Scattering for 3d critical NLS

IAS Fall 03

NLS $\begin{cases} (i\partial_t + \Delta) u = |u|^4 u \\ u(0) = u_0 \in H^1(\mathbb{R}^3) \end{cases}$

$E[u] = \int \frac{1}{2} |\nabla u|^2 + |u|^6 \, dx$, $\int |u|^2 dx$ conserved.

$u_\lambda(t,x) = \frac{1}{\lambda^{1/2}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$; $E[u_\lambda] = E[u]$.

$\boxed{\text{Energy Critical}}$

Standard Tools:

- Conserved quantities $\longrightarrow$ a-priori bounds on $L^\infty_t \dot{H}^1_x$, $L^\infty_t L^6_x$, $L^\infty_t L^2_x$
- LWP Theory, Strichartz spacetime estimates, bootstraps.
  analysis hinges on $L^{10}_{t \in [0,T], x \in \mathbb{R}^3}$.
- $L^{10}_{x,t}$ bound $\longrightarrow$ GWP, Scattering.

$$M(E) = \sup_{I_*, u : E[u] \le E} \left\{ \|u\|_{L^{10}_{t \in I_*, x \in \mathbb{R}^3}} \right\}.$$

<u>Goal</u>: Show $M(E) < \infty$ $\forall$ $0 < E < \infty$.

$\boxed{\text{Assume Not.}}$ Then $\exists$ $E_{crit} < \infty$ s.t. $M(E') < \infty$ $\forall$ $E' < E_{crit}$

but $M(E_{crit}) = +\infty$. E_{crit} is the minimal energy for L^{10}-blowup solution.

We may choose to have $E[u_0] = E_{crit}$. Decompose $u = v + w$ with

$E[v] = E_{crit} - n$, $E[w] = n$ so $t \mapsto v(t)$, $t \mapsto w(t)$ global

absent any v, w interactions. Control interactions to get $M(E_{crit}) < \infty$. (c!).

$\boxed{\text{Direct Approach}}$

The solution $u(t)$ has its conserved energy density

- frequency dispersed at some time $t = t_0 \in I_*$

or

- frequency non-dispersed $\forall$ $t \in I_*$.

In the frequency non-disper

Direct approach to proving L^{10}-bounds.

The solution $u(t)$ has its $\dot{H}^1$ frequency density satisfying:

- dispersed at some $t_0 \in I_*$
- non-dispersed/localized $\forall t \in I_*$ near a fixed dyadic scale N
- non-dispersed/localized $\forall t \in I_*$ near a moving dyadic scale $N(t)$.

Thus, there are three situations:

I. widely dispersed at some time.

II. localized near fixed scale. $N(t) \sim N_{fixed}$

III. localized near moving scale $N(t) \nearrow$.

I. widely dispersed $\Longrightarrow$ L^{10}-bounded.

separate into u_{lo}, u_{med}, u_{hi}.

$E[u_{lo}]$, $E[u_{hi}] > \eta \Longrightarrow$ global if no interaction.

perturbation theory to control interaction exploiting Strichartz refinement.

II. frequency localized around

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$\boxed{\text{energy critical}}$

- a priori bounds of energy type:

$$L^\infty_t L^6_x, \quad L^\infty_t \dot{H}^1_x, \quad L^\infty_t L^2_x$$

- Local well-posedness theory:

  hinges on $L^{10}_{t\in[0,T],\, x\in\mathbb{R}^3}$ bounds.

  Strichartz estimates

- GWP, Scattering $\Longleftarrow$ $\underbrace{\text{global } L^{10}_{t,x} \text{ bound}}_{\text{This is our goal.}}$.

$$M(E) = \sup_{I_*;\, u:\, E[u]\le E} \left\{ \|u\|_{L^{10}_{t\in I_*},\, \mathbb{R}^3_x} \right\}.$$

For (c!) $\longrightarrow$ Assume $\exists$ E_{crit} s.t. $M(E') < \infty$ $\forall$ $E' < E_{crit}$ but $M(E_{crit}) = +\infty$.

E_{crit} is the minimal energy for an L^{10} blowup solution.

<u>Perturbation Idea</u>: Suppose $E = E_{crit}$. $M(E-\eta) < \infty$ $\forall$ $\eta > 0$.

So prove $M(E) < \infty$.

setup

"Third order" scheme is 3 "layers" deep.

We will use

$$0 < \eta_3 = \eta_3(\eta_2, \eta_1, \eta_0) \ll \eta_2 = \eta_2(\eta_1, \eta_0) \ll \cdots \ll \eta_0 = \eta_0(E) \ll 1,$$

and prove

$$M(E) \le C(E, \eta_0, \eta_1, \eta_2, \eta_3).$$

- Frequency truncated interaction Morawetz inequality:

  $\exists$ $0 < \varepsilon = \varepsilon(E) \ll 1$ s.t.

  If $\|P_{\le N/2} u(t)\|_{L^\infty_t \dot{H}^1_x} \le \varepsilon$ then $\|P_{>N} u\|_{L^4_{t\in[0,T]},\, \mathbb{R}^3_x} \lesssim N^{-3}$.

  $[0,T]$

  energy-type bound L^∞_t

  integrated norm in t. L^4_t

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η-Frequency dispersed $\Rightarrow$ L^{10}-bound

If $\eta = \eta(E)$ small enough and $\frac{N_{hi}}{N_{lo}} = (\eta)$ is large enough

and $\|P_{\leq N_{lo}} u(t)\|_{\dot{H}^1} \geq \eta^{100}$, $\|P_{> N_{hi}} u(t)\|_{\dot{H}^1} \geq \eta^{100}$

then

$$\|u\|_{L^{10}_{t\in I_*}, \mathbb{R}^3_x} \leq c(\eta).$$

The idea here is that the solution can be decomposed using hi + low freq. parts. Multilinear improvements to Strichartz inequalities are leveraged using the lower bounds.

$\eta_0 = \eta_0(E)$ condition.

We choose $\eta_0 = \eta_0(E)$ s.t. η_0-Freq. dispersed $\Rightarrow$ L^{10}-bound.

$$N(t) := \sup_{N \text{ dyadic}} \left\{ N : \|P_{\leq c(\eta_0)N} u(t)\|_{\dot{H}^1} \leq \eta_0^{100} \right\}$$

Thus

$$\|P_{\leq c(\eta_0) N} u(t)\|_{\dot{H}^1} \geq \eta_0^{100}$$

and we may assume, for $\frac{C(\eta_0)}{c(\eta_0)} \gtrsim \frac{N_{hi}}{N_{lo}}(\eta_0)$,

$$\|P_{\geq C(\eta_0) N} u(t)\|_{\dot{H}^1} \leq \eta_0^{100}$$

since, otherwise, we are frequency dispersed.

<u>Assumption</u>: Energy is not η_0-frequency dispersed.

$\forall t \in I_*$

$$\|P_{\leq c(\eta_0) N(t)} u(t)\|_{\dot{H}^1} \leq \eta_0^{100} \quad \Longrightarrow \quad \|P_{c(\eta_0)N(t) < \cdot < C(\eta_0)N(t)} u(t)\|_{\dot{H}^1} \gtrsim E_{crit}.$$

$$\|P_{\geq C(\eta_0) N(t)} u(t)\|_{\dot{H}^1} \leq \eta_0^{100}.$$

How does $N(t)$ behave?

$u_0 \in \mathcal{S} \Longrightarrow u(t) \in \mathcal{S} \quad \forall t \in I_*$. Thus, $u(t) \in L^2$.

$$\| P_{C_0 N(t) < \cdot < C_0 N(t)} u(t) \|_{\dot{H}^1} \leq C_0 N(t) \|u_0\|_{L^2}$$

$$\Longrightarrow \quad \frac{E}{C_0 \|u_0\|_{L^2}} \leq N(t), \quad \forall t \in I_*.$$

Thus $\quad N_{min} = \inf_{t \in I_*} N(t) > 0$ and $\exists\, t_{min} \in I_*$ when

$N(t_{min}) = N_{min}$. $\quad \boxed{L^2 \text{ Conserv.} \Rightarrow N(t) \text{ is lower bounded}}$

$$N(t) \geq N(t_{min}) = N_{min} > 0.$$

Q: Does freq. localized Morawetz $\Longrightarrow$ $N(t)$ is upper bounded?

Does $N(t)$ upper bounded $\Longrightarrow$ global L^{10} bound?

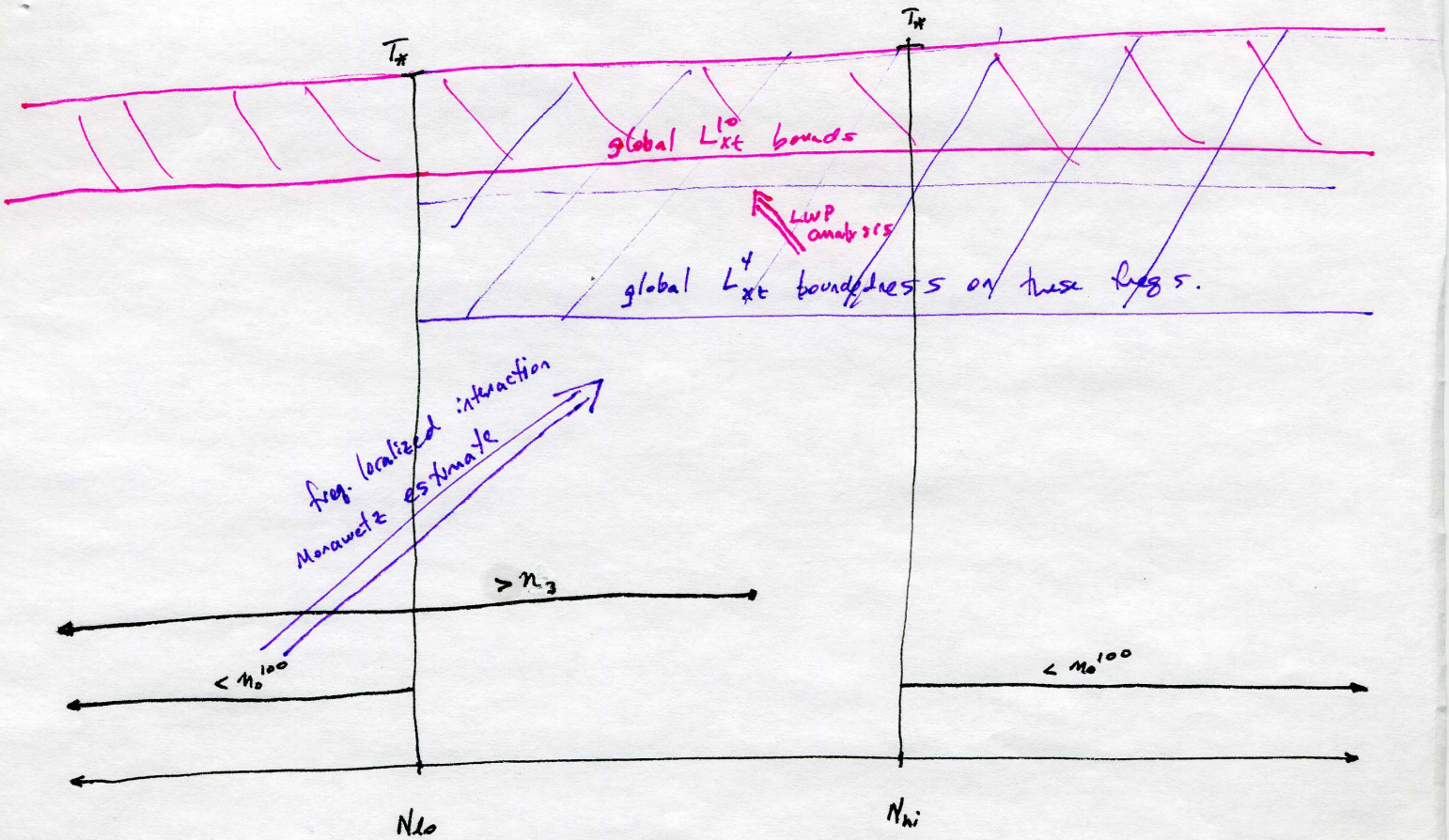
Strichartz bootstrap?

$$\|u\|_{L^{10}_x} \sim \|P_N u\|_{L^{10}_x} \sim N^{\frac{3}{2} - \frac{3}{10}} \|P_N u\|_{L^2_x}$$

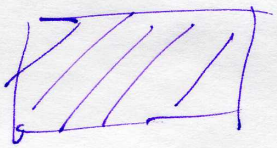
$$\sim N^{\frac{1}{2} - \frac{3}{10}} \|P_N u\|_{\dot{H}^1_x}$$

Q: What is the role of the small frequency bound hypothesis in proof of freq localized interaction Morawitz estimate?

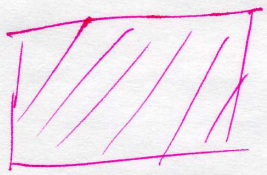
$I_* = [0, T_*]$.



$L^\infty_t \dot{H}^1_x$



"high" freq. localized L^4_{xt} bound.



global L^{10} bound