Scattering & NLS critical NLS

\[ \begin{align*}
(\partial_t + D) u &= i u \nabla u, \\
0 &= u_0 \in H^1(\mathbb{R}^2) \\
\|u(t, x)\|_{L^6} &= \frac{1}{\sqrt{3}} \|u(\frac{x}{\lambda}, \frac{t}{\lambda^3})\|_{L^6} \quad \text{for} \lambda > 0
\end{align*} \]

\[ E[u] = \int_\mathbb{R} \left( |u|^2 + |u_t|^2 \right) dx, \quad \int |u|^6 dx \text{ conserved}. \]

Energy Critical

\[ u_t(t, x) = \frac{1}{\lambda^2} u(\frac{x}{\lambda}, \frac{t}{\lambda^3}) \quad \text{if} \ E[u] = E[0]. \]

Standard Tools:

- \text{Conserved quantities} \quad \Rightarrow \text{a-priori bounds on} \quad L^\infty_t L^6_x, L^{12}_t L^6_x, L^6_t L^\infty_x
- \text{Liouville theory, Strichartz space-time estimates, bootstraps.}
- \text{Analysis hinges on} \quad L^{10}_t L^{60}_x, x \in \mathbb{R}^3.
- L^{10}_x \text{ bound} \Rightarrow \text{Liouville, Scattering.}

\[ M(E) = \sup \left\{ \nu \in \mathbb{R} : \|u\|_{L^6_t L^\infty_x, x \in \mathbb{R}^3} \leq E \right\}. \]

Goal: Show \( M(E) < \infty \) \forall 0 < E < \infty.

\[ \text{Assume Not.} \quad \text{Then} \quad \exists E_{\text{crit}} < \infty \text{ s.t.} \quad M(E') < \infty \quad \forall \ E' < E_{\text{crit}}. \]

but \( M(E_{\text{crit}}) = +\infty \). \( E_{\text{crit}} \) is the minimal energy for \( L^6 \)-blowup solutions.

We may choose to have \( E[u] = E_{\text{crit}} \). Decompose \( u = v + w \) with \( E[v] = E_{\text{crit}} \) and \( E[w] \Rightarrow E_{\text{crit}} \) global.

We may choose to have \( E[v] = E_{\text{crit}} - \epsilon \) and \( E[w] \Rightarrow 0 \) \epsilon \to \infty \) global. Control interactions to get \( M(E_{\text{crit}}) < \infty \).

Direct Approach

The solution \( u(t) \) has its energy density:

- Frequency dispersed at some time \( t = t_0 \in \mathbb{R} \)
- Frequency non-dispersed \( \forall t \in \mathbb{R} \).

In the frequency non-dispersed.
Direct approach to proving $L^1$-bounds.

The solution $u(t)$ has its $H^1$ frequency density satisfying:

- dispersed at some time $t$
- non-dispersed/localized $\forall t \in I_0$ near a fixed dyadic scale $N$
- non-dispersed/localized $\forall t \in I_0$ near a moving dyadic scale $N(t)$.

Thus, there are three situations:

I. widely dispersed at some time.
II. localized near fixed scale. $N(t) \sim N_{\text{fixed}}$
III. localized near moving scale $N(t) \uparrow$.

I. widely dispersed $\Rightarrow L^1$-bounded.

- separate into $u_0$, $u_{\text{med}}$, $u_1$.
- $E[u_0], E[u_1] \to \infty \Rightarrow$ global if no interaction.
- Perturbation theory to control interaction exploiting Strichartz refinement.

II. frequency localized around
\[ \begin{align*} 
\{ \mathbf{M}_4, \mathbf{V}_4 \} & \in W^{1,1}(\mathbb{R}^4), \\
\text{energy} & = \int_{\mathbb{R}^4} \frac{1}{2} |\mathbf{u}|^2 + 1/4 |\mathbf{V}|^2 \, dx 
\end{align*} \]

\[ \begin{align*} 
\mathbf{v}_4(x, \mathbf{x}) & = \frac{1}{4\pi} \mathbf{V}(\frac{x}{4\pi}, \mathbf{x}) 
\end{align*} \]

Energy critical

- a priori bounds of energy type:
  \[ L^6_t L^6_x, L^{10}_t L^2_x \]

- Local well-posedness theory:
  hinges on \( L^{10}_t E(\mathbf{v}), \mathbf{x} \in \mathbb{R}^3 \) bounds.
  Strichartz estimates

- GWP, Scattering \iff global \( L^{10}_t L^2_x \) bound.

This is our goal.

\[ M(E) = \sup_{\mathbf{v}} \left\{ \begin{array}{c} E(\mathbf{v}) \\ \int_{\mathbb{R}^3} |\mathbf{v}(x)|^2 \, dx \leq 1 \\ \int_{\mathbb{R}^3} |\mathbf{v}(x)|^2 \, dx \geq \frac{1}{2} \end{array} \right\} \]

For \( C \) \rightarrow Assume \( E \in \text{Crit} \) s.t.
\[ M(E) < \infty \quad \forall \mathbf{E} \in \text{Crit} \text{ but } M(E_{\text{crit}}) = +\infty. \]

Setup

\[ E_{\text{crit}} \text{ is the minimal energy for an } L^2 \text{ blowing up solution.} \]

Perturbation Idea:
Suppose \( E = E_{\text{crit}} \).
\[ M(E - E_{\text{crit}}) < \infty \quad \forall \mathbf{E} > 0. \]

So prove \( M(E) < \infty \).

\[ \text{"Third order"} \]

\[ \text{scheme is} \]
\[ 3 \text{ layers} \]

\[ \text{we will use} \]
\[ \mathbf{M}_3 \leq M_3(\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_0) \leq \cdots \leq \mathbf{m}_0 = \mathbf{m}_0(E) \ll 1. \]

and prove
\[ M(E) \leq C(E, M_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3). \]

- Frequency truncated interaction Morawetz inequality:

\[ E \leq E = E_0 < 1 \text{ s.t.} \]
\[ \left\| \mathbf{P}_E \mathbf{v}(t) \right\|_{L^6_t L^2_x} \leq E \quad \text{then} \quad \left\| \mathbf{P}_E \mathbf{v}(t) \right\|_{L^{10}_t L^2_x} \leq N^{-3}. \]

Energy type bound
\[ L^{10}_t \]
Scattering & 3d Critical $N=3$

\[ E[u] = \int_{\mathbb{R}^3} \frac{1}{2} |u|^2 + |\nabla u|^2 \, dx \]

\[ u_1(t, x) = \frac{i}{\alpha} v_1^0(\frac{t}{\alpha}, \frac{x}{\alpha}) \]

\[ E[u_1] = E[u]. \]

- a priori bounds of energy type:
  \[ L^4 \rightarrow L^4, \quad L^6 \rightarrow L^6, \quad L^\infty \rightarrow L^\infty \]

- Local well-posedness theory:
  hinges on \( L^{10}_t L^2_x \) bounds.
  Strichartz estimates
  \[ \text{GWP, Scattering} \iff \text{global } L^{10}_t L^2_x \text{ bound.} \]

This is our goal.

\[ M(E) = \sup_{\Omega \subset E, \| E \|_{L^{10}_t L^2_x} = E} \| \nabla u \|_{L^{10}_t L^2_x} \]

For \( E \)

\[ \text{Assume } \exists E_{\text{crit}} \text{ s.t. } M(E') \leq M(\text{Ecrit}) \quad \forall \text{E} \leq E_{\text{crit}} \quad \text{but } M(E_{\text{crit}}) = +\infty. \]

\[ E_{\text{crit}} \text{ is the minimal energy and an } L^{10}_t L^2_x \text{ blowup solution.} \]

\[ \text{Perturbation Idea: } \text{Suppose } E = E_{\text{crit}}, \quad M(E') < \infty \quad \forall \text{ } M > 0. \]

So prove \( M(E) < \infty \).

\[ \text{For } M(E) \]

\[ \text{We will use} \]

\[ 0 < M_3 = M_3 (M_2, M_1, M_0) \ll M_2 = M_2 (M_1, M_0) \ll \cdots \ll M_0 = M_0 (E) \ll 1. \]

\[ \text{and prove} \]

\[ M(E) \leq C(E, M_0, M_1, M_2, M_3). \]

- Frequency truncated interaction Morawetz inequality:
  \[ \| P_{\neq E} v(t) \|_{L^6([0, T], L^2)} \leq C \quad \text{then} \]
  \[ \| P_{\neq E} v(t) \|_{L^6([0, T], L^2)} \leq N^{-3}. \]

\[ \text{Energy-type bound} \]

\[ \| \frac{1}{\alpha} v(t) \|_{L^6([0, T], L^2)} \leq N^{-3}. \]

\[ \text{Integrated norm in } t. \]

\[ \| \frac{1}{\alpha} \|_{L^6([0, T], L^2)} \leq N^{-3}. \]
If \( m = m(E) \) is small enough and \( \frac{N_{\text{hi}}}{N_{0}} = (a) \) is large enough

\[
\| P_{\leq c(a) N(t)} u(t) \|_{L^2} \geq \| P_{N_{\text{hi}}} u(t) \|_{L^2} \geq n_{100}
\]

Then
\[
\| u \|_{L^1_t L^\infty_x} \leq C(a).
\]

The idea here is that the solution can be decomposed using 
low freq. parts. Multilinear improvements to Strichartz

inequalities are leveraged using the lower bounds.

We choose \( m_0 = m_0(E) \) s.t. 
irref. dispersed \( \Rightarrow \) \( L^1 \) bound.

\[
N(t) := \sup_{N \leq \text{ dyadic}} \left\{ N : \| P_{\leq c(a) N} u(t) \|_{L^2} \leq n_{100} \right\}
\]

Thus
\[
\| P_{c(a) N(t)} u(t) \|_{L^2} \geq n_{100}
\]

and we may assume, for
\[
\frac{c(a)}{c(a_0)} \geq \frac{N_{\text{hi}}}{N_{0}} (a_0)
\]

\[
\| P_{\geq c(a) N(t)} u(t) \|_{L^2} \leq n_{0}^{100}
\]

since, otherwise, we are frequency dispersed.

Assumption: Energy is not \( m_0 \)-frequency dispersed.

\[
\forall t \in \mathbb{R}
\]

\[
\| P_{c(a) N(t)} u(t) \|_{L^2} \leq n_{0}^{100}
\]
How does $N(t)$ behave?

$u_0 \in A \implies u(t) \in A \quad \forall t \in I_+$. Thus $u(t) \in L^2$.

$$\| P_{N(t)} u(t) \|_{H^1} \leq C_0 N(t) \| u(t) \|_{L^2}$$

So

$$\frac{E}{C_0 \| u(t) \|_{L^2}} \leq N(t), \quad \forall t \in I_+.$$ 

Thus

$$N_{\min} = \inf_{t \in I_+} N(t) > 0 \quad \text{and} \quad \exists \; \epsilon_{\min} \in I_+ \quad \text{when}$$

$N(\epsilon_{\min}) = N_{\min}$. 

$L^2$ Cons. $\implies N(t)$ is lower bounded

$N(t) \geq N(\epsilon_{\min}) = N_{\min} > 0$.

Q: Does freq. localized Morawetz $\implies N(t)$ is upper bounded? Does $N(t)$ upper bounded $\implies$ global $L^1$ bound?

$$\| u \|_{L^1} \sim \| u \|_{H^1} \sim N^\frac{3}{2} \| P_{N(t)} u \|_{L^2}$$

$\implies N^\frac{1}{2} \| P_{N(t)} u \|_{H^1}$

Q: What is the role of the small frequency bound hypothesis in the proof of freq. localized interaction Morawetz estimate?
\[ T_n = [0, T_n] \]

Global \( L^\infty \) bounds

Local \( L^q \) bounds on these ranges.

\( H_X \)

High \( H_X \).

"High" localized \( L^q \) bound.

Global \( L^\infty \) bound.