

7 June 2004

Pseudo-conformal transformation

$$(y, s) = \left(\frac{x}{t+1}, \frac{1}{t+1} \right) ; \quad (x, t) = \left(\frac{y}{s}, \frac{1}{s} - 1 \right).$$

$$-t \in [0, 1)$$

$$s \in [1, \infty)$$

$$v = p[u] = s^{-d/2} e^{i|y|^2/4s} u(t, x)$$

$$u = p^{-1}[v] = (1+t)^{-d/2} e^{-i|x|^2/4(t+1)} v(s, y).$$

$$v_s = s^{-\frac{d}{2}-2} e^{i \frac{|y|^2}{4s}} \left[-\frac{1}{2} s u - i \frac{|y|^2}{4} u - u_t - y \cdot \nabla u \right]$$

$$v_{y_j} = s^{-\frac{d}{2}-2} e^{i \frac{|y|^2}{4s}} \left[\frac{i y_j}{2} s u + s u_{x_j} \right]$$

$$\Delta v = s^{-\frac{d}{2}-2} e^{i \frac{|y|^2}{4s}} \left[-\frac{|y|^2}{4} u + i u s \frac{1}{2} + i y^j u_{x_j} + \Delta u \right]$$

 \Rightarrow

$$i v_s + \Delta v = s^{-\frac{d}{2}-2} e^{i \frac{|y|^2}{4s}} [-i u_t + \Delta u]$$

Suppose that $i \partial_t u + \Delta u = \pm |u|^{p-1} u$. Then, writing

$u = s^{\frac{d}{2}} e^{-i \frac{|y|^2}{4s}} v$ and substituting reveals that

$$i v_s + \Delta v = \pm s^{(p-1)\frac{d}{2}-2} |v|^{p-1} v.$$

The power of s , vanishes when $p-1 = \frac{4}{d}$, that is, in the L^2 -critical case.

pseudocritical energy

$$H[p_c[v]] = H[v] = \int_{\mathbb{R}_y^d} v_{y_j} \bar{v}_{y_j} \pm \frac{2}{p+1} |v|^{p+1} dy.$$

$$v_{y_j} \bar{v}_{y_j} = s^{-d-2} \left[v_{x_j} \bar{v}_{x_j}(t, x) + \frac{|y|^2}{4} |v|^2(t, x) \right]$$

$$\frac{2}{p+1} |v|^{p+1} = \frac{2}{p+1} s^{-\frac{d}{2}(p+1)} |v|^{p+1}.$$

$$H[p_c[v]] = \int_{\mathbb{R}_y^d} s^{-d-2} \left(|\nabla v|^2(t, x) + \frac{|y|^2}{4} |v|^2(t, x) \right) \pm \frac{2}{p+1} s^{-\frac{d}{2}(p+1)} |v|^{p+1}(t, x) dy.$$

$$y = sx \\ dy = s^d dx$$

$$= \int_{\mathbb{R}_x^d} s^{-2} \left[|\nabla v|^2(t, x) + \frac{s^2 |x|^2}{4} |v|^2(t, x) \right] \pm \frac{2}{p+1} s^{-\frac{d}{2}(p+1)} |v|^{p+1} dx$$

$$= \int_{\mathbb{R}_x^d} \frac{1}{4} |x|^2 |v|^2(t, x) dx + \int_{\mathbb{R}_y^d} s^{-2} |\nabla v|^2(t, x) \pm \frac{2}{p+1} s^{-\frac{d}{2}(p+1)} |v|^{p+1} dx.$$

In L^2 -critical case, $p+1 = \frac{4}{d} \implies$

$$H[p_c[v]] = \int_{\mathbb{R}_x^d} \frac{1}{4} |x|^2 |v|^2(t, x) dx + s^{-2} \int_{\mathbb{R}_y^d} |\nabla v|^2(t, x) \pm \frac{2}{p+1} |v|^{p+1} dx$$

$$= \frac{1}{4} \int_{\mathbb{R}_x^d} |x|^2 |v|^2(t, x) dx + \underbrace{s^{-2}}_{s \in [1, \infty)} H[v].$$

Remark: [B98] claims that $H(p \in L^2) = \| |x| v(x) \|_{L^2}^2$ and I
 don't yet see this. I want to validate the intuition
 that $p \in L^2 \iff \hat{p}$ so that $v_0 \in H^s \iff \hat{v}_0 \in L^2 \cap \{ |R|^{2s} \hat{v}_0 \in L^2 \}$
 is $p \in L^2$ transformed into $v_0 \in L^2 \cap \{ |x|^{2s} v_0 \in L^2 \}$.
 Thus, $p \in L^2$ should transport H^s GWP results for L^2 control
 NLS into GWP results in the class $v_0 \in \{ (1+|x|)^s f \in L^2 \}$.

Pseudoconformal transformation

$(x, t) \quad (y, \tau)$

$u(t, x)$ given

$$e[u](\tau, y) = |\tau|^{-d/2} e^{i \frac{|y|^2}{4\tau}} u\left(-\frac{1}{\tau}, \frac{y}{\tau}\right)$$

$$(i\partial_\tau + \Delta_y) e[u] = \tau^{-\frac{d}{2}-2} e^{i \frac{|y|^2}{4\tau}} (i\partial_\tau + \Delta_x) u.$$

If $u \in NLS_p(\mathbb{R}^d)$, $(i\partial_\tau + \Delta)u = \pm |u|^{p-1}u = -\lambda |u|^{p-1}u$ $\lambda > 0$ focus
 $\lambda < 0$ defocus

$$\begin{aligned} |u|^{p-1}u &= \tau^{\frac{d}{2}} \left| e^{-i \frac{|y|^2}{4\tau}} e[u] \right|^{p-1} e^{-i \frac{|y|^2}{4\tau}} e[u] \\ &= \tau^{\frac{d}{2}} |e[u]|^{p-1} e^{-i \frac{|y|^2}{4\tau}} e[u] \end{aligned}$$

$$\Rightarrow (i\partial_\tau + \Delta_y) e[u] = -\lambda \tau^{\frac{d}{2}(p-1)-2} |e[u]|^{p-1} e[u].$$

$e[\cdot]$ is a solution symmetry when $\frac{d}{2}(p-1) - 2 = 0 \Leftrightarrow p-1 = \frac{4}{d}$.

Assume $p-1 = \frac{4}{d}$.

$$H[e[u](\tau)] = \int_{\mathbb{R}_y^d} |\nabla_y e[u](\tau)|^2 - \frac{\lambda^2}{p+1} |e[u](\tau)|^{p+1} dy = H[e[u](\tau_0)].$$

Calculation \rightarrow

$$= \int_{\mathbb{R}_x^d} \left| \frac{x}{2} u(t, x) + it \nabla_x u(t) \right|^2 - \frac{2\lambda}{p+1} t^2 |u(t, x)|^{p+1} dx$$

choose $\tau_0 \leftrightarrow t = 0$

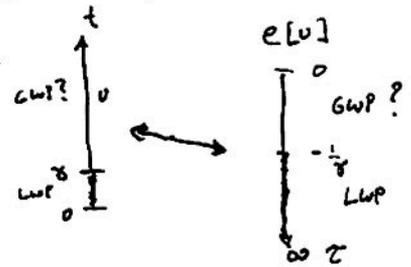
$$= \int_{\mathbb{R}_x^d} \left| \frac{x}{2} u(0, x) \right|^2 dx.$$

Thus, L^2 1st moment in x of u_0 equals the energy of the pseudoconformal image of u .

Presumably I can arrange for $\| |x| u(t') \|_{L^2}$ to appear for any t' .

Proposition If $(1+|x|)v_0 \in L^2_x$ then $v_0 \mapsto u(t)$ solving

$NLS_{\frac{d}{2+1}}(\mathbb{R}^d)$; defocusing is GWP, and scatters.



proof:

$v_0 \in L^2 \Rightarrow NLS$ is LWP.

$\exists \tau > 0$ s.t. $v_0 \mapsto u(t)$ valid for $t \in [0, \tau]$.

Thus, $e[u](t, y)$ is a well-defined solution valid for $t \in [0, \tau]$.

Furthermore, $e[u]$ has finite energy and thus enjoys a priori H^1 boundedness. Using a finite number of time steps we can extend $e[u]$ to a solution valid

also on $t \in [-\frac{1}{8}, 0]$. The extended solution $e[u]$ is bounded in $L^{\frac{4}{d}+2}_{t \in (-\infty, 0], \mathbb{R}^d_y}$.

Thus, by inverting the transformation we have constructed $v_0 \mapsto u(t)$ valid on $[0, \infty)$ and u is bounded in

$$L^{\frac{4}{d}+2}_{t > 0, \mathbb{R}^d_x}.$$

Collapse to the 2d L^2 critical defocusing cubic $NLS_3(\mathbb{R}^2)$.

Proposition If $(1+|x|)^S v_0 \in L^2_x$, $v_0 \mapsto u(t)$ $\textcircled{1}$ $NLS_3(\mathbb{R}^2)$; defocus in GWP and scatters, $\forall S > S_D$.

proof: Bourgain 98: $S > \frac{2}{3}$; $S > \frac{3}{5}$ ✓. Bourgain's proof links S_D with S_{GWP} where $S > S_{GWP} \Rightarrow NLS$ is GWP in $H^s(\mathbb{R}^d)$. [CKSTT] have pushed S_{GWP} down using the I-method.

Q: Does $S_D = S_{GWP}$? Does the I method apply to the slow spatial decay \Rightarrow GWP problem?

Pseud conformal Transformation

Given $u: \mathbb{R}_t \times \mathbb{R}_x^d \rightarrow \mathbb{C}$, not necessarily a solution of any equation.

$$e[u](t, x) = |t|^{-d/2} e^{i \frac{|x|^2}{4t}} u\left(-\frac{1}{t}, \frac{x}{t}\right).$$

$$\begin{aligned} \partial_t e[u] &= -\frac{d}{2} t^{-\frac{d}{2}-1} e^{i \frac{|x|^2}{4t}} u + t^{-\frac{d}{2}} e^{i \frac{|x|^2}{4t}} \frac{-i|x|^2}{4t^2} u \\ &\quad + t^{-\frac{d}{2}} e^{i \frac{|x|^2}{4t}} u_t + \frac{1}{t^2} + t^{-\frac{d}{2}} e^{i \frac{|x|^2}{4t}} u_{x_j} \left(-\frac{x^j}{t^2}\right) \end{aligned}$$

$$= t^{-\frac{d}{2}-2} e^{i \frac{|x|^2}{4t}} \left[\underbrace{-\frac{d}{2} t u}_{\text{mm}} \quad \underbrace{-i \frac{|x|^2}{4} u}_{\text{|||||}} + u_t \quad \underbrace{-x^j u_{x_j}}_{\text{=}} \right]$$

$$\partial_{x_j} e[u] = t^{-d/2} e^{i \frac{|x|^2}{4t}} \frac{i x^j}{2t} u + t^{-d/2} e^{i \frac{|x|^2}{4t}} u_{x_j} \frac{1}{t}$$

~~$$\begin{aligned} \partial_{x_j} \partial_{x_j} e[u] &= t^{-d/2-1} \left[e^{i \frac{|x|^2}{4t}} \left(-\frac{|x|^2}{4t} \right) u + e^{i \frac{|x|^2}{4t}} \frac{id}{2} u + e^{i \frac{|x|^2}{4t}} \frac{i x^j}{2} u_{x_j} \frac{1}{t} \right. \\ &\quad \left. + e^{i \frac{|x|^2}{4t}} \frac{i x^j}{2t} u_{x_j} + e^{i \frac{|x|^2}{4t}} u_{x_j x_j} \frac{1}{t} \right] \\ &= t^{-d/2-2} e^{i \frac{|x|^2}{4t}} \left[-\frac{|x|^2}{4} u + \frac{id}{2} u + \frac{i x^j}{2} u_{x_j} \right] \end{aligned}$$~~

$$\partial_{x_j} e[u] = t^{-d/2-1} e^{i \frac{|x|^2}{4t}} \left[\frac{i x^j}{2} u + u_{x_j} \right]$$

$$\partial_{x_j} \partial_{x_j} e[u] = t^{-\frac{d}{2}-1} e^{i \frac{|x|^2}{4t}} \frac{i x^j}{2t} \left[\frac{i x^j}{2} u + u_{x_j} \right]$$

$$+ t^{-\frac{d}{2}-1} e^{i \frac{|x|^2}{4t}} \left[\frac{id}{2} u + \frac{i x^j}{2} u_{x_j} \frac{1}{t} + u_{x_j x_j} \frac{1}{t} \right]$$

$$= t^{-\frac{d}{2}-2} e^{i \frac{|x|^2}{4t}} \left[\underbrace{-\frac{|x|^2}{4} u}_{\text{|||||}} + \frac{i x^j}{2} u_{x_j} + \frac{id}{2} u \right. \\ \left. + \frac{i x^j}{2} u_{x_j} + u_{x_j x_j} \right]$$

$$[i\partial_t + \Delta] e[u] = t^{-\frac{d}{2}-2} e^{i\frac{|x|^2}{4t}} [i\partial_t + \Delta] u$$

Suppose that

$$NLS_p(\mathbb{R}^d) \quad i\partial_t u + \Delta u = -\lambda |u|^{p-1} u$$

$\lambda > 0$ focus

$\lambda < 0$ defocus.

write

$$u = t^{\frac{d}{2}} e^{-i\frac{|x|^2}{4t}} e[u]$$

and form

$$-\lambda |u|^{p-1} u = -\lambda t^{\frac{d}{2}p} e^{-i\frac{|x|^2}{4t}} |e[u]|^{p-1} e[u]$$

then substitute to get

$$[i\partial_t + \Delta] e[u] = t^{-\frac{d}{2}-2} (-\lambda) t^{\frac{d}{2}p} |e[u]|^{p-1} e[u]$$

$$(i\partial_t + \Delta) e[u] = t^{\frac{d}{2}(p-1)-2} (-\lambda) |e[u]|^{p-1} e[u].$$

Thus, e : solutions of $NLS_p(\mathbb{R}^d) \rightarrow$ solutions of $NLS_p(\mathbb{R}^d)$

in the event that

$$\frac{d}{2}(p-1)-2 = 0 \iff p-1 = \frac{4}{d} \iff L^2 \text{ critical case.}$$

Restrict attention to L^2 critical case.

$$u \text{ solves } NLS_{\frac{4}{d}}(\mathbb{R}^d) \iff e[u] \text{ solves } NLS_{\frac{4}{d}}(\mathbb{R}^d).$$

Thus

$$H[e[u]](t) = H[e[u]](t_0) \quad \forall t, t_0.$$

$$= \int_{\mathbb{R}^d} \left| \frac{1}{2} y \cdot \nabla v(\tau, y) + i \tau \nabla v(\tau, y) \right|^2 - \frac{2\lambda}{p+1} \tau^2 |v(\tau, y)|^{p+1} dy$$

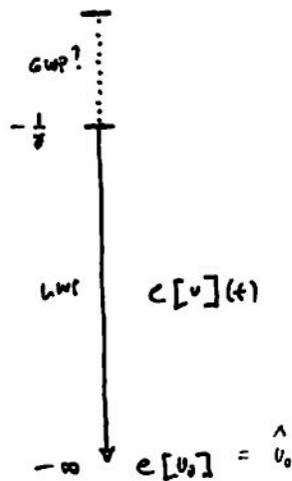
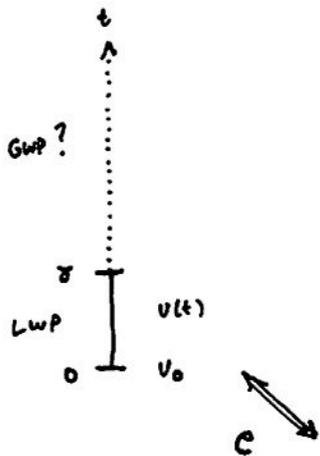
$$= \int_{\mathbb{R}^d} \left| \frac{1}{2} y \cdot \nabla v(0, y) \right|^2 dy.$$

choose

$$\tau(t_0) = 0$$

Thus,

$$H[e[v](t)] = \| |x| v(0) \|_{L_x^2}^2.$$



If $(1+|x|)v_0 \in L^2$ then $H[e[v](t)] < \infty$. For defocusing we thus obtain a priori H^1 control via $H + L^2$ conservation on $e[v]$. The $e[v]$ evolution extends to time interval $[-\frac{1}{2}, 0]$. Correspondingly the $v(t)$ evolution extends to $[\delta, \infty)$.

Express $H[e[v]]$ in terms of v .

$$\partial_{x_j} e[v] = t^{-\frac{d}{2}} e^{i \frac{\lambda x^2}{4t}} \left[\frac{2ix^j}{t} \tilde{v} + \tilde{v}_{x_j} \frac{1}{t} \right]$$

$$= t^{-\frac{d}{2}} e^{i \frac{\lambda x^2}{4t}} \left[\frac{i}{2} \frac{x^j}{t} v(-\frac{1}{t}, \frac{x}{t}) + v_j(-\frac{1}{t}, \frac{x}{t}) \frac{1}{t} \right]$$

$$|\nabla e[v]|_{(t,x)}^2 = t^{-d} \left| \left[\frac{i}{2} \frac{x^j}{t} v(-\frac{1}{t}, \frac{x}{t}) + v_j(-\frac{1}{t}, \frac{x}{t}) \frac{1}{t} \right] \right|^2$$

$$|e[v]|^{p+1} = t^{-\frac{d(p+1)}{2}} \left| v(-\frac{1}{t}, \frac{x}{t}) \right|^{p+1}$$

$$H[e[v]] = \int |\nabla e[v]|^2 - \frac{2\lambda}{p+1} |e[v]|^{p+1} dx$$

$$= \int_{\mathbb{R}^d} t^{-d} \left| \frac{i}{2} \frac{x^j}{t} v(-\frac{1}{t}, \frac{x}{t}) + v_j(-\frac{1}{t}, \frac{x}{t}) \frac{1}{t} \right|^2$$

$$- \frac{2\lambda}{p+1} \left| v(-\frac{1}{t}, \frac{x}{t}) \right|^{p+1} t^{-\frac{d(p+1)}{2}} dx$$

c.o.v. : $\tau = -\frac{1}{t}, \quad y = \frac{x}{t}; \quad t \frac{d}{dy} = dx$

$$= \int_{\mathbb{R}^d} \left| \frac{i}{2} y v(\tau, y) + v_j(\tau, y) (-\tau) \right|^2$$

$$- \frac{2\lambda}{p+1} |v(\tau, y)|^{p+1} t^{d(1-\frac{p+1}{2})} dy$$

$$p-1 = \frac{y}{a}$$

$v(t, x)$ given.

$$e[v](\tau, \gamma) = |\tau|^{-\frac{d}{2}} e^{i \frac{|\gamma|^2}{4\tau}} v\left(-\frac{1}{\tau}, \frac{\gamma}{\tau}\right)$$

A calculation shows

$$(i\partial_x + \Delta_y) e[v] = |\tau|^{-\frac{d}{2}-2} e^{i \frac{|\gamma|^2}{4\tau}} (i\partial_x + \Delta_x) v\left(-\frac{1}{\tau}, \frac{\gamma}{\tau}\right)$$