

# Expository Talk

Fields talk 6 May 2003

①

## Mass critical NLS Blowup.

- 0. mass critical NLS
- I. Overview/pictures
- II. Lemmas  $\Rightarrow$  mass concentration
- III. Squares lemma
- IV. TUBES lemma

This talk will survey some ideas which emerge from

"Refinements of Strichartz..." in IMRN 1998 by J. Bourgain.

$$\text{NLS}_3(\mathbb{R}^2) \quad \begin{cases} i\partial_t u + \Delta u + \lambda |u|^2 u = 0 & \lambda = \pm 1 \\ u(0) = u_0 \in L^2(\mathbb{R}^2) \end{cases}$$

### Conserved Quantities:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \text{mass}$$

$$H[u(t)] = \int |\nabla u(t,x)|^2 - \frac{\lambda}{2} |u(t,x)|^4 dx = H[u_0] \quad \text{energy}$$

### Scaling:

$$u_\lambda(t,x) = \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \quad u_\lambda \text{ @ NLS} \iff u \text{ @ NLS.}$$

$$\|D_x^s u_\lambda\|_{L^2(\mathbb{R}_x^2)} = \left(\frac{1}{\lambda}\right)^{1+s-\frac{2}{p}} \|D_x^s u\|_{L^2(\mathbb{R}_x^2)} \quad ; \quad \boxed{s=0 \text{ critical.}}$$

$$\|u_\lambda\|_{L^p(\mathbb{R}_x^2)} = \left(\frac{1}{\lambda}\right)^{1-\frac{2}{p}} \|u\|_{L^p(\mathbb{R}_x^2)} \quad ; \quad \boxed{p=2 \text{ critical}}$$

### Local well-posedness theory:

#### - 2d Strichartz Estimates:

$$\begin{cases} i\partial_t u + \Delta u = F \\ u(0) = u_0 \end{cases} \quad \frac{2}{q} + \frac{2}{r} = \frac{2}{2} \quad ; \quad 2 < q \quad L^2\text{-admissible}$$

$$\|u\|_{L_t^q L_x^r} \leq \|u_0\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \quad \text{Holder dual}$$

$(q,r)$  -  $L^2$ -admissible

$(\tilde{q}, \tilde{r})$   $L^2$ -admissible

#### - Duhamel Formula

$$u(t) = e^{+it\Delta} u_0 - i\lambda \int_0^t e^{+i(t-t')\Delta} (|u|^2 u(t')) dt'$$

#### - Nonlinear Estimate

$$\text{also } \|u(t) - e^{+it\Delta} u_0\|_{L_t^q L_x^r} \leq \|u\|_{L_t^q L_x^r}^3$$

$$\|u\|_{L_t^q L_x^r} \leq \|u_0\|_{L_x^2} + \| |u|^2 u \|_{L_t^{q/3} L_x^r} \leq \|u_0\|_{L_x^2} + \|u\|_{L_t^q L_x^r}^3$$

LWP Theorem

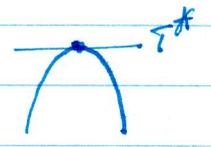
$\exists!$  maximal solution  $u \in C([-T_*, T^*]; L^2(\mathbb{R}^2)) \cap L^4([-T_*, T^*]; L^4(\mathbb{R}^2))$   
with  $T_*, T^* > 0$ .

What happens as  $t \rightarrow T^*$  when, say,  $T^* < \infty$ ?

Thm (Bourgain 1998): Assume  $T^* < \infty$ . ~~Then there is~~

~~nontrivial parabolic concentration of mass. In particular, If~~

$\|u_0\|_2 = M \quad \exists \quad \eta(M) > 0 \quad \text{s.t.}$



(conc)

$$\limsup_{t \rightarrow T^*} \sup_{\substack{\Omega \subset \mathbb{R}^2 \\ \text{side}(\Omega) < (T^* - t)^{1/2}}} \left( \int_{\Omega} |u(t, x)|^2 dx \right)^{1/2} > \eta(M) > 0.$$

PARABOLIC MASS CONCENTRATION

Remarks

① The theorem is also relevant in the defocusing case. The best known (work-in-progress) GWP result for NLS<sub>3</sub>( $\mathbb{R}^2$ ) is in  $H^s$ ,  $s > \frac{8}{17}$ . The method is based on [CKSTT] "I-method" pushing down from the  $H^1$ -level energy. A different strategy toward GWP is to show that (conc) does not occur as  $t \rightarrow T^*$  thereby proving no finite  $T^*$  exists.

② Bourgain's theorem is the first progress towards an "L<sup>2</sup> theory" of mass critical NLS blowup. It has been developed further by [Merle-Vega]. More is known in the case of data initially in  $H^1$ .

"H<sup>1</sup> Theory" of mass critical NLS blowup results:

$u_0 \in H^1$   
[Merle-Raphael]

- Mass Concentration: (Merle, Tsutsumi) ~~A~~ strong  $L^2$  limit as  $t \rightarrow T^*$ . In radial case, parabolic concentration of at least ground state mass into  $x=0$ .
- Self-similarity: (Weinstein) If  $\exists$  exactly one blowup point then, up to translations and dilations and phase, strong  $H^1$  convergence to  $R$ . - see [LV]
- Uniqueness of minimal blowup solutions: (Merle) Blowup solutions of critical mass are, up to invariances, pseudconformal images of  $R$ .

3 open problems

H<sup>s</sup>-theory of blowup

1. Show that (radial, finite variance) initially H<sup>s</sup>, 0 ≤ s < 1, blowup solutions of mass critical NLS (parabolically) concentrate at least the ground state mass (into x = 0).

GROUND STATE MASS CONCENTRATION

2. Show that (radial, finite variance) initially H<sup>s</sup>, 0 ≤ s < 1, blowup solutions of mass critical NLS are pseudoconformal images of R.

Mass Critical Blowup characterization

3. Prove corresponding results for mass-super critical NLS blowups. SUPER-CRITICAL BLOWUP.

Exercise Prove Bourgain's 1998 Theorem in the L<sup>2</sup>-critical 1d problem NLS<sub>5</sub>(R). ~~Note: The [mvv] stuff is done for 1d in [Vargas-Vega].~~ I believe the H<sup>1</sup> inputs are in place for 1d.

I. Pictorial overview of Bourgain's proof.



[0, T\*) = ∪\_{j ≥ 1} I\_j, ||u||\_{L^4\_{t ∈ I\_j, x ∈ R^2}} = γ << 1. (fixed)

LWP Theory ⇒ I\_j = [t\_j, t\_{j+1})

||u(t) - e^{itΔ}u(t\_j)||\_{L^4\_{t ∈ I\_j, x ∈ R^2}} ≤ γ^3

||e^{itΔ}u(t\_j)||\_{L^4\_{t ∈ I\_j, x ∈ R^2}} ~ γ.

∃ (u\_j)\_{j ≥ 1} defined via u\_j(x) = u(t\_j, x) s.t.

||u\_j||\_{L^2\_x} = ||u\_0||\_{L^2\_x} = m > 0, ||e^{itΔ}u\_j||\_{L^4\_{tx}} ≥ γ > 0 } ∀ j.

Squares Lemma

- $f \in L^2(\mathbb{R}^2), \varepsilon > 0.$   $\exists (\hat{f}_r)_{1 \leq r \leq R(\varepsilon)}$  s.t.
- $\text{supp } \hat{f}_r \subset \tau_r \subset \subset \mathbb{R}^2, \tau_r \text{ square side } 4r, \text{ center } \xi_r$
  - $|\hat{f}_r| \leq \frac{1}{4r}$  ← "L<sup>2</sup>-invariant hypothesis"
  - $\|\hat{f}_r\|_{L^2} \geq \varepsilon'(r).$

and

$$\|e^{it\Delta} f - \sum_{r \geq 1} e^{it\Delta} \hat{f}_r\|_{L_{xt}}^4 < \varepsilon.$$

Tubes Lemma

- $g: \bullet \text{supp } \hat{g} \subset \tau \subset \subset \mathbb{R}^2, \tau \text{ square side } A, \text{ center } \xi_0.$
- $|\hat{g}| \leq \frac{1}{A}.$

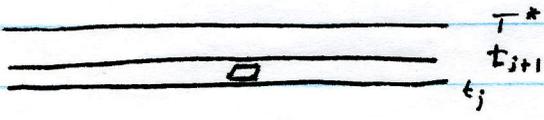
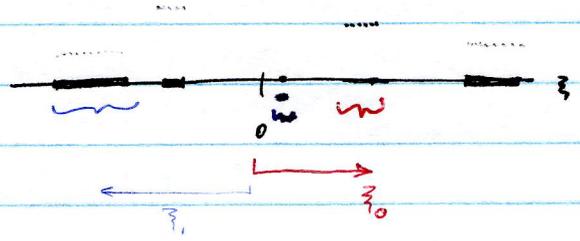
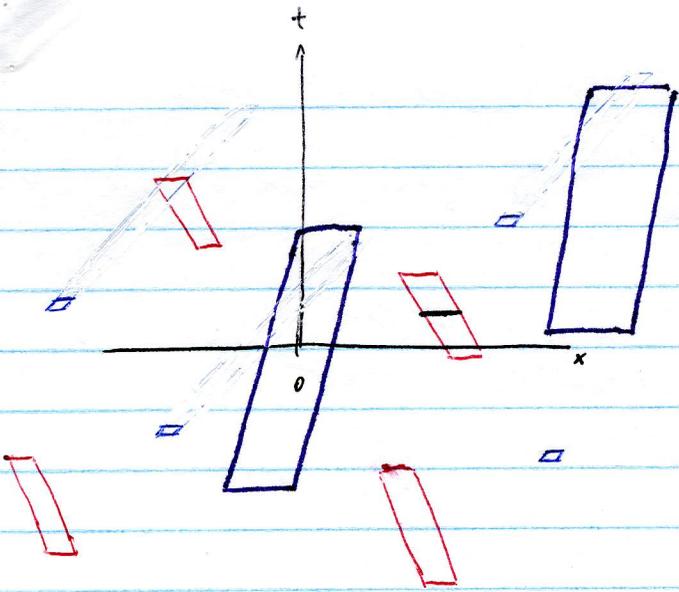
$\forall \varepsilon > 0 \exists$  tubes  $(Q_s)_{1 \leq s \leq S(\varepsilon)}$  of form

$$Q_s = \{ (t, x) \in \mathbb{R}^3 : x + 2t\xi_0 \in \tau_s, t \in J_s \}$$

s.t.  $\text{side}(\tau_s) = A^{-1}, |J_s| = A^{-2}.$  (location not specified)

and

$$\left( \int_{\mathbb{R}^3 \setminus \bigcup_s Q_s} |e^{it\Delta} g|^4 dx dt \right)^{\frac{1}{4}} < \varepsilon.$$



I.

**Lemmas  $\Rightarrow$  mass concentration**

$I_j = [t_j, t_{j+1}) \subset [0, T^*)$ . Let  $u(t_j) = \phi$ .

$$\gamma^4 = \int_{I_j} \int |u(t)|^4 dx dt = \int_{I_j} \int u(t) \left[ e^{i(t-t_j)\Delta} \phi \right] \overline{\left[ e^{i(t-t_j)\Delta} \phi \right]^2} dx dt + o(\gamma^6)$$

squares lemma to  $\phi$  w.  $\varepsilon = \gamma^2$

$$\rightarrow = \sum_{r_1, r_2, r_3 \in R} \int_{I_j} \int u(t) \left[ e^{i(t-t_j)\Delta} f_{r_1} \right] \overline{\left[ e^{i(t-t_j)\Delta} f_{r_2} \right]} \overline{\left[ e^{i(t-t_j)\Delta} f_{r_3} \right]} dx dt + o(\gamma^5)$$

**EXTRACT 3 SQUARES, SPAWN 4th SQUARE**

Thus,  $\exists$  choice  $r_1, r_2, r_3 \in R$  s.t.

$$\int_{I_j} \int_{P_\tau} u(t) \left[ \dots \right] \overline{\left[ \dots \right]} \overline{\left[ \dots \right]} dx dt > \frac{\gamma^4}{R(\gamma^2)^3} \equiv \eta.$$

$\text{supp}(\widehat{f_{r_j}}) \subset \tau_{r_j}$  of side  $A_{r_j}$ . Assume  $A_{r_1} \geq A_{r_2} \geq A_{r_3}$ .

By evolution analysis,  $\exists$  square  $\tau$  of side  $3A_{r_1}$  s.t.

We can replace  $u$  by  $P_\tau u$ ; ( $P_\tau u = \widehat{u} \chi_\tau$ ).

By Hölder

$$\eta < \int_{I_j} \int_{P_\tau} \underbrace{u}_{L^2} \overline{\underbrace{[ \dots ]}_{L^4}} \overline{\underbrace{[ \dots ]}_{L^4}} dx dt \leq \left( \int_{I_j} \int_{P_\tau} |u|^2 \underbrace{[ \dots ]^2}_{L^4} \right)^{\frac{1}{2}} \left( \int_{I_j} \int_{P_\tau} \underbrace{[ \dots ]^4}_{L^2} \right)^{\frac{1}{4}} \left( \int_{I_j} \int_{P_\tau} \underbrace{[ \dots ]^4}_{L^2} \right)^{\frac{1}{4}}$$

$$\leq \left( \dots \right)^{\frac{1}{2}} < \|f_{r_1}\|_{L^2} \|f_{r_3}\|_{L^2}$$

lower bounded, can divide.

$\Rightarrow$

$$c \eta^2 < \int_{I_j} \int |P_\tau u|^2 |e^{i(t-t_j)\Delta} f_{r_1}|^2 dx dt$$

Apply Tubes lemma to  $g = e^{-it_j\Delta} f_{r_1}$  with  $\varepsilon = \eta^{10}$ .

$$\iint_{[\mathbb{R}^3 \setminus \cup Q_s] \cap [I_j \times \mathbb{R}^2]} |P_\tau v|^2 |e^{it\Delta} g|^2 \leq \|P_\tau v\|_{L^4_{I_j \times \mathbb{R}^2}}^2 \eta^{20} \ll C \eta^2.$$

### EXTRACT SPACETIME $L^4$ TUBE

Thus,  $\exists$  choice of some  $Q \in (Q_s)_{1 \leq s \leq S(\tau)}$  such that

$$\iint_{Q \cap [I_j \times \mathbb{R}^2]} |P_\tau v|^2 |e^{i(t-t_j)\Delta} f_{r_1}|^2 > \frac{\eta^2}{S(\tau)} \equiv \eta_1.$$

$\Rightarrow$

$$\iint_{\{(x,t) \mid x+2t\tau_0 \in \mathbb{Z}, t \in J \cap I_j\}} |P_\tau v|^4 dx dt > C \eta_1^2.$$

$\uparrow$  side  $A^{-1}$        $\uparrow$  side  $A^{-2}$ .

Note that

$$\|P_\tau v(t)\|_{L^\infty_x} \leq \|P_\tau v(t)\|_{L^2_x} \leq |\tau|^{1/2} \|v(t)\|_{L^2_x} \leq CA \|v(t)\|_{L^2} \leq CA.$$

Thus, peeling out two factors, we have

$$CA^2 \int_{I_j} \|P_\tau v(t)\|_{L^2}^2 dt > \iint_{\{ \dots \}} |P_\tau v|^4 dx dt > \eta_1^2$$

$$CA^2 [t_{j+1} - t_j] >$$

$$\Rightarrow A > C \frac{\eta_1}{(t_{j+1} - t_j)^{1/2}}. \quad (\text{Thus } J \subset I_j \text{ essentially})$$

III.

Proof of squares Lemma

$L^2$  scale invariance  $\rightarrow$  may assume  $\text{supp } \hat{f} \subset B(0,1)$ .

[MvV] Strichartz refinement (following Bourgain 1992):

for  $j = 1, 2, 3, \dots$  denote with

$\tau$  a square of side  $2^{-j}$ .

$\mathcal{E}_j$  a grid of such squares covering  $\mathbb{R}^2$  w. disjoint interiors.

Theorem ([MvV] 1999):  $\forall \frac{12}{7} < p < 2$ , if  $\text{supp } \hat{f} \subset B(0,1)$  then

$$\| e^{it\Delta} f \|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^2)} \lesssim \| f \|_{X_p}$$

All we use is  $\exists p_* < 2$  s.t. estimate holds for  $p_* < p < 2$ .

nontrivial HA input

where

$$\| f \|_{X_p} = C \left[ \sum_{j=1}^{\infty} \sum_{\tau \in \mathcal{E}_j \cap B(0,1)} (2^{-j})^4 \left\{ \frac{1}{(2^{-j})^2} \int_{\tau} |\hat{f}|^p \right\}^{\frac{4}{p}} \right]^{\frac{1}{4}}$$

Write  $\delta = 2^{-j}$ ,  $\mathcal{E}_{\delta} \ni \tau$ .

$\epsilon^4 \leq$   
WLOG

$$\begin{aligned} \| f \|_{X_p}^4 &= \sum_{\delta} \sum_{\tau \in \mathcal{E}_{\delta}} \delta^{2p-4} \delta^{(p-2)\frac{2}{p}(2-p)} \left( \int_{\tau} |\hat{f}|^p \right)^2 \left( \int_{\tau} |\hat{f}|^p \right)^{\frac{4}{p}-2} \\ &\leq \left[ \sum_{\delta} \sum_{\tau \in \mathcal{E}_{\delta}} \delta^{2p-4} \left( \int_{\tau} |\hat{f}|^p \right)^2 \right] \max_{\delta} \max_{\tau \in \mathcal{E}_{\delta}} \left( \delta^{p-2} \int_{\tau} |\hat{f}|^p \right) \\ &\leq \iint \frac{|\hat{f}(x)|^p |\hat{f}(y)|^p}{|x-y|^{2(2-p)}} dx dy \end{aligned}$$

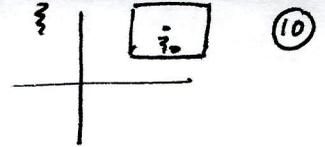
$$4 - \frac{8}{p} = 2p - 4 + (p-2)\frac{2}{p}(2-p)$$

$$\begin{aligned} &\leq \| |\hat{f}|^p \|_{L^{\frac{2}{p}}} \| |\hat{f}|^p \|_{L^{\frac{2}{p}}} \leq \| \hat{f} \|_{L^2}^{2p} \leq C. \\ &\stackrel{\text{HLS}}{\leq} \frac{p}{2} + \frac{p}{2} + \frac{2(2-p)}{2} = 2 \end{aligned}$$

$$\begin{aligned} \text{RK: } &\leq \sum_{\delta} \delta^{4-\frac{8}{p}} \left( \int_{\tau} |\hat{f}|^p \int_{\tau} |\hat{f}|^p \right)^{\frac{2}{p}} \\ &\leq \sum_{\delta} \left( \delta^{2p-4} \int_{\tau} |\hat{f}|^p \int_{\tau} |\hat{f}|^p \right)^{\frac{2}{p}} \\ &\leq \left( \sum_{\delta} \dots \right)^{\frac{2}{p}} \leq C \| f \|_{L^2}^4. \end{aligned}$$



# IV Proof of Tbes Lemma



## ① Normalization

$g$ :  $\text{supp } \hat{g} \subset \tau$  square w. center  $\bar{z}_0$ , side  $A$ .

$$\begin{aligned}
 |e^{it\Delta} g(x)| &= \left| \int_{|\bar{z}-\bar{z}_0| < A} \hat{g}(\bar{z}) e^{ix \cdot \bar{z}} e^{it|\bar{z}|^2} d\bar{z} \right| \\
 &= \left| \int_{|\bar{z}| < A} \hat{g}(\bar{z}_0 + \bar{z}) e^{ix \cdot \bar{z}_0} e^{i(x+2t\bar{z}_0) \cdot \bar{z}} e^{it|\bar{z}|^2} e^{it|\bar{z}_0|^2} d\bar{z} \right| \\
 &= \left| \int_{|\bar{z}| < A} \hat{g}(\bar{z}_0 + \bar{z}) e^{i(x+2t\bar{z}_0) \cdot \bar{z}} e^{it|\bar{z}|^2} d\bar{z} \right|
 \end{aligned}$$

Rescaling and translating via

$$\hat{g}'(\bar{z}') := A \hat{g}(\bar{z}_0 + A\bar{z}') \quad ; \quad t' = A^2 t$$

recasts the integration on  $B(0,1)$  and  $|\hat{g}'| \leq 1$ .

## ② level set analysis

Theorem (Bourgain 1995)

$$\exists \delta_A < 4 \text{ s.t. } \forall g \in (\delta_A, 4]$$

$$\left\| \int_{B(0,1)} F(\bar{z}) e^{i(x \cdot \bar{z} + t|\bar{z}|^2)} d\bar{z} \right\|_{L^8(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \|F\|_{L^\infty_{\bar{z}}}$$

$\Rightarrow$

$$\|e^{it'\Delta} g'\|_{L^8} \leq C$$

$$\int_{[|\epsilon^{i_m}| < \delta]} |e^{i_m}|^\delta + \int_{[> \delta]} |1|^\delta < C. \quad \delta \text{ t.b.d.} \quad (11)$$

sublevel superlevel.

sublevel

$$\int_{[< \delta]} |1|^\delta \frac{1}{|\delta^{4-\delta}|} < C \Rightarrow \int_{[< \delta]} |1|^\delta < C \delta^{4-\delta}$$

$$\left( \int_{[< \delta]} |1|^\delta \right) \frac{1}{\delta^{4-\delta}} < C$$

Thus, by choosing  $\delta = \delta(\epsilon)$  sufficiently small, we have

$$\int_{[< \delta]} |e^{i t' \Delta} g'|^4 < \epsilon^4.$$

superlevel

$$\delta^2 |[> \delta]| < C \Rightarrow |[> \delta]| < \delta^{-2}$$

; superlevel set has bounded (depending on  $\epsilon$ ) 2d Lebesgue measure.

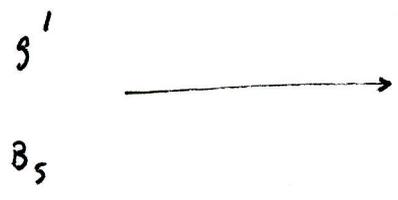
•  $\text{supp } \hat{g} \subset B(0,1), |\hat{g}'| < 1$   
 $\rightarrow e^{i t' \Delta} g(x')$  is Lipschitz w.r.t.  $t', x'$ .

$\Rightarrow |[> \delta]|$  may be covered by  $S(\epsilon)$  unit cubes  $B_S \subset \mathbb{R}^3$ .

Thus,

$$\int_{\mathbb{R}^3 \setminus \cup B_S} |e^{it'\Delta} g'|^4 \lesssim \Sigma^4$$

③ Undo the scaling.



proving the tubes lemma.