GWP + Scattering for cubic defocusing NLS below energy.

Stanford / AIM [CKSTT]

NLS $\begin{cases} i\partial_t u + \Delta u = |u|^2 u & u : [-T_*, T^*] \times \mathbb{R}^3_x \longmapsto \mathbb{C} \\ u(0,x) = \phi(x) , & \phi \in H^s_x = \{ f : \|f\|_{L^2} + \|D_x^s f\|_{L^2} < \infty \} . \end{cases}$

Conserved quantities

$$Q = \left(\int |u(t,x)|^2 dx \right)^{\frac{1}{2}} \qquad L^\infty_t L^2_x$$

$$H = \int \tfrac{1}{2} |\nabla u(t,x)|^2 + \tfrac{1}{4} |u(t,x)|^4 \, dx \qquad L^\infty_t \dot{H}^1_x , \quad L^\infty_t L^4_x .$$

Scaling invariance

time $\sim \sigma^2$, space $\sim \sigma$.

If u solves NLS so does $u_\sigma(t,x) = \sigma \, u(\sigma^2 t, \sigma x)$.

$$\|D_x^s u_\sigma\|_{L^2_x} = \sigma^{1+s-\frac{3}{2}} \|D_x^s u\|_{L^2_x}$$

$\Longrightarrow$ $\|D_x^{\frac{1}{2}} u_\sigma\|_{L^2_x}$ is scaling invariant. $\dot{H}^{\frac{1}{2}}$

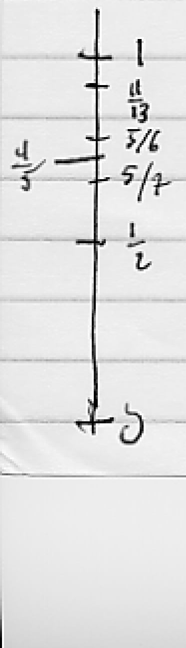
well-posedness

NLS is locally well-posed in $H^s(\mathbb{R}^3)$ if $\exists$ lifetime $T = T(\|\phi\|_{H^s}) > 0$ and uniquely defined cts. map

$$H^s_x \ni \phi \longmapsto u \in X_T$$

so that u solves NLS on time interval $[0,T]$ and $X \subset C([0,T]; H^s_x)$.

If T may be taken arbitrarily large, we say GWP holds.



Known results

	LWP	GWP	Scatter
	[CW] : $s \geq \frac{1}{2}$	$s \geq 1$	
	[GV]		[GV]: $s \geq 1$
		[B]: $s > \frac{11}{13}$	
		[B]: $s > \frac{5}{7}$ (radial)	[B]: $s > \frac{5}{7}$ (radial)
		[CKSTT] : $s > \frac{5}{6}$	
New		$s > \frac{4}{5}$	$s > \frac{4}{5}$
		$s \overset{?}{\geq} \frac{1}{2}$	$s \overset{?}{\geq} \frac{1}{2}$

Talk outline

I. LWP $s > \frac{1}{2}$, GWP $s > 1$ Strichartz Estimates

II. Morawetz Estimate

III. H^1 Scattering

IV. H^s GWP + Scattering

I. linear homogeneous problem

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u(0) = \phi \end{cases}$$

has explicit solution

$$u(t,x) = S(t)\phi(x) = \int e^{ix\cdot\xi} e^{-it|\xi|^2} \hat{\phi}(\xi)\, d\xi = \frac{C_d}{(it)^{d/2}} \int e^{i\frac{|x-y|^2}{4t}} \phi(y)\, dy.$$

multiplier convolution

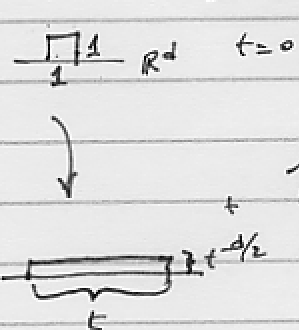
spatial norm decay estimates

$$\|S(t)\phi\|_{L^\infty_x} \leq t^{-d/2} \|\phi\|_{L^1} \qquad L^1 - L^\infty \text{ decay}$$

$$L^p \qquad t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} \qquad L^{p'}$$

$$\|S(t)\phi\|_{L^2_x} = \|\phi\|_{L^2_x} \qquad L^2 - L^2 \text{ unitarity}$$

[linear homogeneous Schrödinger wave decays in amplitude like $t^{-d/2}$ but preserves its L^2_x mass.]



so it spreads out and disperses.

This suggests the possibility that we can prove space-time norm estimates on Schrödinger waves.

Spacetime estimates a.k.a. Strichartz estimates

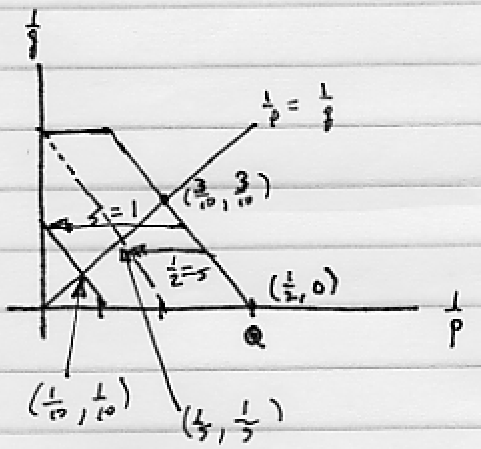
$$\|S(t)\phi(x)\|_{L^q_t L^p_x} \overset{?}{\le} \|\phi\|_{L^2_x} \quad H^s_x$$

$$\phi \underset{\text{rescales}}{\rightsquigarrow} \phi_\alpha(x) = \phi(\alpha x) \rightsquigarrow S(t)\phi_\alpha(x) = S(\alpha^2 t)\phi(\alpha x)$$

$$\alpha^{-\frac{2}{q}-\frac{d}{p}} \sim \alpha^{-\frac{d}{2}+s}$$

(q,p) is L^2-Strichartz-admissible if $2<q$ and

$$\boxed{\frac{1}{q} = \left(-\frac{d}{2}\right)\frac{1}{p} + \frac{d}{4}} - \frac{s}{2}.$$



H^s-Strichartz-Admissible: $2<q$ and

$$\frac{1}{q} = \left(-\frac{d}{2}\right)\left(\frac{1}{p}\right) + \left(\frac{d}{4} - \frac{s}{2}\right)$$

[Denote an arbitrary H^s-admissible $L^q_t L^p_x$ by X_s.]

[A linear homogeneous Schrödinger wave is bounded in certain $L^q_t L^p_x$ spacetime norms.]

$$\|u\|_{X_s} = \sup_{(q,p):\,H^s\text{-admissible}} \|u\|_{L^q_t L^p_x}.$$

Linear inhomogeneous problem

$$\begin{cases} i\partial_t v + \Delta v = f \\ v(0) = 0 \end{cases} \qquad f: \mathbb{R}_t \times \mathbb{R}^d_x \longrightarrow \mathbb{C}. \text{ (given)}$$

has explicit solution via Duhamel's formula

$$v(t,x) = -i \int_0^t S(t-t') f(t',x)\, dt'.$$

We are naturally interested in

$$\left\| \int_0^t S(t-t') f(t',x)\, dt' \right\|_{L^q_t L^p_x} \lesssim \|f\|_{L^{\tilde{q}'}_t L^{\tilde{p}'}_x}.$$

$\Updownarrow$ $\uparrow$ any X_0. e.g. $L^{10/7}_{xt}$

$$\|v\|_{L^q_t L^p_x} \lesssim \|(i\partial_t + \Delta) v\|_{L^{\tilde{q}'}_t L^{\tilde{p}'}_x}$$

Scales $\quad \sigma^{-\frac{2}{q} - \frac{d}{p}} \sim \sigma^{2 - \frac{2}{\tilde{q}'} - \frac{d}{\tilde{p}'}}.$

If (q,p) is any L^2-Strichartz-admissible pair

$$-\frac{2}{q} - \frac{d}{2} = -\frac{d}{2}$$

Then

$$-\frac{d}{2} = 2 - \frac{2}{\tilde{q}'} - \frac{d}{\tilde{p}'}$$

$$= \cancel{2} - 2(\cancel{1} - \frac{1}{\tilde{q}}) - d(1 - \frac{1}{\tilde{p}})$$

$$\frac{d}{2} = \frac{2}{\tilde{q}} + \frac{d}{\tilde{p}}$$ so $(\tilde{q}, \tilde{p})$ are L^2-admissible.

$\boxed{\text{Strichartz Flexibility}}$

LWP, $s > \frac{1}{2}$.

$$u(t) = S(t)\phi - i \int_0^t S(t-t') (|u|^2 u)(t')\, dt'.$$

Take any H^s - Strichartz - norm. Sobolev lets us go back to L^2 - Strichartz line if we put on D_x^s. Therefore, letting X_{sT} denote $X_s, t \in [0,T]$

$$\|u\|_{X_{sT}} \leq \|\phi\|_{H^s} + \| \underset{\frac{10}{3}}{D_x^s u}\ \underset{5}{\bar{u}}\ \underset{5}{u} \|_{L_T^{10/7} L_x^{10/7}}$$ 1

$$\leq \|\phi\|_{H^s} + \|u\|_{X_{sT}} \left(T^\alpha \|u\|_{X_{sT}} \right)^2.$$

Höldering in time.

This ultimately yields $\exists\ T = T(\|\phi\|_{H^s}) > 0$ and we find a solution to NLS on $[0,T]$.

GWP, $s > 1$

$$T = T(\|\phi\|_{H^s}) > T(\|\phi\|_{H^1}) > c.$$

energy conservation $\Longrightarrow \|u(t)\|_{H^1} \leq C \quad \forall t$.
iterate local theory w. lower bound on successive lifetimes
$\Longrightarrow$ global theory.

solutions of NLS exist locally for initial data in H_x^s, $s \geq \frac{1}{2}$.
solutions of NLS exist globally for $s \geq 1$.
Q: What happens to local solutions in range $\frac{1}{2} \leq s < 1$?
Q: What is the long time behavior of solutions?

II. Morawetz Estimates

Morawetz Action at O

$$M_0(t) = \operatorname{Im} \int_{\mathbb{R}^3} \bar{u}(t,x)\, \partial_r u(t,x)\, dx$$

intuition: Let ψ be a wave packet with wave vector $\underline{k}$.

$$\int \overline{\psi(x)}\, \partial_{x_j} \psi(x)\, dx \sim i \underbrace{k_j \, \|\psi\|_{L^2}}_{\text{momentum}_j}$$

So M_0 measures "outward momentum from $x = 0$".

A calculation shows

$$\partial_t M_0(t) = 4\pi^2 |u(t,0)|^2 + \int \frac{2}{r} |\not\nabla_0 u(t,x)|^2\, dx + \int \frac{1}{r} |u(t,x)|^4\, dx$$

so $M_0 \nearrow$, and

$$M_0(T) - M_0(0) = \int_0^T 4\pi^2 |u(t,0)|^2\, dt + \int_0^T \int \frac{2}{r} |\not\nabla_0 u(t,x)|^2\, dx\, dt + \int_0^T \int \frac{1}{r} |u(t,x)|^4\, dx\, dt.$$

Since $|M_0(t)| \leq \|u(t)\|^2_{\dot{H}^{1/2}}$, we have the standard Morawetz estimates.

[nonlinear defocusing Schrödinger wave is repulsed from origin $x = 0$]

Morawetz action at y

$$M_y(t) = \operatorname{Im} \int_{\mathbb{R}^3} \bar{u}(t,x) \left(\frac{x-y}{|x-y|} \cdot \nabla \right) u(t,x)\, dx \rightsquigarrow M_y(t) \nearrow$$

and

$$\int_0^T \int \frac{1}{|x-y|} |u(t,x)|^4\, dx \leq \sup_{t \in [0,T]} \|u(t)\|^2_{\dot{H}^{1/2}}.$$

[nonlinear defocussing Schrödinger wave is repulsed from $x = y$.]

Morawetz Interaction Potential

$$M(t) = \int_{\mathbb{R}^3} |u(t,y)|^2 M_y(t) \, dy .$$

$$|M(t)| \lesssim \|u(t)\|_{L^2_x}^2 \, \|u(t)\|_{\dot H^{1/2}_x}^2 .$$

A calculation shows that

$$\partial_t M(t) \geq 4\pi^2 \int |u(y)|^4 \, dy + \iint |u(y)|^2 |u(x)|^4 \frac{dx\,dy}{|x-y|} .$$

so $M \nearrow$. We learn that there is an <u>L^4_{xt} estimate</u>

$$\int_0^T \int |u(t,y)|^4 \, dy\, dt \lesssim \|u(0)\|_{L^2_x}^2 \sup_{t\in[0,T]} \|u(t)\|_{\dot H^{1/2}_x}^2 .$$

$\dot H^1$

L^2

used energy

H^1- Scattering

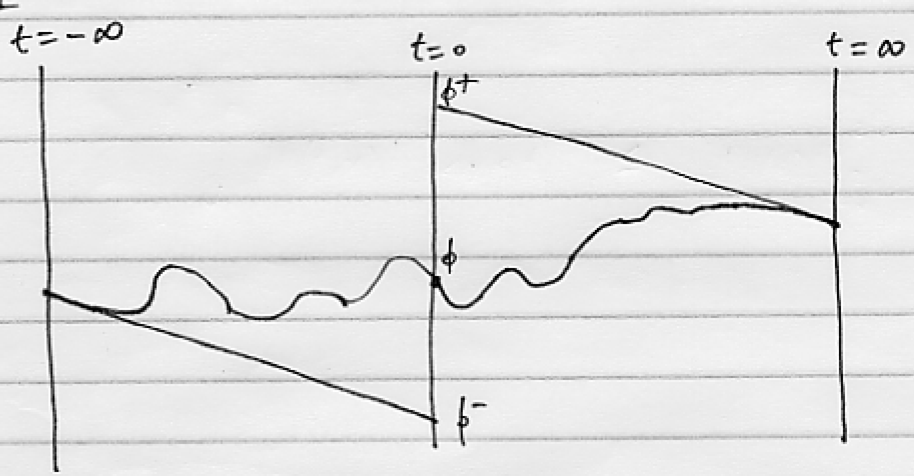
Does the nonlinear solution $\phi \longmapsto u(t)$ eventually behave like the linear solution?

Manifestations of "behave like":

- decays in L^∞ at rate $t^{-d/2}$,
- bounded in global-in-time spacetime norms,
- scattering holds.

H^1-Scattering: Given $\phi \longmapsto u(t)$ solving NLS $\exists\ \phi^\pm$ such that

$$\lim_{t \to \pm\infty} \| S(t)\phi^\pm - u(t) \|_{H^1_x} = 0.$$



How to construct ϕ_+?

$$S(t)\phi^+ - \left[S(t)\phi - i \int_0^t \underbrace{S(t-t')}_{S(t)S(-t')} \left(|u|^2 u(t')\right) dt' \right]$$

$t \to \infty$ to define ϕ^+.

$$S(t) \left\{ \phi^+ - \left[\phi - i \int_0^t S(-t')\left(|u|^2 u\right)(t')\, dt' \right] \right\}.$$

small in H^1 as $t \to \infty$.

We wish to show $\forall\ \varepsilon > 0\ \exists\ T = T(\varepsilon)$ such that

Main issue for scattering

$$\sup_{t^* \in [T, \infty]} \left\| \int_T^{t^*} S(-t')\left(|u|^2 u\right)(t')\, dt' \right\|_{H^1} < \varepsilon.$$

This follows by combining LWP arguments + interaction Morawetz L^4_{xt} estimate.

Global H^1-Strichartz bounds

interaction Morawetz L^4 estimate + L^2 + energy conservation $\Longrightarrow$ global spacetime L^4_{xt}-norm is bounded. Hence, $\forall\ \varepsilon > 0\ \ \exists\ T = T(\varepsilon)$ s.t.

$$\|u\|_{L^4_{x,\, t\in[T,\infty)}} < \varepsilon .$$

Define

$$z_1(t^*) = \|u\|_{X_{1,\, t\in[T,t^*)}} = \sup_{(q,p):\ H^1\text{-admissible}} \|u\|_{L^q_{t\in[T,t^*]} L^p_x} .$$

We bound $z_1(t^*)$.

$$\|u\|_{X_{1,[T,t^*]}} \leq \|u(T)\|_{H^1} + \left\| \int_T^t S(t-t')(|u|^2u)(t')\,dt' \right\|_{X_{1,\, t\in[T,t^*]}}$$

$$\leq C + \Big\| \underset{\frac{10}{3}}{D_x^1 u}\ \ \underset{5}{\bar{u}}\ \ \underset{5}{u} \Big\|_{L^{10/7}_{t\in[T,t^*]} L^{10/7}_x} \qquad 2.$$

$$z_1(t^*) \leq C + z_1(t^*)\, \|u\|^2_{L^5_{x,\, t\in[T,t^*]}}$$

L^5 between L^{10} and L^4

$$\leq C + \big(z_1(t^*)\big)^{1+2\gamma}\, \varepsilon^{2(1-\gamma)} \Longrightarrow z_1(t^*) \text{ bounded.}$$

Scattering

$$\sup_{t^* \in [T,\infty)} \left\| \int_T^{t^*} S(-t')\,(|u|^2 u)(t')\,dt' \right\|_{\dot{H}^1} < \varepsilon .$$

D_x^1 $\qquad\qquad\qquad\qquad\qquad\qquad$ L^2

$$\left\langle g(x),\; D_x^1 \int_0^{t^*} S(-t')\,(|u|^2 u)(t')\,dt' \right\rangle$$

$$\int_T^{t^*} \left\langle S(t')\,g,\; D_x^1 u \;\bar{u}\; u \right\rangle (t')\,dt' .$$

$L^{10/3}_{xt}$ $\qquad$ $L^{10/3}$ $\qquad$ L^5 $\quad$ L^5 $\qquad\qquad$ 3.

L^{10} $\quad$ energy

$$\leq \left(Z_1(t^*) \right)^2 \|u\|^2_{L^5_{x,\,t \in [T,t^*]}}$$

L^4

$$\leq Z_1(t^*)^{2+\frac{2}{3}}\, \varepsilon^{\frac{4}{3}} \lesssim \tilde{\varepsilon} .$$