

GWP + Scattering for NLS_S(R³)

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Joint work [CKSTT]

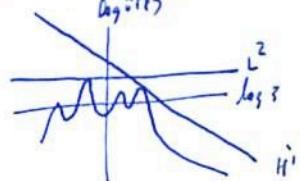
NLS $\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0) = u_0 \end{cases}$ $u: [-T_*, T^*] \times \mathbb{R}^3 \rightarrow \mathbb{C}$ $E[u] = \int_{\mathbb{R}^3} |\partial u|^2 + \frac{1}{3} |u|^6 dx$ $\int_{\mathbb{R}^3} |u|^2 dx$ conserved.

scaling $u_\lambda(t, x) = \frac{1}{\lambda} u(\frac{t}{\lambda}, \frac{x}{\lambda})$

Q1: Do \exists global-in-time classical solutions of NLS?

A1: Yes for small H^1 data.

A2: Yes for radial data.



Bourgain 1999

Grillakis 2000

- ↳ also obtained scattering + complete description of long-time behavior
- ↳ established robust strategy "induction on energy" towards the general theorem with other applications.

Following Bourgain 1999
and with a new Morawetz

Inequality + L^2 mass freezing property

A3: YES.

Theorem: $E[u_0] < \infty \Rightarrow \exists! u \in C(\mathbb{R}_t; H^1(\mathbb{R}^3_x)) \cap L^{\infty}_{tx}(\mathbb{R}_t \times \mathbb{R}^3_x)$ ② NLS

s.t. $\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t, x)|^{10} dx dt \leq C(E[u_0]).$

scattering, asymp. completeness, uniform regularity

may assume solvability
in $x \neq t$ if
constants depend only
upon E .

In this talk, I'll describe some features of our proof.

Two background results to start.

- ① Induction on Energy
- ② Localization control.
"minimal energy blowup solution is irreducible."
- ③ Freq. localized interaction Morawetz
- ④ No concentration via mass freezing.
(Comments on FLIM)

Strichartz Estimates

(g, r) admissible if $\frac{2}{g} + \frac{2}{r} = \frac{2}{2}$, $2 \leq g, r \leq \infty$.

$$\|v\|_{S^0(I \times \mathbb{R}^3)} = \left\{ \sum_{N \in 2\mathbb{Z}} \|P_N v\|_{L_t^{g'} L_x^{r'}(I \times \mathbb{R}^3)}^2 \right\}^{\frac{1}{2}}.$$

Littlewood-Paley onto
 $|N| \sim N$

$\|v\|_{S^0(I \times \mathbb{R}^3)}$

$I \subseteq \mathbb{R}_t$

e.g. (2, 6)

Keel-Tao endpoint

$$\|v\|_{S^k} = \|\nabla^k v\|_{S^0}.$$

Lemma: If $v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ solves $i' v_t + \Delta v = \sum_{j=1}^J F_j$ then

$\forall k \geq 0 \quad \forall t_0 \in I \quad \forall$ admissible (g_j, r_j)

$$\|v\|_{S^k} \lesssim \|v(t_0)\|_{H^k} + \sum_{j=1}^J \|F_j\|_{L_t^{g_j'} L_x^{r_j'}}.$$

Hölder dual.

e.g. $(2, 6/5)$

Perturbation Lemma • Suppose $\exists \tilde{v}: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ ④ $\tilde{NLS}_{\epsilon} \left\{ \begin{array}{l} i \tilde{v}_t + \Delta \tilde{v} = i \tilde{v}' \tilde{v} + \epsilon \\ \tilde{v}(t_0), \quad t_0 \in I. \end{array} \right.$

\exists bdd $\tilde{v} \otimes \tilde{NLS}$ $\|\tilde{v}\|_{L_t^{10} (I \times \mathbb{R}^3)} \leq M, \quad \|\tilde{v}'\|_{L_t^{\infty} H_x^1} \leq E.$

• Suppose

$$\|\nabla \epsilon\|_{L_t^2 L_x^{\infty}} \leq \epsilon$$

?

$$\exists v(t_0) \text{ s.t. } \|\tilde{v}(t_0) - v(t_0)\|_{H^1} \leq E' \text{ and}$$

$$\|e^{i(t-t_0)\Delta} (\tilde{v}(t_0) - v(t_0))\|_{L^{\infty}(I \times \mathbb{R}^3)} \leq \epsilon$$

$\forall \delta \leq \epsilon < \epsilon_1(M, E, E')$ small enough.

Then $\exists v: I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ ④ NLS w. data $v(t_0)$ and

$$\|v - \tilde{v}\|_{L^{\infty}(I \times \mathbb{R}^3)} \leq C(M, E) \epsilon$$

$$\|v - \tilde{v}\|_{S^1(I \times \mathbb{R}^3)} \leq C(M, E) (E' + \epsilon).$$

\exists bdd $v \otimes NLS$.

$$\|v\|_{S^1(I \times \mathbb{R}^3)} \leq C(M, E) (E + E' + \epsilon).$$

① Induction on Energy

$$\forall E \geq 0 \text{ set } M(E) = \sup_{\substack{I^* \\ 0 \in I^* \subset R_t}} \left\{ \|v\|_{L^{\infty}_{t,x}(I^* \times \mathbb{R}^3)} \right\}$$

I^*, v

$v \text{ (NLS on } I^*, E[v] \leq E)$

Contradiction Hypothesis $M(E)$ is not always finite.

$\{E : M(E) < \infty\}$ is open, connected contains $0 \Rightarrow \exists 0 < E_{\text{crit}} < \infty$ s.t.

$M(E) < \infty \quad \forall E < E_{\text{crit}}$

$M(E_{\text{crit}}) = +\infty$

↑
minimal energy
for blow-up.

Induction on Energy Hypothesis

$$E[V(t_0)] \leq E_{\text{crit}} - n, n > 0 \Rightarrow \exists v : \mathbb{R}_t \times \mathbb{R}^3_x \rightarrow \mathbb{C} \text{ (NLS w. data } V(t_0) \text{ s.t.)}$$

$\|v\|_{L^{\infty}_{t,x}(\mathbb{R}_t \times \mathbb{R}^3_x)} \leq M(E_{\text{crit}} - n) = C(n)$

$\|v\|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^3)} \leq C(n).$

Strategy Comments

The proof invokes the induction hypothesis at 6 in "levels":

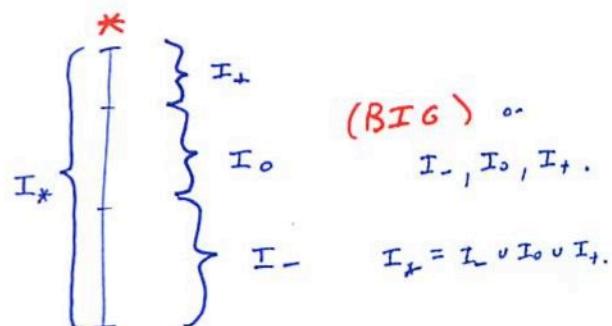
energy concentration \downarrow energy localization \downarrow low(FIM) freq. \downarrow high(FIM) freq. \downarrow highest freq.
 $m_0 > m_1 > m_2 > m_3 > m_4 > m_5 > m_6 > 0$

$\underbrace{m_0 > m_1 > m_2 > m_3}_{\text{makes some bootstrap work}}$ $\underbrace{m_4 > m_5 > m_6}_{\text{freq. localization}}$ $\underbrace{m_0 > m_1 > m_2 > m_3}_{\text{spatial localization}}$ $\underbrace{m_4 > m_5 > m_6}_{\text{non-concentration}}$ $\underbrace{m_0 > m_1 > m_2 > m_3}_{\text{quantifies } M(E_{\text{crit}}) = \infty}$

We assume $E[v_0] = E_{\text{crit}}$ and

$$\|v\|_{L^{\infty}(I^* \times \mathbb{R}^3)} \geq \frac{1}{m_6}.$$

ridiculously large



Notation:

- E_{crit} dependence is implicit
- $0 < c(m_j) \ll 1 \ll C(n_j)$, etc.

No control on sizes
of $I-, I0, I+$.

② Localization Control

Freq. delocalized \Rightarrow spacetime bounded.

Let $n > 0$. Suppose $\exists N_{\text{lo}} \in \mathbb{Z}^{\mathbb{Z}}$ and $\exists t_0 \in \mathbb{I}_{\text{lo}}$ s.t.

$$\| P_{\leq N_{\text{lo}}} u(t_0) \|_{H^1} \gtrsim n^{100}, \quad \| P_{\geq K(n) N_{\text{lo}}} u(t_0) \|_{H^1} \gtrsim n^{100}$$

with $K(n)$ large enough. Then

$$\| u \|_{L_{t,x}^{10}(\mathbb{I}_{\text{lo}} \times \mathbb{R}^3)} \leq C(n).$$

Idea of proof. At time t_0 , decompose $u(t_0)$ into $v(t_0)$ supported on low freqs and $w(t_0)$ supported on high freqs. Both $v(t_0)$ and $w(t_0)$ have non-trivial energy content so both have energy $< E_{\text{crit}} - c(n)$. Thus $v(t_0) \mapsto v$, $w(t_0) \mapsto w$ (5) NLS exist w. bounds by induction hypothesis. wide freq. separation (obtained by pigeonhole using large $K(n)$) via Strichartz refinements \rightarrow interaction control. $\tilde{v} = v + w$, etc.

We apply freq. delocalized statement for $m = m_0, \dots, m_5$ and obtain L^{10} bounds violating (BIG) so we may assume

Frequency Localization $\forall t \in \mathbb{I}_{\text{lo}} \exists N(t) \in \mathbb{Z}^{\mathbb{Z}}$ s.t. for $\overbrace{m_0 \leq m \leq m_5}$

$$\| P_{\leq C(m) N(t)} u(t) \|_{H^1} \leq n^{10}, \quad \| P_{C(m) N(t)} u(t) \|_{H^1} \gtrsim 1, \quad \| P_{\geq C(m) N(t)} u(t) \|_{H^1} \leq n^{10}.$$

Similar arguments establish spatial localization properties.

Spatial localization

Assume $u \in NLS$ and is (BIG).

So, assume u is a minimal energy blowup solution of NLS.

$\forall t \in I_0 :$

- $\|u(t)\|_{L_x^6} \gtrsim n_1$ (potential energy lower bounded)

- $\exists x(t)$ about which $u(t)$ concentrates

$$\int_{|x-x(t)| \leq \frac{C(n_1)}{N(t)} R} |u(t, x)|^p dx \gtrsim c(n_1) N(t)^{\frac{p}{2}-3} \quad p=6 \text{ special}$$

Heisenberg

$$\int_{|x-x(t)| \leq \frac{C(n_1)}{N(t)}} |\nabla u(t, x)|^2 dx \gtrsim c(n_1).$$

- Kinetic energy is localized

$$\int_{|x-x(t)| > \frac{C(n_1)}{n_2 N(t)}} |\nabla u(t, x)|^2 dx \lesssim n_2.$$

\Rightarrow Reverse Sobolev Inequality $\forall t_0 \in I, \forall x_0 \in \mathbb{R}^3, \forall R \geq 0$

$$\int_{B(x_0, R)} |\nabla v(t_0, x)|^2 dx \lesssim n_2 + C(n_1, n_2) \int_{B(x_0, C(n_1, n_2) R)} |v(t_0, x)|^6 dx.$$

Remark

Localization control statements follow from induction hypothesis + perturbation lemma using various results to control interaction.

These statements do not ^{heavily} rely on defocusing or Morawetz estimates so this reduction may be relevant in other problems.

pseudocovariant identity is used in proving localization.

Two solution scenarios remain which could be (BIG). (6)

- $0 < N_{\min} \leq N(t) \leq N_{\max} < \infty$

"pseudosoliton"

- $0 < N_{\min} \leq N(t)$, unbounded.

"energy concentration"

why?

→ energy must leak from low freqs.

By freq. localization assumption,

$$\|P_{c(M_0)} N(t)\|_{H^1} \leq C(M_0) \|N(t)\|_{L_t^\infty L_x^2},$$

$$\Rightarrow N(t) \geq c(M_0) \|u\|_{L_t^\infty L_x^2}^{-1} \quad \forall t \in I_0.$$

$$N(t) \geq \inf_{t \in I_0} N(t) := N_{\min} > 0.$$

$\exists t_{\min} \in I_0$ when
 $N(t_{\min}) = N_{\min}$.

③ FLIM

Proposition Let $u: I_* \times \mathbb{R}^3 \rightarrow \mathbb{C}$ (5) NLS satisfy freq. + spatial localization (so u is any remaining (BIG) candidate). $\forall N_* < c(M_3) N_{\min}$ [localization minimal energy blowup solution]

$$\int_{I_*} \int_{\mathbb{R}_y^3} |P_{\geq N_*} u(t, y)|^4 dy dt \lesssim n_*^{-3} N_*^{-3}.$$

Compare with

• Independent of I_*

• N_*^{-3} mandated by scaling.

24 Feb. talk (Interaction Morawetz)
 Fields Working group talks (Spring '03)

$$\int_{I_*} \int_{\mathbb{R}^3} |u|^4 dy dt \lesssim \|u\|_{L_t^\infty L_x^2}^3 \|u\|_{L_t^{\infty} H_x^1} \|u\|_{L_t^{\infty} H_x^1}^3$$

[is morally follows:
 should compare
 with 3-6 localized
 Morawetz as
 well.]

- not an a priori estimate,
 and specific to quintic case
 (false for linear solution)
- small constant n_* used in bootstrap

• valid & defocusing problems

• a priori estimate

• scaling blows up L^2 on RHS, but small freqs...

(Return to this later)

(7)

FLIM kills pseudosoliton

Sketch

- By localization

$$\int_{\mathbb{R}^3} |P_{>c(m_3)N_{\min}} v(t, x)|^4 dx \underset{\substack{\uparrow \\ \text{freq.} \\ \text{local.}}}{\sim} \int |v(t, x)|^4 dx \underset{\substack{\uparrow \\ \text{L}^4 \text{ concentration}}}{\gtrsim} c(m_3) N(t)^{-1}$$

$$N_x^{-3} \geq \int_{I_0} \dots \geq \int_{I_0} N(t)^{-1} dt \geq I_0 N_{\max}^{-1}.$$

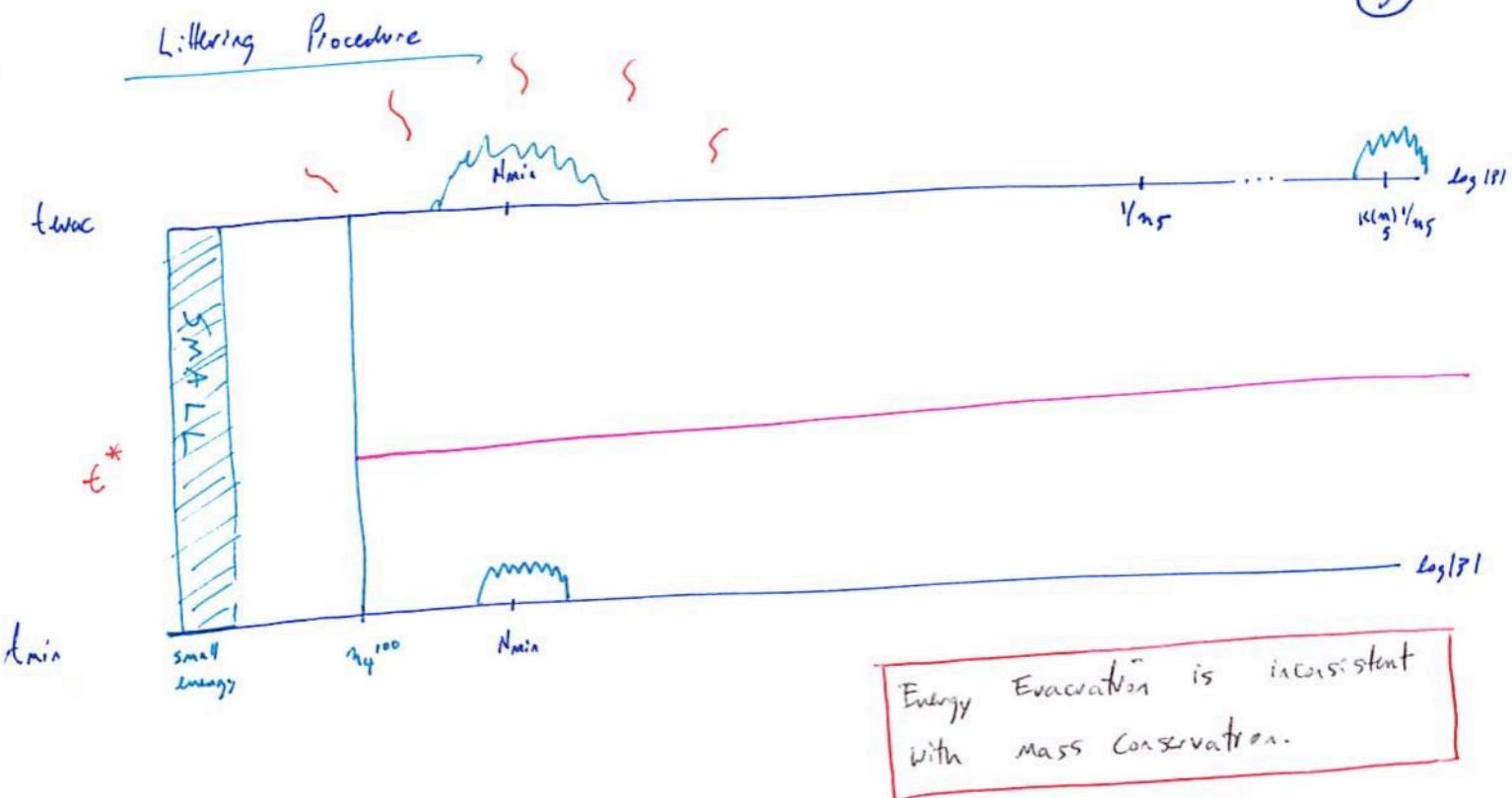


$|I_0|$ bounded.

- L_x^4 control + Freq. localization $\xrightarrow{\text{(Burstein)}}$ L_x^{10} control.

- Local-in-time theory $\Rightarrow L^{10}(I_0 \times \mathbb{R}^3)$ control violating (BIG).

⑨



$$P_{>m_y^{100}} v(t) := v_{hi}(t).$$

$$L(t) = \int |v_{hi}(t, x)|^2 dx.$$

$$L(t_{min}) \geq c_0.$$

Suppose we introduce $t_* \in (t_{min}, t_{wac}]$ s.t.

$$\inf_{t_{min} \leq t \leq t_*} L(t) \geq \frac{1}{2} c_0$$

and we prove that

$$\inf_{t_{min} \leq t \leq t_*} L(t) \geq \frac{3}{4} c_0 \implies t_* = t_{wac}.$$

But $\forall t$ using energy boundedness

$$\|P_{>M} v(t)\|_{L^2_x} \leq \frac{1}{M} \quad \text{so mass remains near } N_{min}.$$

(C!) evacuation hypothesis

Time? Yes \rightarrow IAS talk p 11.
No? stay here.

(10)

FLIM Remarks

We apply the interaction Morawetz argument to $P_{\geq N_k}(\text{NLS})$, with spatially localized weight function a . New terms occur since

- $P_{\geq N_k}(\text{NLS})$ is not L^2 -mass or momentum conserving. \rightarrow flux terms.
- derivatives fall on spatial cutoffs. \rightarrow cutoff terms.

$$P_{hi} = P_{\geq N_k}$$

key steps

$$\left\{ \begin{array}{l} i \partial_t u_{hi} + \Delta u_{hi} = P_{hi} (|u_{hi}|^4 u_{hi}) \\ u_{hi}(t_0) = P_{hi} v(t_0). \end{array} \right.$$

For $y \in \mathbb{R}^3$ fixed define $M^y(t) = \int_{\mathbb{R}^3} a_j 2 \operatorname{Im} (\bar{v}_{hi} \partial_j v_{hi}) dx$ $\curvearrowright L^2 + \text{energy bounded.}$

where $a(x) = |x-y| \chi\left(\frac{|x-y|}{R}\right)$, $\chi \sim \chi_{[0,1]}$ smooth bump.

Define

$$M^{\text{interact}}(t) = \int_{\mathbb{R}^3} |v_{hi}(t,y)|^2 M^y(t) dy.$$

\curvearrowright bounded.

$$\partial_t M^{\text{interact}} = \dots, \text{ then } \int_0^t \text{ and rearrange}$$

Unravel algebra to obtain

(11)

$$\forall R \geq 1$$

$$\int_{I_0} \int_{\mathbb{R}^3} |v_{hi}|^4 dx dt + \int_{I_0} \int_{|x-y| \leq 2R} \frac{|v_{hi}(t, y)|^2 |v_{hi}(t, x)|^6}{|x-y|}$$

~~(*)~~

$$\lesssim \|v_{hi}\|_{L_t^\infty L_x^2}^3 \|v_{hi}\|_{L_t^\infty H_x^1} + \text{(nicely bounded)}$$

$$+ O \left(\int_{I_0} \int_{\mathbb{R}_y^3} \int_{\mathbb{R}_x^3} |v_{hi}(t, y)|^2 \Delta \left(\chi \left(\frac{|x-y|}{R} \right) \right) |x-y| |v_{hi}(t, x)|^2 dx dy dt \right)$$

$\frac{1}{R^4} \nabla \left(\frac{|x-y|}{R} \right) R$

$$+ O \left(\int_{I_0} \int_{\mathbb{R}_y^3} \int_{\mathbb{R}_x^3} |v_{hi}(t, y)|^2 |x-y| D^2 \left(\chi \left(\frac{|x-y|}{R} \right) \right) \left[|\nabla v_{hi}(t, x)|^2 + |v_{hi}(t, x)|^6 \right] dx dy dt \right)$$

$\frac{1}{R^2} \nabla \left(\frac{|x-y|}{R} \right)$

localizes to

$$B_R \cdot B_y$$

Reverse Sobolev

Now, drastically decompose B_{2R} in annuli:

$$(*) + B_1 + A_2 + \dots + A_{2R} \leq (\text{OK}) + A_{2R}.$$

Replacing R by $R+1$ yields

$$(*) + B_1 + A_2 + \dots + A_{2R} + A_{2(R+1)} \leq (\text{OK}) + A_{2R} + A_{2(R+1)}$$

(4) Energy nonconcentration

Proposition $\forall t \in I_0, N(t) \leq C(n_5) N_{\min}.$



All solutions of NLS satisfy

$$\|u\|_{L^{\infty}(I_0 \times \mathbb{R}^3)} \leq C(n_0, \dots, n_5),$$

c! (BIG)

so E_{cont} d/n exist.

nontrivial mass near $181 \sim 1$.

Proof sketch

At time t_{\min} \exists nontrivial energy near $181 \sim 1$, after rescaling $N_{\min} = 1$.

Suppose proposition fails and (BIG) holds. Then $\exists t_{\text{vac}} \in I_0$

when $N(t_{\text{vac}}) > k(n_5)^{1/n_5}$, nontrivial energy near $N(t_{\text{vac}})$ at time.

Since (BIG) holds, we have the evacuation hypothesis

"all" energy evacuates
low frequencies.

$$\|P_{\leq \frac{1}{n_5}} u(t_{\text{vac}})\|_{H^1} \leq n_5^{100}.$$

Energy Evacuation is inconsistent with mass conservation