

GWP below energy for NLS via the I-method

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[CKSTT]:

- method runs GWP below conserved regularity
- flexible/robust/sometimes sharp
- insight into frequency dynamics

Goal of talk: Describe these ideas in $NLS_3(\mathbb{R}^2)$ setting.

Outline

① $NLS_3(\mathbb{R}^2)$ i.v.p.

② Problem: local but not (yet?) global solutions.

③ The I-method

implementations {
 ④ $\tilde{E} = H[Iv]$
 ⑤ $\tilde{E}_{j+1} = \tilde{E}_j + A_{2j+2}(\tau_{2j+2})$ j multilinear corrections.

Connection w. Normal Forms?

①

$$\begin{cases} i\partial_t u + \Delta u - |u|^2 u = 0 & u: \mathbb{R}_t \times \mathbb{R}_x^2 \rightarrow \mathbb{C} \\ u(0) = u_0 \in H^s(\mathbb{R}_x^2) \end{cases}$$

defocusing

Conserved Quantities:

$$\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2} \quad \forall t \quad \text{mass}$$

$$H[u(t)] = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t)|^2 + \frac{1}{4} |u(t)|^4 dx \quad \begin{matrix} \text{kinetic} \\ \text{potential} \end{matrix} \quad \text{energy}$$

Dilation Symmetry

$$u_\lambda(t, x) = \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad u: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{C} \quad \text{solves NLS}$$

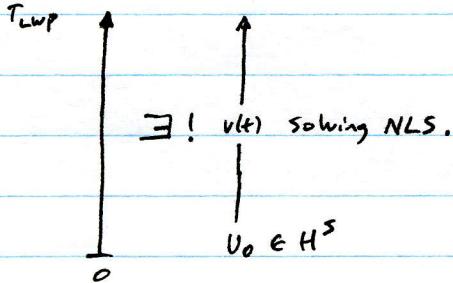
$$\uparrow \downarrow \quad u_\lambda: [0, T\lambda^2] \times \mathbb{R}^2 \rightarrow \mathbb{C} \quad \text{solves NLS.}$$

$$\|D_x^\sigma u_\lambda\|_{L_x^2} = \left(\frac{1}{\lambda}\right)^\sigma \|D_x^\sigma u\|_{L_x^2} \quad ; \quad \sigma = 0 \quad \text{is critical.}$$

L²-critical.

LWP Theory: $[cw]$, $[cv]$, $[T]$

$s > 0$



$\exists ! v(t)$ solving NLS.

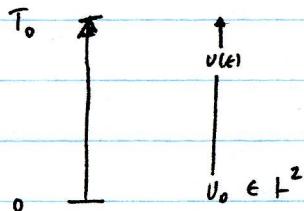
$$\exists T_{LWP} = T_{LWP}(\|u_0\|_{H^s}) > 0$$

Space-time
control on
slab.

$$u \in C([0, T_{LWP}]; H^s) \cap C^1_{[0, T_{LWP}] \times \mathbb{R}^2}$$

$$\textcircled{C} \sup_{t \in [0, T_{LWP}]} \|u(t)\|_{H^s} \leq 2 \|u_0\|_{H^s}.$$

$s = 0$



$$\exists T_0 = T_0(u_0) > 0$$

$$u \in C([0, T_0]; L^2) \cap C^1_{[0, T_0] \times \mathbb{R}^2}$$

\textcircled{C} L_x^2 is conserved.

$s < 0$

ill-posed. $[KPV]$, $[CCT]$

Remarks:

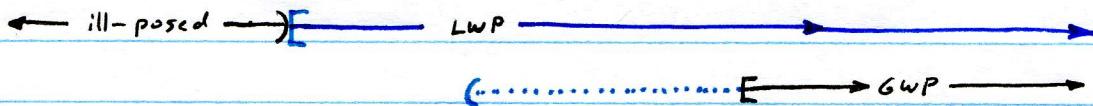
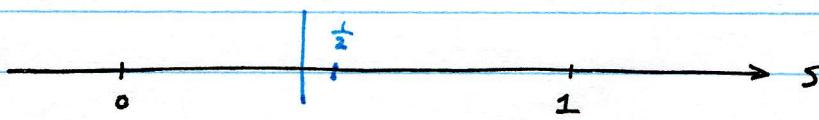
- H^s may double on (local) $[0, T_{LWP}]$.

Iterations of H^s LWP involve exponentially shrinking time steps.

- Mass, Energy Conservation $\Rightarrow \|u(t)\|_{H^1} \leq C(u_0) \quad \forall t$
 \Rightarrow No doubling so no shrinkage of timesteps \Rightarrow GWP in H^1 .
- A variant LWP theory may be constructed on nonuniform slabs $I_j \times \mathbb{R}^2$ defined by $\|u\|_{L^4_{t \in I_j, x \in \mathbb{R}^2}} \sim \gamma \ll 1$.
I_j have varying lengths.

(3)

(2)

 $S_{\text{GWP}}?$ 

Q: For $0 \leq s < 1$ are the local-in-time H^s solutions global-in-time?

Conjecture [B98]: $\text{NLS}_3(\mathbb{R}^2)$ is GWP in $H^s(\mathbb{R}^2)$, $s \geq 0$.

progress $s > \frac{2}{3}$, $s > \frac{3}{5}$ GWP [B98] (Breakthrough)

$s > \frac{4}{7}$ [CKSTT]

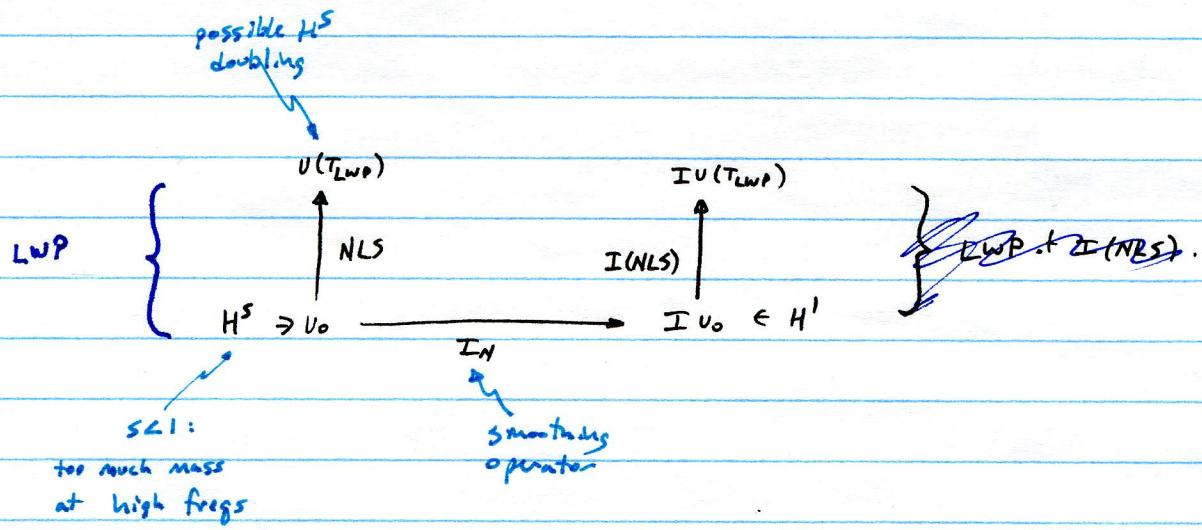
$s \geq \frac{1}{2}$ Best possible result using [B98] method.

" $s > \frac{2}{5}$ " [CKSTT; in progress].

[B98] Idea: Nonlinear flowmap is "essentially unitary" on H^s for high frequencies due to nonlinear smoothing in Duhamel term.

Progress on S_{GWP} \implies Dynamical insight into NLS.

(3) The I - Method (Scheme).



Find "modified energy" $\tilde{E}(Iu)$:

$$\exists t_g \in [\frac{1}{2}T_{LWP}, T_{LWP}] \text{ s.t.}$$

$$\|u(t_g)\|_{H^S} \leq \tilde{E}[Iu(t_g)] = \tilde{E}[Iu_0] + O(N^{-\alpha})$$

↑ ↑
 Modified Energy AC Law
 controls $\|u\|_{H^S}$. (parametrized
 by decay property)

No doubling of $\|u(t)\|_{H^S}$
occurs until $t \sim N^\alpha T_{LWP}$.

Iterate with H^S control.

$\nearrow \infty$ with $N \nearrow \infty$.

What is I_N ? How to find \tilde{E} ? How to implement method?

Remark: The role of the Hamiltonian in soliton stability, blowup behavior, ... could be (almost) played by \tilde{E} .

Role of $H[u]$ elsewhere may (almost) be played by \tilde{E} .

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I "Integration" "Identity-like"

$$(I_N f)^{\wedge}(\beta) = m_N(\beta) \hat{f}(\beta)$$

m_N smooth, radial, nonincreasing in $|\beta|$,

$\swarrow I = Id$
on low freq.

$$m_N(\beta) = \begin{cases} 1 & |\beta| < N \\ N^{1-s} |\beta|^{s-1} & |\beta| > 2N \end{cases}$$

\swarrow improves H^s decay
into H^1 decay on
high freqs.

④ First Implementation. ($\tilde{E} = H[Iv]$)

Fix $T \gg 1$. $u_0 \in H^s$.

GOAL: $u_0 \mapsto u(t) \oplus \text{NLS } \forall t \in [0, T]$.

Rescale to small initial modified energy

$$u_0, \lambda \mapsto u_\lambda(t) \quad [0, \lambda^2 T].$$

Assume $N \gg 1$ given. ($N = N(T)$ chosen.)

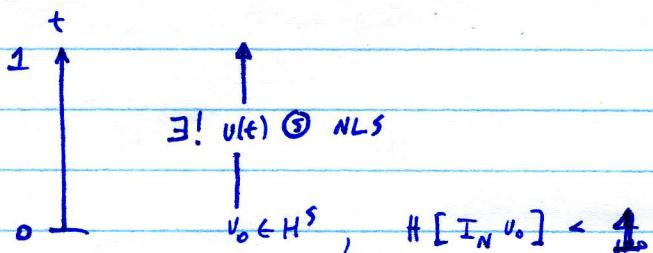
Consider

$$H[I_N u_0, \lambda] = \int \frac{1}{2} \| \nabla I_N u_0, \lambda \|^2 + \frac{1}{4} \| I_N u_0, \lambda \|^4 dx.$$

$$\leq \frac{1}{100} \quad \text{if } \lambda = \lambda(N, \|u_0\|_{H^s}) \gg 1 \text{ is chosen.}$$

$$\lambda \sim N^{\frac{1-s}{s}}.$$

Modified LWP Theory



local existence lifetime depends upon modified energy
 $\exists \tilde{T}_{LWP} = \tilde{T}_{LWP}(H[I_N u_0]) \sim 1$
 spatial norms might double.
 (spacetime control)

AC Law

$$u_0 \in H^s, s > S(\alpha), H[I_N u_0] < 1. \quad u_0 \mapsto u(t) \oplus \text{NLS}.$$

Then

$$\left\{ \begin{array}{l} \sup_{t \in [0, \tilde{T}_{LWP}]} H[I_N u(t)] \leq H[I_N u_0] + O(N^{-\alpha}) \\ \alpha = \frac{3}{2} \end{array} \right.$$

BOOKKEEPING

Iterate N^α times, each time with $\mathcal{O}(1)$ -sized time interval.

We need $N^\alpha \geq \lambda^2 T \sim (N^{\frac{1-s}{s}})^2 T$ by choosing N .

We can do it if $s > \frac{2}{2+\alpha} = s(\alpha)$.

Ideas in proof of AC Law

• $u_0 \longmapsto u(t) \text{ (④ NLS)}$.

$$H[u] = \int \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 dx$$

$$\begin{aligned} \partial_t H[u] &= \operatorname{Re} \int [-\Delta u \bar{u}_t + |u|^2 u \bar{u}_t] dx \\ &= -\operatorname{Re} \int \bar{u}_t [i u_t + \Delta u - |u|^2 u] dx = 0. \end{aligned}$$

[Makes no sense]
for $u_0 \in H^{\frac{1}{2}}$.

Q: Is there any flexibility in this choice?

• $Iu_0 \longmapsto Iu(t) \text{ (⑤ I(NLS))}$

$$I(NLS) \left\{ \begin{array}{l} i \partial_t Iu + \Delta Iu = I(|u|^2 u) = 0 \\ Iu(0) = Iu_0 \in H^1. \end{array} \right.$$

$$\begin{aligned} \partial_t H[Iu_0] &= -\operatorname{Re} \int I \bar{u}_t [i I u_t + \Delta I u + |Iu|^2 I u] dx \\ &\quad \underbrace{\qquad\qquad\qquad}_{=} \end{aligned}$$

$$= -\operatorname{Re} \int I \bar{u}_t [I(|u|^2 u) - |Iu|^2 I u] dx$$

commutator!

$$\Rightarrow |H[Iu(t)] - H[Iu_0]| = \left| \operatorname{Re} \int_0^t \int_{\mathbb{R}^2} I \bar{u}_t [I(|u|^2 u) - |Iu|^2 I u] dx dt \right|$$

$$\leq N^{-\alpha} (LWP)^4, \quad \alpha = \frac{3}{2}$$

Harmonic Analysis

Dyadic decomposition

Use MVT on M_N to extract cancellation
(Refined) Strichartz

(6) Multilinear Corrections

Describe "multilinear calculus" for NLS_3 .

[CM]

A calculation:

$$\begin{aligned} \int f_1 \bar{f}_2 f_3 \bar{f}_4 dx &= \int (\int e^{ix \cdot \vec{z}} \hat{f}_1(\vec{z}) dz_1) (\overline{\hat{f}_2}) (\overline{\hat{f}_3}) (\overline{\hat{f}_4}) dx \\ &= \iiint \left(\int e^{ix \cdot (\vec{z}_1 + \vec{z}_2 + \vec{z}_3 + \vec{z}_4)} dx \right) \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_4 d\vec{z}_{1-4} \\ &= \int_{\vec{z}_1 + \vec{z}_2 + \vec{z}_3 + \vec{z}_4 = 0} \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_4 . \end{aligned}$$

Notation:

$$A_n(M_n; f_1, \dots, f_n) = \int_{\vec{z}_n} M_n(\vec{z}_1, \dots, \vec{z}_n) \hat{f}_1 \hat{f}_2 \cdots \hat{f}_n .$$

$$A_n(M_n) = A_n(m_n; f) = A_n(M_n; f, \bar{f}, \dots, \bar{f}) \quad (n \in 2\mathbb{N})$$

Proposition: $v(t) \circledast NLS_3$, $n \in 2\mathbb{N}$, (symmetric) M_n :

$$\partial_t A_n(M_n; v(t)) = A_n(M_n \times_n; v(t)) + A_{n+2}(\underbrace{X_2[M_n]}_{n+2 \text{ mult.}}, v(t))$$

$$\alpha_n = -i(z_1^2 - z_2^2 - \cdots - z_n^2)$$

elagation operator
stretch.

explicit.

Example:

$$H[v] = A_2\left(\frac{z_1 z_2}{2}\right) + A_4\left(\frac{1}{4}\right).$$

$$\begin{aligned} \partial_t H[v] &= A_2\left(\frac{z_1 z_2}{2} \alpha_2^0\right) + A_4\left(\underbrace{X_2\left[\frac{z_1 z_2}{2}\right] + \frac{1}{4} \alpha_4}_{=0}\right) + A_6\left(X_2\left[\frac{1}{4}\right]\right)^0 \\ &= 0 \end{aligned}$$

Seeing energy conservation pointwise in frequency space.

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Formal Iteration

$$\Lambda_2(\tilde{v}_2) = \|\mathcal{I} \nabla v\|_{L^2}^2$$

$$\tilde{E}_1 = \Lambda_2(m_2) + \Lambda_4(v_4) ; \quad m_2 = m(\beta_1)\beta_1 m(\beta_2)\beta_2 , \quad v_4 \text{ t.b.d.}$$

$$\partial_t \tilde{E}_1 = \Lambda_2(m_2 \overset{\alpha_2}{\cancel{\alpha}}) + \Lambda_4(x_2[m_2] + v_4 \alpha_4) + \Lambda_6(x_2[v_4])$$

want: $v_4 = -\frac{x_2[M_2]}{\alpha_4} \sim \Delta^{-1} \circ u^2$.

$$\Rightarrow \partial_t \tilde{E}_1 = \Lambda_6(m_6) ; \quad m_6 = x_2 \left[\frac{-x_2[M_2]}{\alpha_4} \right]$$

$$\tilde{E}_2 = \tilde{E}_1 + \Lambda_6(v_6) ; \quad v_6 \text{ t.b.d.}$$

$$\partial_t \tilde{E}_2 = \Lambda_6(m_6) + \Lambda_6(v_6 \alpha_6) + \Lambda_8(x_2[v_6])$$

want: $v_6 = -\frac{m_6}{\alpha_6}$.

$$\Rightarrow \partial_t \tilde{E}_2 = \Lambda_8(m_8) . . .$$

Remark: Iteration progresses with well-defined v_j for KdV.
Perhaps generally for integrable equations.

For NLS₃

$$v_4 \sim \frac{m^2(\beta_1)|\beta_1|^2 - m^2(\beta_2)|\beta_2|^2 + m^2(\beta_3)|\beta_3|^2 - m^2(\beta_4)|\beta_4|^2}{|\beta_1|^2 - |\beta_2|^2 + |\beta_3|^2 - |\beta_4|^2 \underset{\alpha_4}{\sim}}$$

$$\text{On } \kappa_4: \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 ,$$

$$m=1, \rightarrow v_4 \rightarrow H[u]$$

$$\alpha_4 \sim (\beta_1 + \beta_2) \cdot (\beta_3 + \beta_4).$$

Note: $m(\beta_1)\beta_1 + m(\beta_2)\beta_2 + m(\beta_3)\beta_3 + m(\beta_4)\beta_4 \neq 0$ on κ_4
so numerator & ln factor so nicely.

Interactions among frequencies $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4$ such that :

$$\tilde{\gamma}_1 + \tilde{\gamma}_2 \neq 0, \quad \tilde{\gamma}_1 + \tilde{\gamma}_4 \neq 0, \quad \tilde{\gamma}_1^* > 10N$$

$$\text{and } (\tilde{\gamma}_1 + \tilde{\gamma}_2) \perp (\tilde{\gamma}_1 + \tilde{\gamma}_4)$$

are resonant, thus small divisor problem in NLS_3 .

[CKSTT] are extending the I-method scheme to cover this resonance by excising the resonant region from the 4-linear cancellation and carrying along the resonant interaction.

some difficulties emerge

$$"\zeta > \frac{\pi}{2}"$$