

# Global well-posedness and scattering for $NLS_5(\mathbb{R}^3)$ .

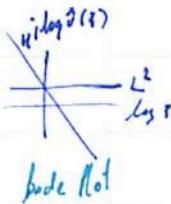
J. Colliander

December 2003 (IAS Talk.)

joint work: M. Keel, G. Staffilani, H. Takaoka, T. Tao.

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0) = u_0 \end{cases} \quad x \in \mathbb{R}^3$$

$$E[u] = \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{3} |u|^6 dx \quad \text{energy}$$



Scaling:  $u_\lambda(t, x) = \frac{1}{\lambda^{1/2}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$ . invariant energy critical  
Also  $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$  is conserved but supercritical.

Theorem If  $E[u_0] < \infty \exists! u \in C^0(\mathbb{R}^+ \times \mathbb{H}^1(\mathbb{R}^3)) \cap L^{\infty}_{t,x}$  solving NLS

with

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t, x)|^{10} dx dt < C(E[u_0]).$$

finite, huge

May assume  $u_0 \in \mathcal{S}$ .  
"Schwartz solutions" are Schwartz in  $x \forall t$ .

scattering, asymptotic completeness, uniform regularity.

Bourgain 1999: • radial case

• established a strategy which we follow.

Grillakis 2000. • smooth radial data evolves into smooth global solution.

Main Ingredients in our proof:

① Standard stuff: Strichartz estimates, perturbation, bootstraps, ...

① Bourgain's induction on energy strategy

② Localization Control

heuristic: Minimal energy blow-up solution must be irreducible.

Freq. or spatial delocalized at one time  $\implies$  global spacetime bdd.

3 Refined Strichartz  
+ perturbation

pseudoconformal law  
+ perturbation

Reduction to localized case.

③ Localized Morawetz estimate  $\longleftrightarrow$  technical

④ Preventing energy concentration

Almost Conservation

freezes  $L^2$  mass near  $|\text{frequency}| \sim 1$ .

(2)

## ① Strichartz Estimates

$(\theta, r)$  admissible if  $\frac{2}{\theta} + \frac{3}{r} = \frac{3}{2}$ ,  $2 \leq \theta \leq 8$ ,  $r \leq \infty$ .

$$\|v\|_{S^\theta}^{\circ} = \sup_{(\theta, r) \text{ admissible}} \left( \sum_{N \in 2^{\mathbb{Z}}} \|P_N v\|_{L_t^{\theta} L_x^r(I \times \mathbb{R}^3)}^2 \right)^{\frac{1}{2}}$$

↑  
L<sup>2</sup>-Strichartz norm

↑  
Littlewood-Paley Projection onto  $|B| \sim N$ .

$$\|v\|_{S^K}^{\circ} = \|\nabla^K v\|_{S^0}^{\circ}.$$

Lemma Suppose  $v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$  solves  $i v_t + \Delta v = \sum_{m=1}^M F_m$ .

Then  $\forall K \geq 0$ ,  $\forall t_0 \in I$ ,  $\forall$  admissible  $(\theta_m, r_m)$

$$\|v\|_{S^K(I \times \mathbb{R}^3)}^{\circ} \lesssim \|v(t_0)\|_{H^K(\mathbb{R}^3)}^{\circ} + \sum_{m=1}^M \|F_m\|_{L_t^{\theta_m'} L_x^{r_m'}(I \times \mathbb{R}^3)}^{\circ} \quad \text{Holder's law}$$

Perturbation Lemma Suppose  $\tilde{v}: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$  solves  $\tilde{NLS}_e \begin{cases} i v_t + \Delta v = \tilde{v}^4 \tilde{v} + e \\ \tilde{v}(t_0), \quad t_0 \in I \end{cases}$  with bounds

$$\|\tilde{v}\|_{L_t^{\infty} L_x^6(I \times \mathbb{R}^3)} \leq M \quad (\text{spacetime control})$$

$$\|\tilde{v}\|_{L_t^{\infty} H_x^1(I \times \mathbb{R}^3)} \leq E \quad (\text{energy}).$$

Suppose  $\tilde{v}$  is a near-solution to  $NLS$ :

- $\|\nabla e\|_{L_t^2 L_x^{6/5}} \leq \varepsilon$  (small forcing)
- $\exists v(t_0)$  s.t.  $\|\tilde{v}(t_0) - v(t_0)\|_{H^1} \leq E'$  (data perturbation)

$$\|e^{i(t-t_0)\Delta} (\tilde{v}(t_0) - v(t_0))\|_{L_t^{\infty} (I \times \mathbb{R}^3)} \leq \varepsilon \quad (\text{linear flows close in spacetime})$$

$\forall 0 \leq \varepsilon < \varepsilon_1(M, E, E')$ ,  $\varepsilon_1$  small enough.

Then  $\exists v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$  solving  $NLS$  w. data  $v(t_0)$  and

$$\|v - \tilde{v}\|_{L_t^{\infty} (I \times \mathbb{R}^3)} \leq C(M, E) \varepsilon \quad (\text{retain spacetime control})$$

$$\|v - \tilde{v}\|_{S^1(I \times \mathbb{R}^3)}^{\circ} \leq C(M, E)(E' + \varepsilon).$$

① Induction on Energy

$\forall$  energy  $E \geq 0$  define  $M(E) = \sup_{I_x, v} \left\{ \|v\|_{L_{t,x}^{10}(I_x \times \mathbb{R}^3)} \right\}$

where  $I_0 \in I_x \subset \mathbb{R}_t$ , (Schwarz) solutions  $v$  with  $E[v] \leq E$ .

Goal:  $M(E) < \infty \quad \forall E$ .

Assume, for contradiction, that  $M(E)$  is not always finite.

By Perturbation Lemma,  $\{E : M(E) < \infty\}$  is open, also connected and contains 0. The contradiction hypothesis implies  $\exists 0 < E_{\text{crit}} < \infty$ , the critical energy, s.t.

$$M(E) < \infty \quad \forall E < E_{\text{crit}}$$

$$M(E_{\text{crit}}) = +\infty.$$

Induction on Energy Hypothesis

Let  $v(t_0) \in \mathcal{S}$  satisfy  $E[v(t_0)] \leq E_{\text{crit}} - n$  for some  $n > 0$ .

Then  $\exists$  Schwarz global solution  $v : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}$  of NLS w. data  $v(t_0)$ :

$$\|v\|_{L_{t,x}^{10}(\mathbb{R} \times \mathbb{R}^3)} \leq M(E_{\text{crit}} - n) = C(n)$$

and

$$\|v\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^3)} \leq C(n).$$

[This statement follows from the definition of  $E_{\text{crit}}$  and well-posedness theory.]

Our argument involves seven  $n$  parameters:

$$I \gg n_0 \gg n_1 \gg n_2 \gg n_3 \gg n_4 \gg n_5 \gg n_6 > 0.$$

freq local      spatial Concentration      spatial local  
 localized Morawetz      low freq. Strich.      evac hypothesis

quantities

$$\begin{bmatrix} I_t \\ I_0 \\ I \end{bmatrix}$$

We start with  $u_0$ ,  $E[u_0] = E_{\text{crit}}$ , and assume  $M(E_{\text{crit}}) = +\infty$

(BIG)

$$\|u\|_{L_{t,x}^{10}(I_* \times \mathbb{R}^3)} > \frac{1}{n_6}. \quad (\text{ridiculously large})$$

[Decompose  $I_* = I^- \cup I_0 \cup I^+$  s.t. (BIG) holds on  $I_\pm^\pm$ ]

- Suppose  $\exists t_0 \in I_*$  when  $u(t_0)$  is " $n_0$ -frequency-delocalized".

Separate  $u$  into two pieces w. weak interaction.

Perturbation  $\rightarrow \|u\|_{L_{t,x}^{10}(I_* \times \mathbb{R}^3)} \leq C(n_0)$  C!

$\rightsquigarrow$  Henceforth, we may assume  $u(t)$  is " $n_0$ -freq.-localized"  $\forall t$ .

- Suppose  $\exists t_0 \in I_*$  when  $u(t_0)$  is " $n_1$ -spatially-delocalized".

Separate, perturbation  $\rightarrow \|u\|_{L_{t,x}^{10}(I_* \times \mathbb{R}^3)} \leq C(n_0, n_1)$  C!

$\rightsquigarrow$  May assume  $u(t)$  is " $n_1$ -spatially-localized"  $\forall t$

- The only remaining candidates for (BIG) are localized solutions.

Localized solutions are shown to satisfy:

- hi-freq-localized interaction Morawetz inequality
- Various Strichartz estimates
- Energy non-evacuation from low frequencies  $\Rightarrow$  No energy concentration
- $\|u\|_{L_{t,x}^{10}} \leq C(n_0, \dots, n_5)$ .



(BIG) does not occur.  $E_{\text{crit}}$  fails to exist.

We turn to some details...

② Localization Control

Proposition Frequency delocalization at one time  $\Rightarrow$  spacetime bounded.

Let  $n > 0$ . Suppose  $\exists N_{\leq 0} \in 2^{\mathbb{Z}}$ ,  $t_0 \in I$  s.t.

$$\| P_{\leq N_{\leq 0}} u(t_0) \|_{\dot{H}_1} \geq n^{100}$$

and

$$\| P_{\geq K(n)N_{\leq 0}} u(t_0) \|_{\dot{H}_1} \geq n^{100}.$$

If  $K(n)$  is big enough depending upon  $n$  then

$$\| u \|_{L_{t,x}^{10}(I_* \times \mathbb{R}^3)} \leq C(n).$$

Corollary (Freq. localization of energy  $\forall t$ ).

$\forall t \in I_* \exists N(t) \in 2^{\mathbb{Z}}$  s.t.

$$\| P_{\leq c(n_j)N(t)} u(t) \|_{\dot{H}_1} \leq n_j^{100},$$

small energy or freq.  
 $\ll N(t)$

$$\| P_{\geq c(n_j)N(t)} u(t) \|_{\dot{H}_1} \leq n_j^{100}$$

small ...  $\gg N(t)$

and

$$\| P_{c(n_j)N(t) \ll c(n_j)N(t)} u(t) \|_{\dot{H}_1} \sim 1$$

large energy near  $N(t)$ .

for every  $0 \leq j \leq 5$  provided  $0 < c(n_j) \ll 1 \ll c(n_j) < \infty$   
depending upon  $n_j$ .

The corollary follows from the delocalized statement  
since otherwise (BIG) is violated.

(6)

proof of freq. delocalized proposition.

## Spatial Localization

$\forall t \in I_0$

- $L_x^6$  is lower bounded:

$$\|u(t)\|_{L_x^6} \gtrsim n_1.$$

If not  $\exists t_0$   $\|u(t_0)\|_{L_x^6} \leq n_1$  but

$$\|e^{i(t-t_0)D_p} u(t_0)\|_{L_x^\infty} \sim 1$$

$$+ \| \cdot \|_{L_x^6} \leq C(n_0) \rightarrow z(t_0, x_0)$$

where  $L_x^6$  big.

- $\exists$  "concentration point"  $x(t)$  about which

$$-\int_{|x-x(t)| \leq c(n_1)/N(t)} |u(t, x)|^P dx \gtrsim C(n_1) N(t)^{\frac{P}{2}-3} \quad (L_x^P \text{ concentrates})$$

$$-\int_{|x-x(t)| \leq c(n_1)/N(t)} |\nabla u(t, x)|^2 dx \gtrsim C(n_1). \quad (\text{kinetic energy concentrates})$$

- Energy is spatially localized about  $x(t)$ :

$$\int_{|x-x(t)| > c(n_1, n_2)/N(t)} |\nabla u(t, x)|^2 dx \lesssim n_2.$$

$\Rightarrow$  (Reverse Sobolev Inequality)

$\forall t_0 \in I, x_0 \in \mathbb{R}^3, \forall R \geq 0$

$$\int_{B(x_0, R)} |\nabla u(t_0, x)|^2 dx \lesssim n_2 + C(n_1) \int_{B(x_0, c(n_1, n_2)R)} |u(t_0, x)|^6 dx.$$

Thus, up to  $n_2$  error, potential energy locally controls kinetic energy.

### Summary

Any minimal energy ( $E_{\min}$ ) blow-up solution (BIG)

has kinetic energy frequency localized about  $N(t)$ ,

spatially localized up to Heisenberg about  $x(t)$

and potential energy controls kinetic energy locally in space.

Add blow-up +  
solution scenarios.

Summary

Any solution with energy  $E_{\text{crit}}$  satisfying (BII6) :

- has kinetic energy localized in frequency about  $N(t) \in 2^{\mathbb{Z}}$ .
  - " " " space (up to Heisenberg) about  $x(t)$
  - potential energy controls kinetic energy locally in space.
- 

$\exists$  (at least) two scenarios to be ruled out.

- $N(t) \rightarrow \infty$  as in self-similar blow-up. self-similar, blowup.
- $N(t) \sim N_{\text{fixed}}$ ,  $u \sim \text{soliton}$  pseudosoliton

↙ energy concentration.

To rule at these (and remaining) possibilities, we use  
a new Morawetz inequality, and other stuff.

---

(9)

### ③ Frequency Localized Interaction Morawetz

Recall the Interaction Morawetz Estimate:

$$\int \int_{I_0 \times \mathbb{R}^3} |v(t, y)|^4 dy dt + (\geq 0) \lesssim \|v\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^3)}^3 \|v\|_{L_t^\infty H_x^1}.$$

Not  $L_{t,x}^6$ .

But we are frequency localized!

we don't expect motion of energy into low freqs. to be a source of blowup  
so this issue should somehow be irrelevant.

not energy controlled.

e.g. Scaling invariance explodes this norm by pushing kinetic energies toward zero frequency.

$N_{\min}$

$$\begin{aligned} \|P_{c(n_0)N(t)} v(t)\|_{H^1} &\leq c(n_0) N(t) \|v\|_{L_t^\infty L_x^2} \\ \Rightarrow N(t) &\geq c(n_0) \|v\|_{L_t^\infty L_x^2}^{-1} \quad \forall t \in I_* \\ \forall t \in I_* &\Rightarrow N(t) > 0 \Rightarrow N_{\min} = \inf_{t \in I_*} N(t) > 0. \end{aligned}$$

Proposition (FLIM) Assume  $v$  satisfies the frequency and spatial localization properties required for  $v$  to satisfy (BIG).  
 $\forall N_* < c(n_3) N_{\min}$

$$\int \int_{I_0 \times \mathbb{R}^3} |P_{\geq N_*} v(t, y)|^4 dy dt \lesssim \underbrace{n_2 N_*^{-3}}_{\bullet \text{ independent of } I_0}.$$

The proof is long + technical.  
 $I'$ ll explain it later.

- $N_*$  dependence mandated by scaling

$$\int |P_{>c(n_3)N_{\min}} v(t, x)|^4 dx \underset{\substack{\uparrow \\ \text{freq. localization}}}{\sim} \int |v(t, x)|^4 dx \underset{\substack{\uparrow \\ L^p\text{-concentration}}}{\gtrsim} c(n_1) N(t)^{-1}.$$

Thus, we  $\int_{I_0} (\dots) dt$  to get

$$\text{Corollary : } \int_{I_0} N(t)^{-1} dt \lesssim c(n_1, n_2, n_3) N_{\min}^{-3}.$$

We exclude the pseudosolution.

Suppose  $N_{\min} \leq N(t) \leq N_{\max} < \infty \quad \forall t \in I \subseteq I_0$ . Rescale  $N_{\min} = 1$ .

Then

$$|I| \leq C(n_1, n_2, n_3, N_{\max}).$$

Decompose  $I = \bigcup_{j=1}^{\sqrt[10]{N_{\max}}} I_j$ ,  $|I_j| \sim \delta$ ,  $t_j \in I_j$ .

$$\| P_{\geq C(n_0) N_{\max}} v(t_j) \|_{H^1} \leq n_0.$$

Set  $\tilde{v}(t) = e^{i(t-t_j)\Delta} P_{\leq C(n_0) N_{\max}} v(t_j)$  so

$$\| \tilde{v}(t_j) - v(t_j) \|_{H^1} \leq n_0.$$

$$\| \tilde{v}(t) \|_{L_x^{10}} \lesssim C(n_0, N_{\max}) \quad \forall t \in I_j.$$



$$\| \tilde{v} \|_{L_{t,x}^{10}(I_j \times \mathbb{R}^3)} \leq C(n_0, N_{\max}) \delta^{\frac{1}{10}}. \Rightarrow \text{Interaction controlled}$$

By Perturbation lemma

~~$$\| v \|_{L_{t,x}^{10}(I_j \times \mathbb{R}^3)} \leq 1$$~~

so  $v$  is  $L_{t,x}^{10}$  bounded on  $I \times \mathbb{R}^3$ .

Proposition (Nonconcentration of energy  $\Rightarrow$  space-time bounded.)

Let  $I \subseteq I_0$  and suppose  $N_{\min} \leq N(t) \leq N_{\max} \quad \forall t \in I$ .

Then

$$\| v \|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \leq C(n_1, n_2, n_3, N_{\max}/N_{\min})$$

..  $s_1$  ..

(4) Energy concentration does not occur.

The remaining energy is a solution of NLS whose kinetic energy is frequency localized about  $N(t)$  with  $N(t)$  unbounded for  $t \in I_0$ .

Proposition  $\forall t \in I_0$

$$N(t) \leq C(n_5) N_{\min}.$$

Assuming the proposition, we have that all solutions of NLS with  $E[u_0] \leq E_{crit}$  satisfy

$$\|u\|_{L_{t,x}^{10}(I_0 \times \mathbb{R}^3)} \leq C(n_0, \dots, n_5)$$

which contradicts (BIG) and thus the existence of  $E_{crit}$  and the theorem is proved.

proof of proposition (ideas anyway)

~~Assume the proposition fails~~

~~Rescale  $N_{\min} = 1$ . Since  $N(t) \in \mathbb{Z}^+$  s.t.  $N(t_{\min}) = N_{\min} = 1$ .~~

~~At time  $t_{\min}$  we have low frequencies~~

~~little energy on  $H_1 < n_0$~~

(12)

### Idea of proof

$\forall t \in I_0$ , we have frequency localized kinetic energy:

$$\exists N(t) \in \mathbb{Z}^+ \text{ s.t.}$$

$$\| P_{\leq c(n_0)N(t)} u(t) \|_{H^1} \leq n_0^{100}, \quad \| P_{c(n_0)N(t) < \cdot < C(n_0)N(t)} u(t) \|_{H^1} \gtrsim 1, \quad \| P_{\geq C(n_0)N(t)} u(t) \|_{H^1} \leq n_0^{-100}.$$

Scaling



$$\exists \text{ time } t_{\min} \in I_0 \text{ when } N(t_{\min}) = N_{\min} = 1.$$

Suppose proposition fails and (BIG) holds. Then  $N(t)$  gets big

so  $\exists$  time  $t_{\text{vac}} \in I_*$  when  $N(t) > K(n_5)^{1/n_5}$ .

Since (BIG) holds, we must have evacuation hypothesis

$$\| P_{< 1/n_5} u(t_{\text{vac}}) \|_{H^1} \leq n_5.$$

Otherwise we would have a frequency delocalized wave at  $t_{\text{vac}}$ .

Thus, "all" the energy has evacuated from  $L^2 \sim 1$  at  $t_{\text{vac}}$ .

This behavior (energy evacuation) is inconsistent with mass conservation.

At time  $t_{\min}$ , we know from the energy localization property (Note: we are not using any Schwartz property besides the existence of  $N_{\min}$  to make this  $L^2$  claim) that

$$\| P_{c(n_0)N(t) < \cdot < C(n_0)N(t)} u(t) \|_{L_x^2} \gtrsim c(n_0).$$

Let  $P_{hi} := P_{>n_y^{100}}, \quad u_{hi} = P_{hi} u.$

Suppose we have  $t_* \in (t_{\min}, t_{\max}]$  satisfying

$$\inf_{t_{\min} \leq t \leq t_*} \|u_{hi}(t)\|_{L^2} \geq \frac{1}{2} c_0.$$

Using an almost conservation law analysis on the quantity

$$L(t) = \int |u_{hi}(t, x)|^2 dx$$

we prove

$$\inf_{t_{\min} \leq t \leq t_*} \|u_{hi}(t)\|_{L^2} \geq \frac{3}{4} c_0 \implies t_* = t_{\max}.$$

Thus, there is nontrivial  $L^2$  mass in  $\{|z| > n_y^{100}\}$ .

But,  $\forall t$ , using energy boundedness

$$\|P_M u\|_{L_x^2} \lesssim \frac{1}{M}$$

so this mass must remain near  $N_{\min}$ . But this contradicts  $(e!)$  evaporation hypothesis.

spacetime bounds to prove this: FLIM, very low freq. Strichartz implied by frequency localization at all times,