

Variations on a theme by Bourget

joint work
[CKSTT]

J. Colliander

semilinear NLS

$$\text{GNLS} \quad \begin{cases} i\partial_t u + \Delta u = F(|u|^2)u \\ u(0, x) = f(x) \end{cases} \quad x \in \mathbb{R}^d \quad u: [-T_0, T^*] \times \mathbb{R}^d \rightarrow \mathbb{C}$$

Q: As $t \rightarrow \infty$ evolves, what happens?

$F' \geq 0$ defocus: decay / scattering

$F' \leq 0$ focus: solitons / blowup

Goal: Systematize and extend a collection of identities and inequalities which underpin fundamental results into long time behavior of GNLS solutions in specific settings.

Conservation Structure

$$\alpha[u] = \int |u(t, x)|^2 dx \quad \text{conserved.}$$

$$H[u] = \int |\nabla u|^2 + F(|u|^2) dx \quad \text{conserved.}$$

$$\text{mass density} \quad T_{00} := |u|^2$$

$$\text{mass current / momentum} \quad T_{0j} := T_{j0} = 2 \operatorname{Im}(u \bar{u}_j)$$

$$\text{momentum current} \quad T_{jk} := T_{kj} = 4 \operatorname{Re}(u_k \bar{u}_j) - \delta_{jk} \Delta(|u|^2) + 2 \delta_{jk} G(|u|^2)$$

$$\text{where } G(z) := 2F'(z) - F(z) \sim F(z).$$

$$\partial_t T_{00} = \partial_j T_{0j}$$

$$\partial_t T_{k0} = \partial_j T_{kj}$$

$$\partial_t^2 T_{00} = \partial_j \partial_k T_{kj}$$

space-time localizations of mass density and their evolution.

$$V_0^a(t) := \int_{\mathbb{R}^d} a(t, x) |u|^2(t, x) dx = \int_{\mathbb{R}^d} a(t, x) T_{00}(t, x) dx.$$

Variel
potential associated
to a .

$$M_0^a(t) := \partial_t V_0^a = \int_{\mathbb{R}^d} a_t T_{00} + a \partial_j T_{0j} dx = \int_{\mathbb{R}^d} a_t T_{00} - a_j T_{0j} dx.$$

parameter
action

conservation laws + I.P.P. \Rightarrow

$$\begin{aligned} \partial_t^2 V_0^a(t) &= \partial_t M_0^a = \int_{\mathbb{R}^d} (\alpha_{tt} - \Delta a) |u|^2 dx - 4 \int_{\mathbb{R}^d} a_{tj} \operatorname{Im}(u \bar{u}_j) dx \\ &\quad + 4 \int_{\mathbb{R}^d} a_{jk} \operatorname{Re}(u_k \bar{u}_j) dx + 2 \int_{\mathbb{R}^d} \Delta a G(|u|^2) dx. \end{aligned}$$

Example 1 (Variance identity)

Vlasov-Petrishchev-Tatarski 1971
Gasser 1977

Set $\begin{cases} a(t, x) = |x|^2 = x_j x_j \\ a_{jk} = 2\delta_{jk}, \quad \Delta a = 2d. \\ \Delta \Delta a = 0 \end{cases} \rightarrow \text{identity.}$

$$\begin{aligned} \partial_t^2 V_0^a(t) &= +4 \int 2\delta_{jk} \operatorname{Re}(u_k \bar{u}_j) dx + 4d \int G(|u|^2) dx \\ &\sim |\nabla u|^2 \\ &= 8H + (\leq 0) \\ &\quad \text{assuming focusing sign} \end{aligned}$$

$$\leq 8H < 0$$

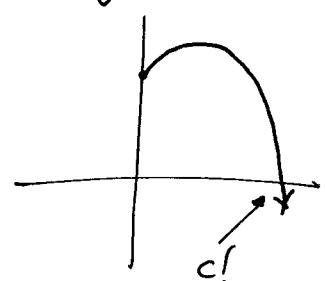
certain data in
critical & supercritical case



$$\boxed{\partial_t^2 \int |x|^2 |u|^2 dx \leq 8H < 0}$$

\leadsto blowup.

$$\int |x|^2 |u|^2$$



Example 2 (Lin-Strauss Morawetz-type identity)

$$\text{Set } a(t, x) = |x|$$

$$a_j = \frac{x_j}{|x|}$$

$$a_{jk} = \left[\delta_{jk} - \frac{x_j x_k}{|x|^2} \right] \frac{1}{|x|}$$

$$\Delta a = \frac{d-1}{|x|} \quad \longrightarrow \text{identity}$$

$$\partial_t M_0^a(t) = - (d-1) \int \Delta \left(\frac{1}{|x|} \right) |u|^2 dx + 4 \int \underbrace{\left[\delta_{jk} - \frac{x_j x_k}{|x|^2} \right] \frac{1}{|x|}}_{\left[|\nabla u|^2 - |\partial_r u|^2 \right] \frac{1}{|x|}} \operatorname{Re}(u_n \bar{u}_j) dx \\ + 2(d-1) \int \frac{G(|u|^2)}{|x|} dx \geq 0.$$

Take $d=3$.

$$\text{On } \mathbb{R}^3 \quad -\Delta \left(\frac{1}{|x|} \right) = 4\pi \delta_0 \quad \Rightarrow \text{ we get}$$

$$\partial_t M_0^a = 8\pi |u(t, \cdot)|^2 + 4 \int_{\mathbb{R}^3} \frac{|\nabla_0 u|^2 + G(|u|^2)}{|x|} dx \\ \rightsquigarrow M_0^a \text{ is } \nearrow$$

Recall

$$M_0^a(t) = - \int_{\mathbb{R}^d} \frac{x_j}{|x|} 2 \operatorname{Im}(u \partial_j) dx; \quad |M_0^a(t)| \lesssim \|u\|_{H^{\frac{1}{2}}}^2 \text{ bounded.}$$

$$\rightarrow \int_0^T |u(t, \cdot)|^2 dx + 4 \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla_0 u|^2 + G(|u|^2)}{|x|} dx dt \leq 2 \sup_{t \in [0, T]} \|u(t)\|_{H^{\frac{1}{2}}}^2$$

"reward"
emerges as a useful
term below.

$\frac{1}{|x|}$ weight introduces technical issues

Example 3 (Bougash, Grillakis' spatial truncation of Lin-Strauss identity)

$$n \sim \chi_{BR}$$

$a(t, x) = |x| n \rightarrow \text{identity}, \text{ collapse to } x \in \mathbb{R}^3, I \text{ time interval}$

$$\int_I |u(t, 0)|^2 dt + \int_I \int_{|x| < R/2} \frac{|Du|^2 + G(|u|^2)}{|x|} dx dt \lesssim R^{-1} |I| E + R E$$

choose $R \sim |I|^{\frac{1}{2}}$

$$\lesssim E |I|^{\frac{1}{2}}.$$

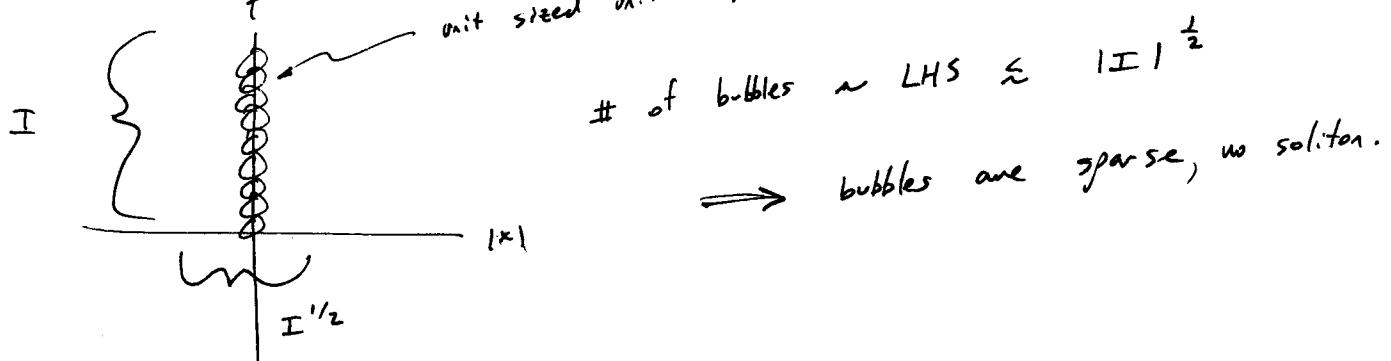
quartic, \mathbb{R}^3 ,
defocusing

$$\int_I \int_{|x| < \frac{1}{2}|I|^{1/2}} \frac{|u|^6}{|x|} dx dt \lesssim |I|^{\frac{1}{2}} \sup_{t \in I} \|u(t)\|_{H^1}^2.$$

Application: Energetic / combinatorial / radial / clever arguments in B, G

include ruling out one solution scenario

unit sized unit height "bubbles" in solution.



Example 4 (Nakanishi spacetime Maranete identity)

Let $\lambda := \sqrt{t^2 + |x|^2}$ $x \in \mathbb{R}^n$.

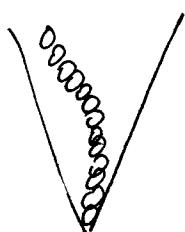
$u(t, x) = \lambda \rightarrow \partial_t^2 V_0^\alpha$ identity, explicit calculations, homogeneity,

algebra \rightsquigarrow

$$\int_1^T \int \frac{|xu + 2xt \nabla u|^2}{\lambda^3} + \frac{(\partial_t u)^2 |x|^2}{\lambda^3} + \frac{G(|u|^2) t^2}{\lambda^3} dx dt \leq C(H, Q)$$

we rule out a solution scenario.

Suppose we had a "soliton" in the light cone $|x| \leq |t|$.



Then $0 < G(|u|^2) \sim O(1)$ along a $O(1)$ wide spacetime channel. On the channel $\frac{\tau^2}{\lambda^3} \sim \frac{1}{\epsilon}$ so

LHS $\sim \log T$ RHS bad.

\Rightarrow No such "soliton channel" can exist.

Remark: \exists similar ideas in work of Ogawa-Tsutsumi, Merle, Nawa who use "localized viriel-type identities" to study the NLS blowup without assuming finite variance initial data.

Example 5 (3D interaction Morawetz; $L^4(\mathbb{R}_t \times \mathbb{R}^3_x)$)

For $a(t, x) = |x|$ w. $x \in \mathbb{R}^3$ we encounter in [L-S]:

$$M_0[u](t) = \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot 2 \operatorname{Im}(v \nabla \bar{v})(t, x) dx; |M_0| \lesssim \|u(t)\|_{H_x^{\frac{1}{2}}}^2.$$

average outward
radial mass current

defocusing: $\partial_t M_0 \geq 0 \Rightarrow$ mass is repelled from spatial origin.

We can center the whole argument at $y \in \mathbb{R}$:

$$M_y[u](t) = \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|} \cdot 2 \operatorname{Im}(v \nabla \bar{v})(t, x) dx.$$

defocusing: mass is repelled from $y \in \mathbb{R}^3 \forall y$.

Thus, we might hope to recover a translation invariant spacetime estimate by averaging the L-S identity w.r.t. the center y .

Define the Morawetz interaction potential

$$M[u](t) = \int_{\mathbb{R}^3_y} |v(t, y)|^2 M_y(t) dy; |M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{H_x^{\frac{1}{2}}}^2$$

(sub) energy bounded

\uparrow
 mass density
 averaged

\uparrow
 outward mass
 from y

We calculate:

$$\partial_t M(t) = \int_{R_y^3} |u(t, y)|^2 \underbrace{\partial_t M_y(t)}_{\text{[L-S] identity}} dy + \int_{R_y^3} \underbrace{\partial_t |u(t, y)|^2}_{\text{mass current}} M_y(t) dy$$

$$= C \int_{R_y^3} |u(y)|^4 dy \stackrel{\text{I}}{=} + \int_{R_y^3} \int_{R_x^3} \frac{2 |u(r)|^2 \left(\frac{1}{t_0} |u(x)|^2 + G(|u|^2)(x) \right)}{|x - r|} dx dy \stackrel{\text{II}}{=} \\ + \int_{R^3} \nabla \cdot 2 \operatorname{Im}(u G) \nabla \bar{v}(r) M_y(t) dy \stackrel{\text{IV}}{=} .$$

One can show that $\text{IV} \geq -\text{II}$. The other terms are ≥ 0 (defocusing) so $\partial_t M \geq 0$.

Integrating in t , we obtain

$$\int_0^T \int_{R_y^3} |u|^4 dy dt \leq \|u\|_{L^2}^2 \sup_{t \in [0, T]} \|u(t)\|_{H^{\frac{1}{2}}}^2 .$$

conserved. L^2 energy bounded

Remark

- This is essentially an $H^{\frac{1}{2}}$ -admissible Strichartz estimate proved via IBP, valid for nonlinear defocusing problems, without relying on explicit representation of linear solution operator.

Tensor Product approach to proving global spacetime bounds w/o weights

(This idea emerged from an observation of Andrew Hassell.)

Let Δ_n denote the Laplacian on \mathbb{R}^n . For $j=1, \dots, J$, consider

$$\begin{cases} i\partial_t v_j + \Delta_d v_j = F'(|v_j|^2) v_j \\ v_j(0) = \phi_j, \quad x_j \in \mathbb{R}^d. \end{cases}$$

Form $\bar{W}(t, x_1, x_2, \dots, x_J) = \prod_{j=1}^J v_j(t, x_j)$.

Then

$$(i\partial_t \bar{W} + \Delta_{Jd} \bar{W}) = \left[\sum_{k=1}^J F'(|v_k|^2) \right] \bar{W}$$

(Revisit) (3d interaction Morawetz; $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ estimate)

Example 5 (3d interaction Morawetz; $L^4(\mathbb{R}_t \times \mathbb{R}_x^3)$ estimate)
 $x_1 \in \mathbb{R}^3, \quad x_2 \in \mathbb{R}^3, \quad (x_1, x_2) \in \mathbb{R}^6$.

$$\bar{W}(t, x_1, x_2) = v_1(t, x_1) v_2(t, x_2).$$

$$\bar{x} = \frac{1}{2}(x_1 + x_2) \in \mathbb{R}^3 \quad \text{center of mass}$$

$$y = (x_1 - \bar{x}, x_2 - \bar{x}) \in \mathbb{R}^6$$

(\bar{x}, y) are coordinates on \mathbb{R}^6 , as are (x_1, x_2) .

Set $a(x) = |x| \rightarrow (\text{identity}), \dots$

$$\partial_t M_a^4(t) = \int_{\mathbb{R}^6} -\Delta \Delta (|x|) |v_1(x_1) v_2(x_2)|^2 dx + (\geq 0).$$

$$= -\frac{8\pi}{2} \int_{\mathbb{R}^3} |v_1(\bar{x}) v_2(\bar{x})|^2 d^3x + (\geq 0)$$

take $\phi_1 = \phi_2 \implies v_1 = v_2 = u$

$\sim H^4$ Strichartz estimate

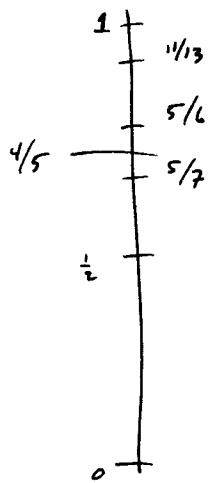
\implies

$$\int_0^T \int_{\mathbb{R}^3} |u|^4 dx dt \lesssim \|u(t)\|_2^2 \sup_{t \in [0, T]} \|u(t)\|_{H^{\frac{1}{2}}}^2$$

No weight!
simplifies
functionality proof

Application

(cubic defocusing NLS on \mathbb{R}^3)



LWP
 $[cw] \quad s \geq \frac{1}{2}$
 $[gv]$

GWP
 $s \geq 1$

Scatter

$[ev]: s \geq 1$

$[B]: s > \frac{11}{13}$

$[B]: s > \frac{5}{7}$ (radial)

$[cksst]: s > \frac{5}{6}$

$[B]: s > \frac{5}{7}$ (radial)

$s > \frac{4}{5}$

$s > \frac{4}{5}$

Example 6 (1d interaction Morawetz; global $L^8(R_t \times R_x)$ estimate.)

$$R^4 = R \times R \times R \times R = \{(x_1, x_2, x_3, x_4) , x_i \in R, i=1, 2, 3, 4\}$$

"space of quadruples of particles"

Given a quadruplet, we form

$$\text{center of mass } \bar{x} = \frac{x_1 + x_2 + x_3 + x_4}{4} \in R$$

$$y = (x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x}, x_4 - \bar{x}) \in R^3.$$

$(\bar{x}, y) \leftrightarrow (x_1, x_2, x_3, x_4)$ as coordinates on R^4 .

Set $a(t, x) = |x| \rightarrow \text{identity}$

$$\partial_t M_0^a(t) = \int_{R^4} (-\Delta a) |u_1(x_1) u_2(x_2) u_3(x_3) u_4(x_4)|^2 dx + (\geq 0)$$

$$= \int_{R^4} |\delta(y)|^2 dy + (\geq 0)$$

$$= \int_R |u_1(\bar{x}) u_2(\bar{x}) u_3(\bar{x}) u_4(\bar{x})|^2 d\bar{x} + (\geq 0)$$

Take $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi, u_i^- = u \rightarrow$

$$\boxed{\int_0^T \int |u|^8 dx dt \leq \|u\|_{L^2}^6 \sup_{t \in [0, T]} \|u\|_{H^{\frac{1}{2}}}^2}.$$

$\sim H^{\frac{1}{2}}$ -Strichartz estimate

Now GWP / Scattering results?
 [Tzirakis], [Nakanishi].