u(t) - \lambda(t)^{-1} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\theta(t)} \to u^*$$

strongly in $L^2(\mathbb{R}^2)$. Typically, $u^* \notin H^s \cup L^p$ for $s > 0, p > 2$!

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- $\forall u_0 \in L^2$ there exists $T_{lwp}(u_0)$ determined by

$$\|e^{it\Delta}u_0\|_{L^4_{tx}}([0, T_{lwp}] \times \mathbb{R}^2) < \frac{1}{100}.$$

$\exists$ unique solution $u \in C([0, T_{lwp}]; L^2) \cap L^4_{tx}([0, T_{lwp}] \times \mathbb{R}^2)$.

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- Define the maximal forward existence time $T^*(u_0)$ by

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  for all $\delta > 0$ but diverges to ∞ as $\delta \downarrow 0$.

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- $\exists$ small data scattering threshold $\mu_0 > 0$

$$\|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_{tx}(\mathbb{R}\times\mathbb{R}^2)} < 2\mu_0.$$

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[Weinstein]

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  [CKSTT]
  $NLS_5^+(\mathbb{R}^1)$ is similarly H^s-GWP for $s > \frac{4}{9}$.
  [Tzirakis]
  $NLS^+_{\frac{4}{d}+1}(\mathbb{R}^d)$ is H^s-GWP for $s > \frac{d+8}{d+10}$.
  [Visan-Zhang]

L^2 Critical Case: Blowup Solution Properties

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Explicit Blowup Solutions

Blowup properties for critical and super-critical NLS at low regularity

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$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}.$$

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All mass in S is conically concentrated into a point.

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- Minimal mass H^1 blowup solution characterization: $u_0 \in H^1, \|u_0\|_{L^2} = \|Q\|_{L^2},\ T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry. [Merle]

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Virial Identity $\implies$ $\exists$ **Many Blowup Solutions**

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$$\partial_t^2 \int_{\mathbb{R}_x^2} |u(t,x)|^2 dx = 8H[u_0].$$

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- How do these solutions blow up?

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[Merle-Tsutsumi]

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- Fantastic recent progress on the H^1 blowup theory. [Merle-Raphaël]

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[Bourgain]

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- For large L^2 data, do there exist tiny concentrations?

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L^2 Theory of Mass Concentration

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  [Bourgain]

  L^2 blowups parabolically concentrate some mass.

- For large L^2 data, do there exist tiny concentrations?
- Extensions in [Merle-Vega], [Carles-Keraani], [Bégout-Vargas].

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- **Are there any general upper bounds?**

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- Same results for $NLS^-_{\frac{4}{d}+1}(\mathbb{R}^d)$ in H^s, $s > \frac{d+8}{d+10}$. [Visan-Zhang]

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- Spacetime norm divergence rate

$$\|u\|_{L^4_{tx}([0,t]\times\mathbb{R}^2)} \gtrsim (T^* - t)^{-\beta}$$

is linked with mass concentration rate

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- No explicit blowup solutions are known.
- Virial identity $\implies$ $\exists$ many blowup solutions.
- $H^1 \cap \{radial\} \ni u_0 \longmapsto u$, $T^* < \infty$ then for any $a > 0$

$$\|\nabla u(t)\|_{L^2_{|x|<a}} \uparrow \infty \text{ as } t \uparrow T^*.$$

Thus, radial solutions must explode at the origin.

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Inner contribution estimated using Gagliardo-Nirenberg by $C(Mass, a)\|\nabla u(t)\|^3_{L^2_{|x|<a}}$. Exterior region estimated by pulling out two factors in L^∞_x then using radial Sobolev to get control by $\|u(t)\|^3_{L^2}\|\nabla u(t)\|_{L^2}$. Absorb the exterior kinetic energy to left side

$$\|\nabla u(t)\|_{L^2}^2 \lesssim C(a, Mass[u_0], H[u_0]) + C(a, Mass[u_0])\|\nabla u(t)\|^3_{L^2_{|x|<a}}.$$

$\square$

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with Hamiltonian

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- **Conjecture:** $NLS_7^+(\mathbb{R}^3)$ is GWP and scatters in $H^{7/6}(\mathbb{R}^3)$. [See discussion by Bourgain, GAFA Special Volume, 2000]

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- Frequency heuristic: Bounded $H^{1/2}$ blowup inconsistent with mass conservation.

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- Contradiction Hypothesis (CH): Assume $\exists\, \Lambda < \infty$ such that
$$\|u\|_{L^\infty_t H^{1/2}_x([0,T^*)\times\mathbb{R}^3)} < \Lambda.$$
- Concentration Property: If $H^1 \cap \{radial\} \ni u_0 \longmapsto u$ solves $NLS_3^-(\mathbb{R}^3)$, $T^* < \infty$ and we assume (CH) then
$$\liminf_{t \uparrow T^*} \|u(t)\|_{L^3_{|x|<(T^*-t)^{1/2-}}} \geq \frac{\sqrt{2}}{\pi^{2/3}} = c^*.$$
The proof follows [Merle-Tsutsumi] with the (CH) upper bound as a proxy for L^2 conservation. Explicit constant from sharp Gagliardo-Nirenberg estimate. [Delpino-Dolbeaut]

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- Frequency level Sets:

$$R_\mu(t) := \sup\{R : \|P_{|\xi|>R}u(t)\|_{\dot{H}^{1/2}_x} > \mu\}$$

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- Fix a number K by the condition

$$K^{1/2}\mu_0 = 3\Lambda.$$

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- Kth Doubling Time after t_0:
$$t_1 := \inf\{t \in (t_0, T^*) : R_M(t_1) > K R_{M-\mu_0}(t_0)\}$$

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- For small γ_0, we can not park this mass inside the gap $R_M(t_0) < |\xi| < R_M(t_1)$ so we have to put it in the high frequency boondox $|\xi| > R_M(t_1)$.
- Since, at time t_1, $R_M(t_1) \geq K R_{M-\mu_0}(t_0)$, we conclude

$$\|u(t_1)\|_{H^{1/2}} \geq 3\Lambda,$$

a **contradiction**.

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- Since L^4_{tx} is $H^{1/4}$-critical and we have $H^{1/2}$ control on u we can control such integrals with some gain:

$$\lesssim \left\{ (t_1 - t_0)^{1/5} \|u\|_{L^5_{t,x}([t_0,t_1]\times\mathbb{R}^3)} \right\}^{5/2} \Lambda^{3/2}.$$

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- Assuming that $\|u\|_{L^5_{tx}([0,t]\times\mathbb{R}^3)} \lesssim (T^* - t)^{-1/5+}$ and the Concentration Property we **contradict** (CH) proving critical norm explosion.

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- Extends to the general mass supercritical case?