

GWP + Scattering for $NLS_5(\mathbb{R}^3)$

20 Jan Fields
Updates in Black

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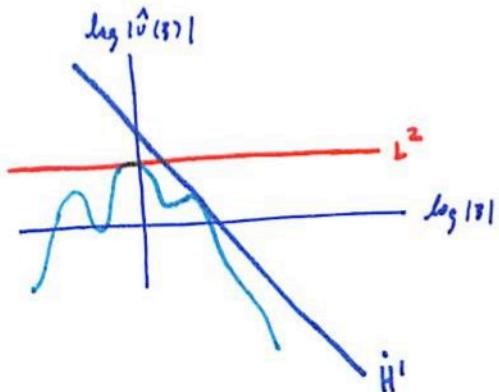
IAS
Talk

joint work [CKSTT]

$$\begin{cases} i\partial_t v + \Delta v = |v|^4 v \\ v(0) = v_0, \quad x \in \mathbb{R}^3 \end{cases}$$

$$E[v] = \int_{\mathbb{R}^3} |\nabla v(t, x)|^2 + \frac{1}{3} |v(t, x)|^6 dx \quad \text{conserved energy}$$

$$v_\lambda(t, x) = \frac{1}{\lambda^{1/2}} v\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \quad \text{scaling} \quad \text{Both pieces are invariant}$$



"Energy Critical".

$$\int_{\mathbb{R}^3} |v(t, x)|^2 dx \quad \text{conserved mass} \quad \text{scaling dependent}$$

Theorem $E[v_0] < \infty \implies \exists! v \in C^0(\mathbb{R}_t; \dot{H}^1(\mathbb{R}^3)) \cap L^{\infty}_{t,x}$ solving NLS

$$\text{s.t. } \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |v(t, x)|^{10} dx dt \leq C(E[v_0])$$

scattering, asymptotic completeness, uniform regularity

suffices to consider
Schwarz solutions.

Bourgain 1999 : • established robust strategy toward this result which we follow.

- proved radial case.

Grillakis 2000 : • proved smooth radial data \Rightarrow smooth global solution.

Main Ingredients in our proof

① Standard stuff: Strichartz Estimates, perturbation, bootstraps, ...

① Bourgain's Induction on Energy Strategy.

② Localization Control. Heuristic: Minimal energy blow-up solution must be irreducible.

③ Frequency Localized Interaction Morawetz Estimate (FLIM) (technical)

④ Energy does not concentrate

AC law freezes L^2 mass near $|x| \sim 1$.

① Standard Stuff

Strichartz Estimates

(δ, r) admissible if $\frac{2}{\delta} + \frac{3}{r} = \frac{3}{2}$, $2 \leq \delta, r \leq \infty$.

$$\|v\|_{S^\delta(I \times \mathbb{R}^3)} = \sup_{(\delta, r) \text{ admissible}} \left\{ \sum_{N \in 2\mathbb{Z}} \|P_N v\|_{L_t^\delta L_x^r(I \times \mathbb{R}^3)}^2 \right\}^{1/2}.$$

L²-Strichartz norm
 $I \subseteq \mathbb{R}_t$
 Littlewood-Paley projection onto $|I| \sim N$.

$$\|v\|_{S^K} = \|\nabla^K v\|_{S^0}$$

Lemma Suppose $v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ solves $i v_t + \Delta v = \sum_{m=1}^M F_m$.

$\forall K \geq 0 \quad \forall t_0 \in I \quad \forall \text{admissible } (\delta_m, r_m)$

$$\|v\|_{S^K} \lesssim \|v(t_0)\|_{H^K} + \sum_{m=1}^M \|F_m\|_{L_t^{\delta_m} L_x^{r_m}}$$

Hölder dual
dual Strichartz exponents

Note: $(2, 6)$ admissible, $(2, 6/5)$ dual Strichartz.

Perturbation Lemma

Suppose $\tilde{v}: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ solves \tilde{NLS} $\begin{cases} i \tilde{v}_t + \Delta \tilde{v} = |\tilde{v}|^4 \tilde{v} + \tilde{c} \\ \tilde{v}(t_0), \quad t_0 \in I \end{cases}$

with bounds

$$\|\tilde{v}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \leq M, \quad \|\tilde{v}\|_{L_t^\infty H_x^1(I \times \mathbb{R}^3)} \leq E. \quad (\text{Energy control})$$

exists \tilde{v}

Suppose \tilde{v} is a near-solution to NLS :

- $\|\nabla \tilde{v}\|_{L_t^2 L_x^{6/5}} \leq \varepsilon$

- $\exists v(t_0) \text{ s.t. } \|\tilde{v}(t_0) - v(t_0)\|_{H^1} \leq E'$,

$$\|e^{i(t-t_0)\Delta} [\tilde{v}(t_0) - v(t_0)]\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \leq \varepsilon$$

$\forall 0 \leq \varepsilon < \varepsilon_1(M, E, E')$, ε_1 small enough.

Then $\exists v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ solving NLS w. data $v(t_0)$ and

$$\|v - \tilde{v}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \leq C(M, E) \varepsilon$$

$\rightarrow \exists$ bdd $\circledast NLS$

$$\|v - \tilde{v}\|_{S^1(I \times \mathbb{R}^3)} \leq C(M, E) (E' + \varepsilon).$$

$$\|v\|_{S^1(I \times \mathbb{R}^3)} \leq C(M, E) (E + E' + \varepsilon).$$

① Induction on Energy

$\forall E \geq 0$ set $M(E) = \sup_{I_*, v} \{ \|v\|_{L^{\infty}_{t,x}(I_* \times \mathbb{R}^3)} \}$ where supremum ranges over all $v \in I_* \subset \mathbb{R}_t$, (Schwarz) solutions v with $E[v] \leq E$.

Theorem $\iff M(E) < \infty \quad \forall E \geq 0$.

Contradiction Hypothesis $M(E)$ is not always finite.

The set $\{E : M(E) < \infty\}$ is open, by Perturbation Lemma.
is connected and contains 0 .

Thus, $\exists 0 < E_{\text{crit}} < \infty$, the critical energy, s.t.

$$M(E) < \infty \quad \forall E < E_{\text{crit}},$$

$$M(E_{\text{crit}}) = +\infty$$

$\exists E_{\text{crit}}$, LWP theory



Induction on Energy Hypothesis

If (Schwarz) $V(t_0)$ satisfies $E[V(t_0)] \leq E_{\text{crit}} - n$, $n > 0$
then \exists (Schwarz) global solution $V : \mathbb{R}_t \times \mathbb{R}^3_x \rightarrow \mathbb{C}$ of NLS w. data $V(t_0)$

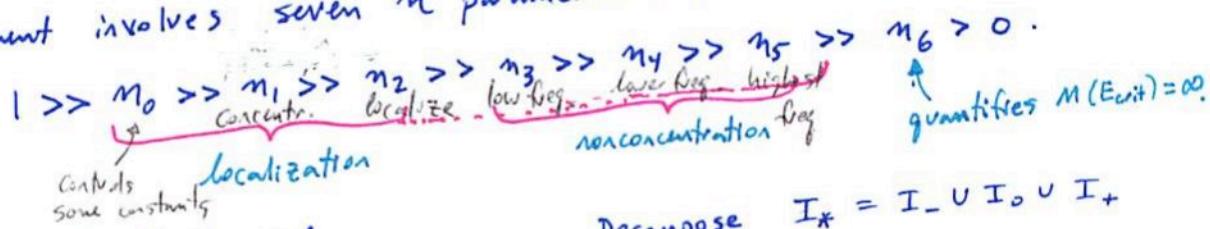
s.t. $\|V\|_{L^{\infty}_{t,x}} \leq M(E_{\text{crit}} - n) = c(n)$

$$\|V\|_{L^{\infty}_{t,x}} \leq M(E_{\text{crit}} - n) = c(n)$$

$$\|V\|_{\dot{S}^1} \leq c(n).$$

Strategy Comments / Proof Overview

The argument involves seven n parameters:



Assume $E[v_0] = E_{\text{crit}}$ and

(BIG)

$$\|v\|_{L^{\infty}_{t,x}(I_* \times \mathbb{R}^3)} > \frac{1}{n_6}.$$

ridiculously large

$$\text{Decompose } I_* = I_- \cup I_0 \cup I_+$$



(BIG) on
all three pieces.

- Suppose $\exists t_0 \in I_*$ when $u(t_0)$ is " n_0 -frequency-delocalized". (4)
 procedure is:
 prove L^{∞} bound using
 delocalized assumption.
 Thus, (BIG) \Rightarrow localized.
 - Separate u into two pieces w. weak interaction, energy subcritical.
 Induction Hypothesis, Perturbation $\Rightarrow \|u\|_{L^{\infty}_{t,x}(I_* \times \mathbb{R}^3)} \leq c(n_0)$ C!
 - Henceforth, assume $u(t)$ is " n_0 -frequency-localized" $\forall t$.
 - Similarly, may assume $u(t)$ is " n_1 -spatial-localized" $\forall t$.
 - Remaining (BIG) candidates are localized. Localized solutions are shown to satisfy
 - hi-freq-localized interaction Morawetz (FLIM).
 - Various Strichartz estimates
 - Nonevacuation of energy from medium frequencies \Rightarrow No concentration
 - $\|u\|_{L^{\infty}_{t,x}(I_* \times \mathbb{R}^3)} \leq c(n_0, \dots, n_5)$
- \downarrow
- (BIG) does not occur. E_{crit} fails to exist.

We turn to some details ...

Notational Remarks

- Implicit dependence on E_{crit} throughout
- $0 < c(n_j) \ll 1 \ll C(n_j) < \infty$.

↑
Small enough
↑
Big enough
- depending on n_j

similarly for
 $c(n_1, n_2)$.

② Localization Control

⑤

Frequency Localization

$\forall t \in I_0 \exists N(t) \in \mathbb{Z}^+$ about which the kinetic energy of $v(t)$ is freq. localized:

$$\| P_{\leq c(m_j)N(t)} v(t) \|_{H^1} \lesssim n_j^{100}, \quad \| P_{c(m_j)N(t)} v(t) \|_{H^1} \gtrsim 1, \quad \| P_{\geq c(m_j)N(t)} v(t) \|_{H^1} \lesssim n_j^{100}.$$

for $j = 0, 1, 2, 3, 4, 5$.

↑ ingredients:
 • Assume not so \exists wide freq. H^1
 • Pigeonhole to find small energy region
 • interaction control via
 Bourgain's Refined Strichartz

Spatial Localization

$\forall t \in I_0$

- potential energy is lower bounded:

$$\|v(t)\|_{L_x^6} \gtrsim n_1$$

- suppose not. $\exists t$ with small L^6
- L^6 must be big + freq. localized
 $\rightarrow L^\infty$ big somewhere in future.
- Decouple concentration from rest.

- \exists "concentration point" $x(t)$ about which:

$$\int_{|x-x(t)| \leq \frac{C(m_1)}{N(t)}} |v(t, x)|^P dx \gtrsim c(m_1) N(t)^{\frac{P}{2}-3}$$

L_x^P concentrates

Note $q=6$ is special

$$\int_{|x-x(t)| \leq \frac{C(m_1)}{N(t)}} |\nabla v(t, x)|^2 dx \gtrsim c(m_1)$$

kinetic energy concentrates

- Kinetic Energy is spatially localized about $x(t)$

$$\int_{|x-x(t)| > \frac{C(m_1, m_2)}{N(t)}} |\nabla v(t, x)|^2 dx \lesssim n_2.$$

- Assume not. wide in space
- pigeonhole
- decouple near and far
- pseudoconformal / which identifies to control interaction

up to n_2 error
 potential energy controls
 kinetic energy locally in space

Reverse Sobolev Inequality

$\forall t_0 \in I, x_0 \in \mathbb{R}^3, \forall R \geq 0$

$$\int_{B(x_0, R)} |\nabla v(t_0, x)|^2 dx \lesssim n_2 + C(m_1) \int_{B(x_0, C(m_1, m_2)R)} |v(t_0, x)|^6 dx.$$

Remark: Localization control steps do not use Morawetz or defocusing. (except for pseudoconformal, at t_0)
 Thus, these reductions should be available for other problems, also focusing.

There remain two solution scenarios which could be (BIG).

- $0 < N_{\min} \leq N(t) \leq N_{\max} < \infty$

"pseudosoliton"

- $0 < N_{\min} \leq N(t), N(t) \rightarrow \infty$

"energy concentration"

justify N_{\min} shortly.

These are ruled out with a Morawetz inequality, among other things.

③ Frequency Localized Interaction Morawetz

Recall Interaction Morawetz Estimate:

Independent of I_0 !

$$\int_{I_0} \int_{\mathbb{R}_y^3} |v(t, y)|^4 dy dt + (\geq 0) \lesssim \|v\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^3)}^3 \|v\|_{L_t^\infty H_x^1(I_0 \times \mathbb{R}^3)}^6$$

not $L_{t,x}^6$, but we are freq. localized.

"angular" derivative term

defocusing nonlinear potential energy

not energy controlled

e.g. scaling invariance explodes this norm by pushing kinetic energies towards zero frequency.

Somehow, small freq. Should not be the problem for us....

Since $1 \lesssim \|P_{c(m_0)N(t)} v(t)\|_{H^1} \lesssim C(m_0) N(t) \|v\|_{L_t^\infty L_x^2}$

$$N(t) \geq c(m_0) \|v\|_{L_t^\infty L_x^2}^{-1} \quad \forall t \in I_0.$$

Since $v \in \mathcal{S}$, $N(t) \geq \inf_{t \in I_0} N(t) = N_{\min} > 0$.

Since $N(t) \in 2^{\mathbb{Z}}$ $\exists t_{\min} \in I_0$ when $N(t_{\min}) = N_{\min}$.

Proposition (FLIM). Assume (Schwarz) $v: I_0 \times \mathbb{R}^3 \rightarrow \mathbb{C}$ solves NLS and satisfies frequency + spatial localization properties (so v has a chance to be (BIG)). $\forall N_* < c(n_3) N_{\min}$

$$\int_{I_0} \int_{\mathbb{R}_y^3} |P_{\geq N_*} v(t, y)|^4 dy dt \lesssim \underbrace{n_2^{-1}}_{\text{Independent of } I_0} N_*^{-3}$$

- N_*^{-3} mandated by scaling

proof is long and technical.

I'll explain it later. Also further remarks.

• very specific to $NLS_5(\mathbb{R}^3)$

• small constant n_2 on RHS used in bootstrap

Note that

$$\int_{\mathbb{R}^3} |P_{>c(n_3)N_{\min}} v(t, x)|^4 dx \underset{\substack{\uparrow \\ \text{freq. localized}}}{\sim} \int |v(t, x)|^4 dx \gtrsim c(n_1) N(t)^{-1}.$$

Now integrate over I_0 to get

Corollary $\int_{I_0} N(t)^{-1} dt \leq c(n_1, n_2, n_3) N_{\min}^{-3}.$

Consequently, the pseudosoliton does not exist.

Proposition (Nonconcentration of energy \Rightarrow spacetime bounded)

Let $I \subseteq I_0$, $N_{\min} \leq N(t) \leq N_{\max} < \infty \quad \forall t \in I$.

Then

$$\|u\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \leq C(n_1, n_2, n_3, N_{\max}/N_{\min})$$

↓
 \leq^1 bounds.

idea of proof: Rescale $N_{\min} = 1$. Corollary $\Rightarrow |I| \leq C(n_1, n_2, n_3, N_{\max})$.

Decompose $I = \bigcup_{j=1}^{1/\delta} I_j$, $|I_j| \sim \delta$, $t_j \in I_j$.

Use perturbation Lemma on I_j with $\tilde{v}(t) = e^{i(t-t_j)\Delta} P_{\leq C(n_0)N_{\max}} v(t_j)$
 taking advantage of Bernstein + short time to
 get $L_{t,x}^{10}(I_j \times \mathbb{R}^3)$.

(4) Energy concentration does not occur.

(5)

The remaining energy is a solution of NLS whose kinetic energy is freq. localized about $N(t)$ w. $N(t)$ unbounded for $t \in I_0$.

Proposition $\forall t \in I_0$

$$N(t) \leq C(m_5) N_{\min}.$$

Assuming the proposition, we have that all solutions of NLS with $E[u] \leq E_{\text{crit}}$ satisfy

$$\|u\|_{L_{t,x}^{10}(I_0 \times \mathbb{R}^3)} \leq C(m_0, \dots, m_5)$$

which contradicts (BIG). Thus E_{crit} does not exist.

Sketch proof: \exists time $t_{\min} \in I_0$ when $N(t_{\min}) = N_{\min} = 1$.

Thus, at time t_{\min} we have nontrivial energy content near $\|u\| \sim 1$.

Suppose proposition fails, and (BIG) holds. Then

$N(t)$ gets big so \exists time $t_{\text{vac}} \in I_0$ when

$$N(t_{\text{vac}}) > K(m_5)^{1/m_5}.$$

scale
↓
mass + energy
are equivalent
here.

Since (BIG) holds, we have the evacuation hypothesis

$$\left\| P_{\leq \frac{1}{m_5}} u(t_{\text{vac}}) \right\|_{H^1}^{100} \leq m_5.$$

Otherwise we are frequency delocalized $\Rightarrow L^{100}$ bound below (BIG).

Thus "all" the energy has evacuated from $\|u\| \sim 1$ at t_{vac} .

(10)

Energy evacuation is inconsistent with mass conservation.

At time t_{\min} , energy localization $\Rightarrow \|P_{>N} v(t_{\min})\|_{L_x^2} \gtrsim c(n_0)$.

Note: The Schwartz assumption implies $\exists N_{\min}$. All quantitative L^2 mass properties used here are deduced from energy.

Let $P_{hi} = P_{>n_y^{100}}$, $v_{hi} = P_{hi} v$.

Suppose we have $t_* \in (t_{\min}, t_{\max}]$ s.t.

$$\inf_{t_{\min} \leq t \leq t_*} L(t) \geq \frac{1}{2} c_0$$

where

$$L(t) = \int |v_{hi}(t, x)|^2 dx$$

and we then prove that

$$\inf_{t_{\min} \leq t \leq t_*} L(t) \geq \frac{3}{4} c_0 \implies t_* = t_{\max}$$

To prove this, we control the L^2 increment using FLIM + very low freq. Strichartz implied by evacuation at all times + other Strichartz bounds.

Small n_2 constant in FLIM is used to close bootstrap.

Thus, \exists nontrivial L^2 mass in $|t| > n_y^{100}$ at time t_{\max} .

But $\forall t$ using energy boundedness

$$\|P_{\leq N} v(t)\|_{L_x^2} \lesssim \frac{1}{N}$$

so mass must remain near N_{\min} . This contradicts (!)
the evacuation hypothesis.

Frequency Localized Interaction Morawetz

Basic Ideas:

- Run Interaction Morawetz on $P_{\geq N_k}(\text{NLS})$. $U_{hi} = P_{\geq N_k} u$
- $P_{\geq N_k}(\text{NLS})$ is not mass or momentum conserving \rightarrow new flux terms
- Control the problem terms by
 - spatially localizing interaction argument $\xrightarrow{\text{cutoff function terms}}$
 - exploiting positive potential energy term via Reverse Sobolev.
 - bootstrap, averaging over radii, Strichartz

(Non) Conservation Identities

$$G_{\text{NLS}} \quad i \partial_t \phi + \Delta \phi = N \quad x \in \mathbb{R}^d.$$

mass conservation

$$\partial_t |\phi|^2 + \partial_j \Im(\bar{\phi} \phi_j) = 2 \{N, \phi\}_M; \quad \{f, g\}_M = \Im(f \bar{g})$$

momentum conservation

$$\partial_t \Im(\bar{\phi} \phi_j) + \partial_k \left[-\partial_j \partial_k |\phi|^2 + 4 \operatorname{Re}(\bar{\phi}_j \phi_k) \right] = 2 \{N, \phi\}_P^j;$$

$$\{f, g\}_P = \operatorname{Re}(f \nabla \bar{g} - \bar{g} \nabla f).$$

momentum bracket.

Remark: In $U(1)$ gauge invariant Hamiltonian case $N = F'(|\phi|^2) \phi$

$$\{F'(|\phi|^2) \phi, \phi\}_M = 0$$

$$\{F'(|\phi|^2) \phi, \phi\}_P = -\nabla G(|\phi|^2), \quad G(z) = z F'(z) - F(z)$$

$$\text{e.g. } \{|\phi|^4 \phi, \phi\}_P = -\frac{2}{3} \nabla |f|^6.$$

Viriel Identity

(12)

$$V_a(t) = \int_{\mathbb{R}^d} a(t, x) |\phi(t, x)|^2 dx.$$

$$M_a(t) = \int_{\mathbb{R}^d} a_t |\phi|^2 + a_j 2 \operatorname{Im}(\bar{\phi} \phi_j) dx ; \quad M_a \neq V_a.$$

$$\begin{aligned} \partial_t M_a &= \int_{\mathbb{R}^d} (\cancel{a_{tt}} - \Delta a) |\phi|^2 + 4 a_{jk} \operatorname{Re}(\bar{\phi}_j \phi_k) + 2 a_j \sum_{N, \ell} \cancel{\mathcal{Z}_p^j} \\ &\quad + 4 \cancel{a_{tj}} \operatorname{Im}(\bar{\phi} \phi_j) + 2 \cancel{a_t} \sum_{N, \ell} \cancel{\mathcal{Z}_m} dx \end{aligned}$$

Interaction Viriel Identity + Morawetz Estimate (spatially localized)

$$\text{Fix } y \in \mathbb{R}^d. \quad \text{Choose } a(x) = |x-y| \underbrace{\chi_{[0,1]}}_{\text{smooth}} \left(\frac{|x-y|}{R} \right).$$

derivatives here
→ Lin-Strauss
terms

here → cutoff
terms in annulus
with $\frac{1}{R}$ factors.

$$M^{\text{interact}}(t) = \int_{\mathbb{R}^d_y} |\phi(t, y)|^2 M^y(t) dy, \quad \text{bounded by } \|\phi(t)\|_{L_x^2}^3 \|d(t)\|_{H_x^1}^3.$$

$\cancel{a_j}$ Momentum $\cancel{\mathcal{Z}_m}$

$\cancel{a_t} + \cancel{a_{tj}} + \cancel{a_{tt}}$

$$\partial_t M^{\text{interact}} = \dots$$

\int_0^t and rearrange to get a spatially localized Morawetz estimate valid for general NLS in general dimension.

Collapse to $P_{\geq N_k}(\text{NLS})$ on \mathbb{R}^3

(13)

We do algebra and obtain: $\forall R \geq 1$

$$\int_{I_0} \int_{\mathbb{R}^3} |v_{hi}|^4 dx dt + \int_{I_0} \int_{|x-y| \leq 2R} \frac{|v_{hi}(t, y)|^2 |v_{hi}(t, x)|^6}{|x-y|} dx dy dt \lesssim X_R$$

↑ momentum bracket.
 2 derivs on $|x-y|$,
 none on x .

where

$$X_R = \|v_{hi}\|_{L_t^\infty L_x^2}^3 \|v_{hi}\|_{L_t^\infty H_x^1} + (\text{Strichartz estimable})$$

$$+ \mathcal{O} \left(\int_{I_0} \int_{\mathbb{R}_y^3} \int_{\mathbb{R}_x^3} |v_{hi}(t, y)|^2 \Delta D \left(\chi \left(\frac{|x-y|}{R} \right) \right) |x-y| |v_{hi}(t, x)|^2 dx dy dt \right)$$

↗ $\frac{1}{R^3}$ $|x-y| \leq 2R$ ①, all derivs
 on x .

$$+ \mathcal{O} \left(\int_{I_0} \int_{\mathbb{R}_y^3} \int_{\mathbb{R}_x^3} |v_{hi}(t, y)|^2 |x-y| D^2 \left(\chi \left(\frac{|x-y|}{R} \right) \right) \left[|\nabla v_{hi}(t, x)|^2 + |v_{hi}(x)|^6 \right] dx dy dt \right)$$

↗ $\frac{1}{R}$ $|x-y| \leq 2R$ ②

controlled
via bootstrap.

RS \Rightarrow can drop $|\nabla v_{hi}|^2$ term
since it is controlled by $|v|^6$ piece.

Averaging in R shows ≥ 0 term on left
dominates $|v|^6$ piece.

so last term goes away.