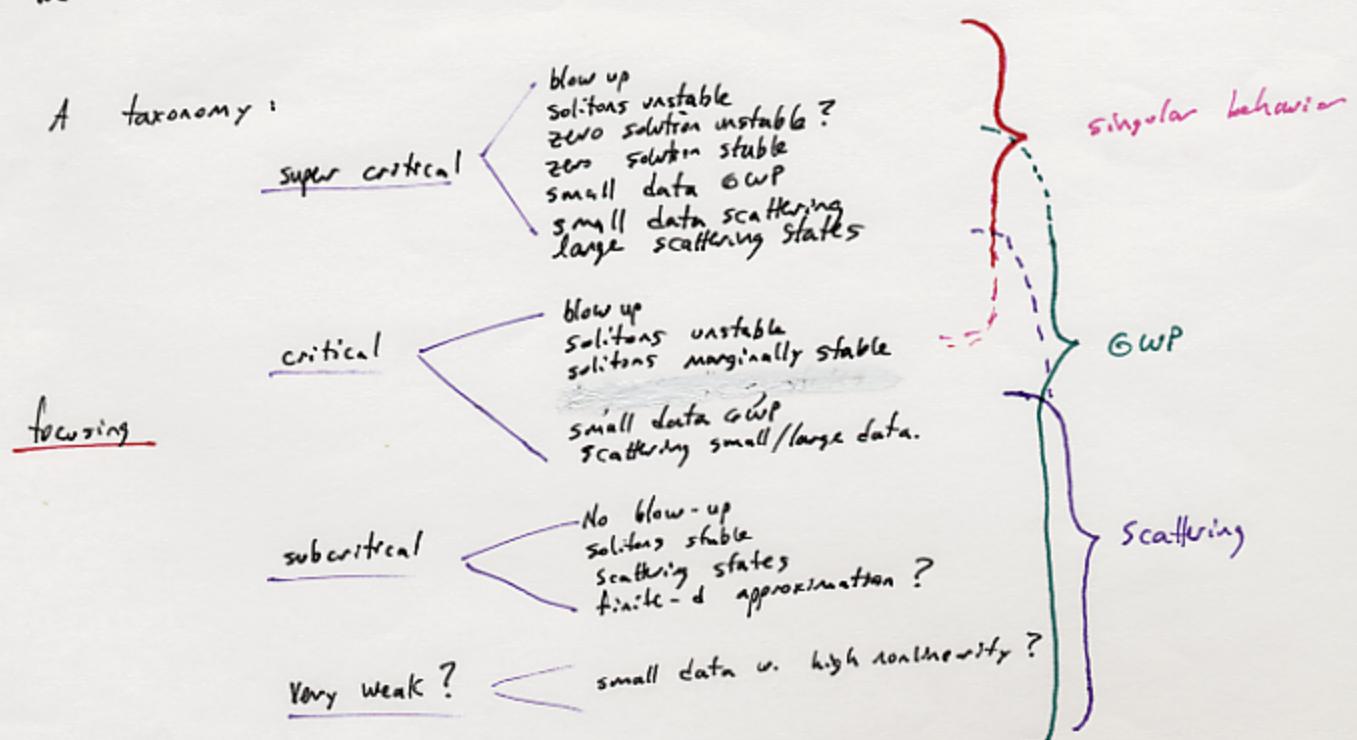


The main question about any initial value problem, such as

$$(NLS) \quad \begin{cases} i\epsilon u_t + \Delta u + f(|u|^2)u = 0 & , \quad \Omega \subset \mathbb{R}_t \times \mathbb{R}_x^d \mapsto \mathbb{C}, \\ u(0, x) = \phi(x) & , \quad \phi \in H^s(\mathbb{R}^d), \end{cases}$$

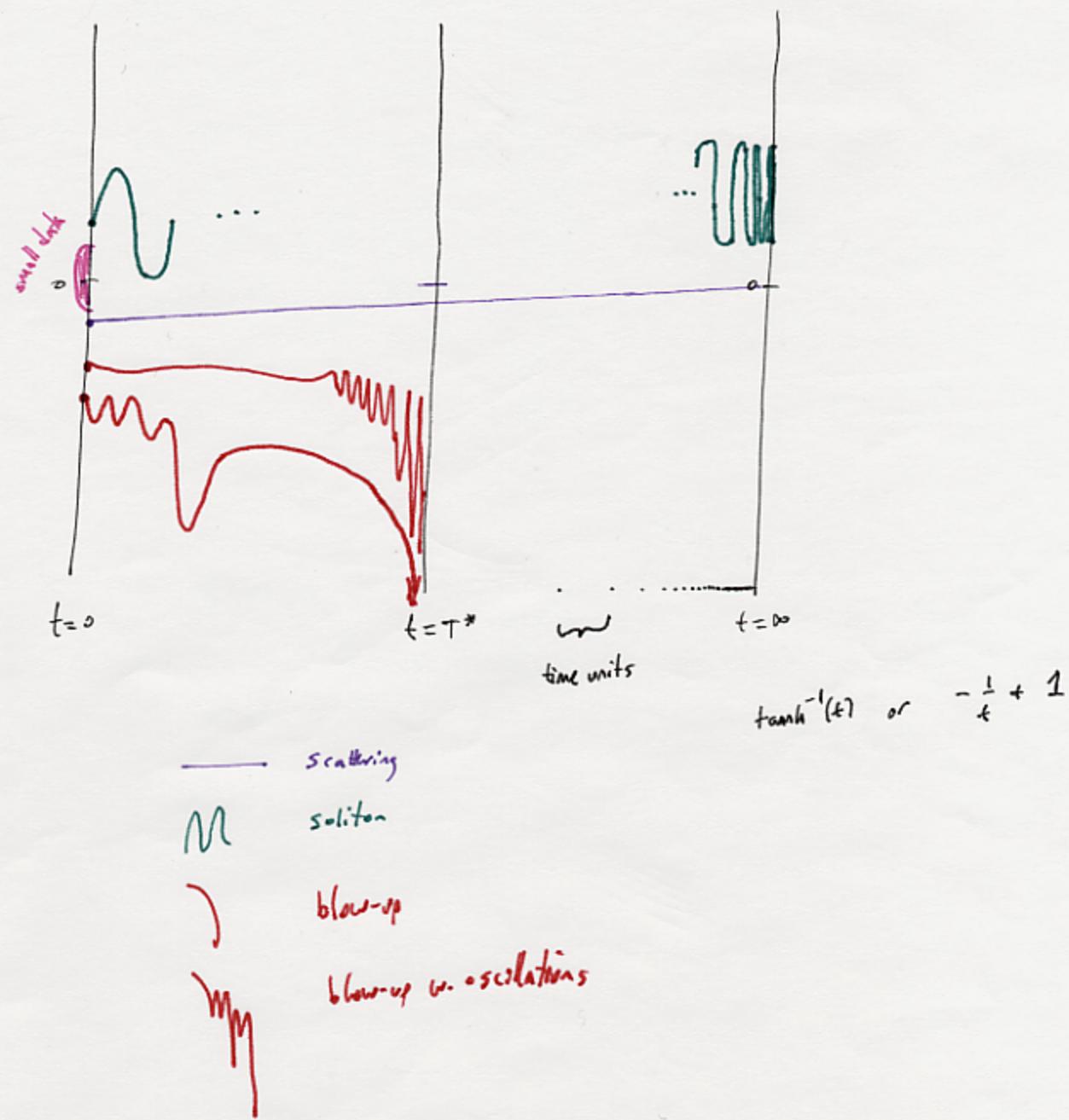
is: What happens? Apart from certain situations, such as scattering, we don't know the answer. A zoology of possible evolutions has been identified using analysis and physical and numerical experiments including scattering, soliton formation and resolution, (creatures, kinks, coherent structures) turbulence, singularity formation, . . .

A classification of families of NLS problems with distinct final outcomes has been developed using the available conserved quantities and various mathematical techniques (linear estimates, approximations, boot strap arguments, fixed point arguments, compactness, ...). The sign of f distinguishes two classes: focusing ($f > 0$), defocusing ($f < 0$). These classes are further stratified by quantifying the "strength" of the nonlinearity. In case $f(z) = z^\sigma$, we can separate various classes of "strength" using σ .



Remark - Similar (and more complete) taxonomies for other problems (like KdV, general dispersion, defocusing, derivative nonlinearities, quasilinear, fully nonlinear, variable coefficient) should be worked out.

Pictorial representation of some evolutions



These can also be combined in one solution



"mixed states"

e.g. multiple solitons,
soliton + scattering, ...

may not be desirable w. scattering
+ solitons.

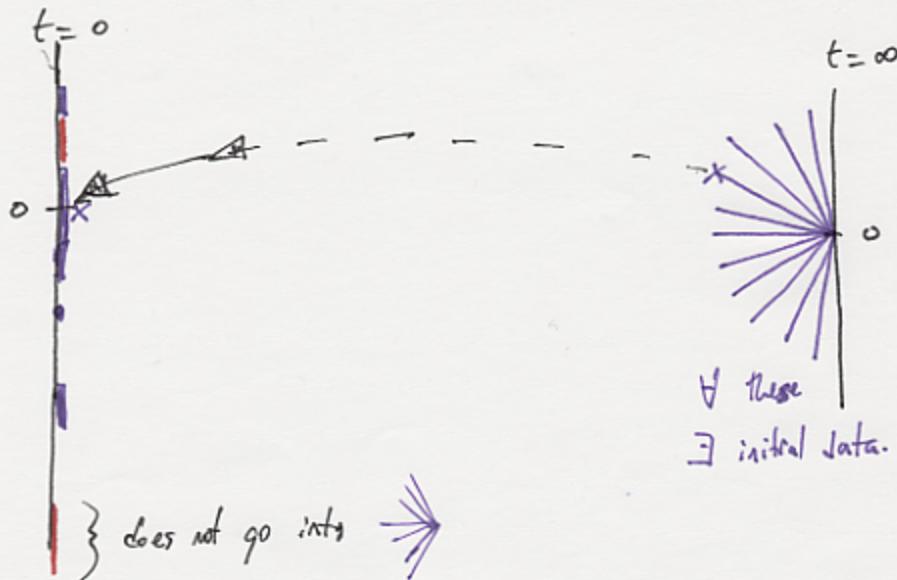
Turbulence

$t=0$

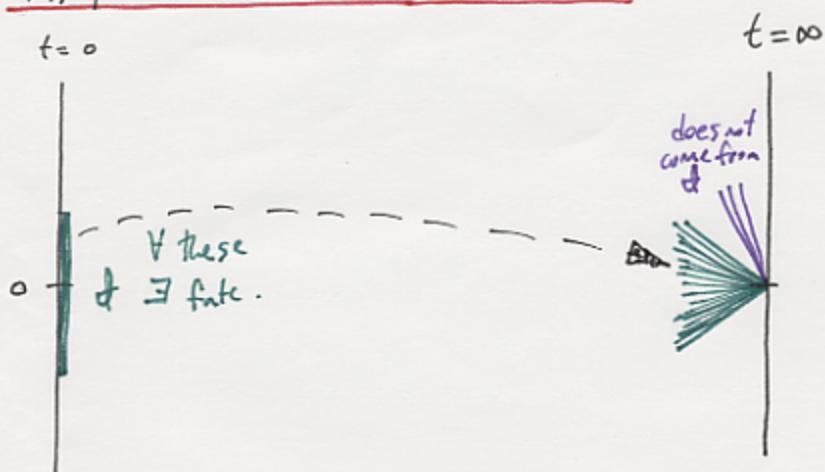


?

Existence of wave operators



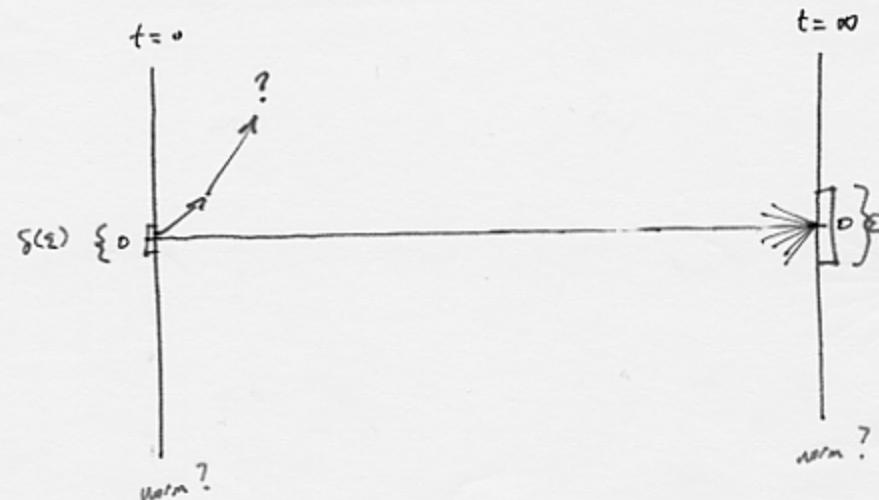
Asymptotic completeness on $\mathcal{F} \subset I$



These constructions provide two distinct answers to the main question. We can ask ourselves what are the possible outcomes and try to show a large family of possible outcomes does occur by constructing a large family of initial data $w^{+(\mathcal{F})}$ which evolves into \mathcal{F} . We can also try to construct large families of initial data whose evolution is completely described.

$$\phi \rightarrow S_{NL}(\phi)$$

Stability properties of the zero solution



Conservation laws:

- $Q[u] = \int |u(t, x)|^2 dx$
- $H[u] = \int |u|^{2\sigma+2} + F(|u|^2) dx ; F' = f.$
 $\pm |u|^{2\sigma+2}$

$\Rightarrow L^2$ stability of zero solution.

defocusing: $L_x^2 + H_x^1 + L_x^{2\sigma+2}$ stability.

focusing: H_x^1 stability if $Q + H$ control H_x^1 -norm.

$$\begin{array}{c}
 H^{\frac{d}{2}} \\
 | \\
 H \\
 | \\
 s(r) \\
 | \\
 H^1 \\
 | \\
 H^0
 \end{array}
 \quad
 \begin{array}{c}
 L_x^0 \\
 | \\
 L_x^{2\sigma+2} \\
 | \\
 L_x^2
 \end{array}
 \quad
 \begin{aligned}
 s(r) &= \Theta \frac{d}{2} + (1-\Theta)^0 \\
 \frac{1}{2\sigma+2} &= \Theta \frac{1}{\infty} + (1-\Theta) \frac{d}{2} \Rightarrow \frac{1}{\sigma+1} = (1-\Theta) \\
 \Theta &= \frac{\sigma}{\sigma+1}
 \end{aligned}$$

$$\Rightarrow s(r) = \frac{\sigma}{\sigma+1} \frac{d}{2}.$$

$$\|u\|_{L^{2\sigma+2}} \sim \|u\|_{H^s(r)} \text{ and } \|u\|_{L^2}^{1-\alpha} \|u\|_{H^1}^\alpha \Rightarrow \alpha = s(r).$$

$$s(r) < 1$$



$$\boxed{r < \frac{2}{d-2}}$$

$$(2\sigma+2)s(r) < 2 \text{ for } \underline{\text{a priori }} H^1 \text{ estimate.}$$

$$2(\sigma+1) \frac{r}{2(\sigma+1)} d < 2$$

$$\boxed{r < \frac{2}{d}}$$

Scaling Considerations

$$\begin{cases} i\partial_t u + \Delta u + |u|^{2^*} u = 0 \\ u(0) = f \in H^s \end{cases} \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^d \longrightarrow \mathcal{Q}$$

$$u_\lambda(t, x) = \lambda^\alpha \phi(\lambda^2 t, \lambda x).$$

$$\alpha + 2 = \alpha [2^* + 1]$$

$$\frac{1}{\sigma} = \alpha$$

$$\|D_x^\sigma u_\lambda\|_{L^2} = \lambda^{\frac{1}{\sigma} + s - \frac{d}{2}} \|D_x^\sigma \phi_\lambda\|_{L^2}$$

$$\boxed{s_c = \frac{1}{2} - \frac{1}{\sigma}}.$$

If $\sigma = \frac{2}{d}$ (critical for H^1 control) then
 scaling invariance is at $s = 0$. L^2 -scaling invariant.

If $\sigma = \frac{2}{d-2}$ then H^1 is scaling invariant.

Consider the problem: Find the smallest constant C such that

$$\|u\|_{L^{2^{\sigma}+2}}^{2^{\sigma}+2} \leq C \|u\|_L^{2(\tau+1)-\sigma d} \|\nabla u\|_{H^1}^{\sigma d}$$

Form, for $v \in H^1$, the functional

$$J[v] = \frac{\|\nabla v\|_{H^1}^{\sigma d} \|v\|_L^{2(\tau+1)-\sigma d}}{\|v\|_{L^{2^{\sigma}+2}}^{2^{\sigma}+2}}$$

Theorem (Weinstein) For $0 < \sigma < \frac{2}{d-2}$,

$$\alpha := \inf_{v \in H^1} J[v]$$

is attained at a function R with the properties:

(i) $R \geq 0$, R is symmetric w.r.t. some origin,

(ii) $R \in H^1 \cap C^\infty$

(iii) R satisfies

$$\Delta R - R + R^{2^{\sigma}+1} = 0.$$

(R is called the ground state.)

(The theorem does not assert that infinizers of $J[v]$ must necessarily be R . Nor does it characterize the infima.)

Thus, NLS_σ , focusing is H^1 stable if $\tau d < 2$ (subcritical)

Summarized by saying
"Hamiltonian $\rightarrow L^2$ -norm control
the $H^1(R^d)$ -norm".

- if $\tau d = 2$ + L^2 -norm is small
(relative to ground state mass.)
- if $\tau d > 2$ + H^1 norm is sufficiently small.

H^s -stability of zero solution

Assume Hamiltonian + L^2 norm control the H^1 norm.

$s > 1$

If $\phi \in H^s(\mathbb{R}^d)$ with $\|\phi\|_{H^s} < \delta(\varepsilon)$, $\phi \mapsto u(t)$, $\|u(t)\|_{H^s} < \varepsilon$?

This is known in integrable case for $s \in \mathbb{N}$.

This is known if $\phi \mapsto u(t) \sim S(t) \phi_\pm$, i.e. if scattering holds.

Otherwise, the best estimates are of the form

$$\|u(t)\|_{H^s} \lesssim \varepsilon t^{x(s)}$$

Bourgain
Staffilani:
CDKS

No examples are known which show $\|u(t)\|_{H^s}$ can grow without bound in the NLS _{σ} family (assuming H^1 control).

$s < 1$

If $\phi \in H^s(\mathbb{R}^d)$ w. $\|\phi\|_{H^s} < \delta(\varepsilon)$, $\phi \mapsto u(t)$, $\|u(t)\|_{H^s} < \varepsilon$?

This probably follows if scattering is known.

Otherwise,

Theorem [CKSTT] For cubic NLS on \mathbb{R}

$$\|u(t)\|_{H^s} \lesssim \varepsilon t^{2s+}$$

(The proof should generalize to other settings.)

NLS_σ

$$\begin{cases} i\partial_t u + \Delta u + |u|^{2\sigma} u = 0 & x \in \mathbb{R}^d \\ u(0, x) = u_0(x). \end{cases}$$

locally power generalized NLS.

Assume σ subcritical w.r.t. blowup:
associated w.

$$\sigma < \frac{2}{d}$$

symmetry

Conserved Quantities

$$Q[u] = \left(\int |u(t, x)|^2 dx \right)^{\frac{1}{2}}$$

$$u \mapsto e^{i\theta} u$$

$$H[u] = \int [|\nabla u(t, x)|^2 - \frac{1}{\sigma+1} |u|^{2\sigma+2}] dx.$$

$$u(t) \mapsto u(t+t_0)$$

$$P[u] = \frac{1}{2} \operatorname{Im} \int [\bar{u}(t, x) \nabla u(t, x)] dx.$$

$$u(t, x) \mapsto u(t, x+x_0)$$

$$M[u] = \frac{1}{2} \operatorname{Im} \int x \times [\bar{u}(t, x) \nabla u(t, x)] dx.$$

$$u(t, x) \mapsto u(t, \theta x).$$

Given a solution $u(t, x)$ of NLS_σ , we can manufacture a parametrized family of solutions by symmetry group action. \exists galilean & (perhaps) "conformal" invariances as well.

Separated Solutions

The ansatz $u(t, x) = e^{iEt} R_E(x)$ leads to solutions of NLS_σ if R_E solves the semi-linear elliptic PDE

$$-ER + \Delta R - |R|^{2\sigma} R = 0. \quad (E > 0, \sigma < \frac{2}{d}).$$

Variational PDE
connected to H + EQ.

(time independent NLS_0)

[Strauss], [Brezis- Lions]: For $E > 0 \exists$ positive, radial, smooth, exponentially decaying solution of (time independent NLS_0) called R , the ground state.

The associated evolving function $e^{iEt} R_E$ is called a solitary wave solution of NLS_σ .

Uniqueness
uniqueness of ground states is understood.
this is known about uniqueness properties of non-positive solutions.

Note: There may (and do) exist solutions of (time independent NLS_σ) which are not ground states. These solutions have nodal structures which may characterize them. ~~These~~

Group Action

DATE _____

The symmetries of the equation applied to the ground state solitary wave trace out a group orbit aka parametrized family of solitons. We focus our attention on a subgroup of the full symmetry group by considering the ground state cylinder. $\Sigma_E \subset H^1(\mathbb{R}^2)$ where

$$\Sigma_E = \left\{ e^{i\theta} R_{E_x}(\cdot + x_0) : x_0 \in \mathbb{R}, e^{i\theta} \in S^1 \right\}.$$

The nonlinear flow of NLS_σ preserves Σ_E .

Remark A rescaling allows us to restrict attention to $E = 1$.

Why? Suppose R solves (time independent NLS_0). Form $R_{x\beta}(x) = \alpha R(\beta x)$, which is a rescaling of R and the independent variable x . Then $R_{x\beta}$ satisfies

$$-\frac{E}{\alpha} (R_{x\beta}) + \frac{1}{\alpha^2 \beta^2} \Delta R_{x\beta} - \frac{1}{\alpha^2 \beta^2} |R_{x\beta}|^2 R_{x\beta} = 0.$$

Now, restrict α, β such that $\beta = \alpha$ and multiply through by α^2 to obtain

$$-(E\alpha^2)(R_\alpha) + \Delta R_\alpha - |R_\alpha|^2 R_\alpha = 0.$$

Let $\Sigma = \Sigma_1$.

Notions of ground state stability

DATE _____

$u_0 \mapsto u(t)$ exists $\forall t$.

Suppose we know that $\forall t$

$$\text{dist}_{H^1}(u(t), \Sigma) \sim \text{dist}_{H^1}(u_0, \Sigma).$$

Then, we say the ground state cylinder is H^1 -orbitally-stable.

Orbital stability implies the existence of geometric parameters $x_0(t)$, $e^{i\theta_0(t)}$ $\forall t$ with the property that

$$\text{dist}_{H^1}(e^{i\theta_0(t)} u(t, x + x_0(t)), R) \sim \text{dist}_{H^1}(u_0, \Sigma).$$

It does not imply a description or representation of the maps $t \mapsto x_0(t)$, $t \mapsto \theta_0(t)$.

We obtain more information about the long time behavior if we can describe the evolutions $t \mapsto x_0(t)$, $t \mapsto \theta_0(t) \forall t$, perhaps as ode solutions, or as ode + weak coupling to pole, or some other description. Moreover, we may be able to show a convergence of the perturbed solution to an explicit modulated soliton in some non-conserved norm. These extensions of the orbital stability result are called asymptotic stability or perturbed solitary wave dynamics.

Quantifications of stability notions(NLS₀-setting)

orbital stability

The ground state cylinder Σ is stable if

$$\forall \delta_0 > 0 \quad \exists \alpha_0 > 0 \quad \text{s.t.} \quad \|u_0 - R\|_{H^1} \leq \alpha_0 \implies$$

 $\forall t > 0 \quad \exists \quad t \mapsto \tilde{\theta}_0(t), \quad t \mapsto \tilde{x}_0(t) \quad \text{s.t.}$

$$\|u(t) - e^{i\tilde{\theta}_0(t)} R(\cdot + \tilde{x}_0(t))\|_{H^1} \leq \delta_0.$$

(data α_0 -close in H^1 forever stays H^1 -close-to- Σ .)

The ground state cylinder Σ is asymptotically stable if

$$\exists \alpha_0 > 0 \quad \text{s.t.} \quad \|u_0 - R\|_{H^1} \leq \alpha_0 \implies$$

 $\forall t > 0 \quad \exists \quad t \mapsto \theta_0(t), \quad t \mapsto x_0(t) \quad \text{s.t.}$

$$e^{i\theta_0(t)} u(t, \cdot + x_0(t)) - R(x) \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in } H^1.$$

implied by proving
 $G \circ u - R$ "scatters".

These notions are adaptations from

[Mather-Mather: ~~stable~~]
ARMA

Orbital Stability: Intuition from the Dirichlet problem

Let $\Omega \subset \mathbb{R}^d$ be a nice domain. Assume $\overline{\pi_1}(\Omega)$ is an interval with 1d Lebesgue measure $< M$. Define

$$\mathcal{A}_g = \{w: \Omega \rightarrow \mathbb{R} \mid w = g \text{ on } \partial\Omega\}.$$

Suppose

$$I[w] = \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx,$$

and $\exists v \in \mathcal{A}_g$ such that $\forall w \in \mathcal{A}_g$

$$I[v] \leq I[w].$$

Theorem If $v \in \mathcal{A}_g$ satisfies

$$I[v] \leq I[u] + \delta$$

energy close to minimizing
↓

then

$$\|v - u\|_{H^1} \leq C\delta^{1/2}.$$

H^1 -close to minimizer.

proof (worked out w. J. Garciá following [Evans, p 449-450].)

Let $v = u + w$. Then $w = 0$ on $\partial\Omega$

$$I[v] = I[u + w] = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + 2\nabla u \cdot \nabla w + |\nabla w|^2 dx$$

$$= I[u] + \underbrace{\int_{\Omega} \nabla u \cdot \nabla w dx}_{0} + \frac{1}{2} \int_{\Omega} |\nabla w|^2.$$

$$= I[u] + \underbrace{\int_{\Omega} (-\Delta u) w dx}_{0} + \frac{1}{2} \int_{\Omega} |\nabla w|^2.$$

since u is
a minimizer.

$$-\Delta u = \frac{8I}{8u}.$$

Thus, for $v = u + w$

$$I[v] - I[u] = \frac{1}{2} \int_{\Omega} |\nabla w|^2 = I[w].$$

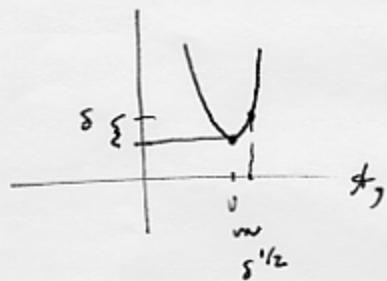
So, energy proximity to minimizer $I[u+w] - I[u]$
equals $(H^1\text{-distance})^2$ from the minimizer.

Poincaré's inequality gives us

$$\|v-u\|_{L^2(\Omega)} \leq M \|\nabla(v-u)\|_{L^2(\Omega)} \leq M\sqrt{s}$$

and this proves the claim. \blacksquare

Pictorially, we have a parabola:



We now begin a major topic. Our goal is to understand the recent breakthroughs by Y. Martel and F. Merle concerning asymptotic stability properties of generalized KdV solitons.

Martel - Merle

2001

"Asymptotic Stability of GKdV" ARMA ($p = 2, 3, 4$)

2000

"A Liouville Theorem..." JMPA (critical case $p = 5$)
"Instability noticed" GAFA ($p = 5$)

KdV_p

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0 & t, x \in \mathbb{R}^+ \times \mathbb{R} \\ u(0, x) = u_0(x), & x \in \mathbb{R} \\ & H^1. \end{cases} \quad 1 < p \in \mathbb{Z}.$$

Wellposedness

If $u_0 \in H^1$ $\exists T^* > 0$ and a unique maximal solution
 $u_0 \mapsto u(t) \in C([0, T^*]; H^1)$. Either $T^* = +\infty$ or
 $T^* < \infty$ and $\|u(t)\|_{H^1} \rightarrow \infty$ as $t \rightarrow T^*$.

$$\forall t \in [0, T^*) \quad \int u^2(t) dx = \int u_0^2 dx.$$

$$H[u] = \frac{1}{2} \int u_x^2(t) dx - \frac{1}{p+1} \int u^{p+1}(t) dx = \frac{1}{2} \int u_{0x}^2 dx - \frac{1}{p+1} \int u_0^{p+1} dx$$

Familiar Sobolev $\iff p = 2, 3, 4$ GWP in H^1
 $p = 5$ critical.

KdV_p has soliton solutions

$$u(t, x) = R_c(x - ct) \rightarrow \text{KdV}_p. \quad (\text{We seek } R_c \in H^1(\mathbb{R}) \text{ s.t.})$$

$$R_{c,xx} + R_c^p = c R_c.$$

$$\Rightarrow R_{cx}^2 + \frac{2}{p+1} R_c^{p+1} = c R_c^2.$$

$$R_c(x) = \left\{ \frac{c(p+1)}{2} \operatorname{sech}^2 \left(\frac{p-1}{2} \sqrt{c} x \right) \right\}^{\frac{1}{p-1}}.$$

One might expect that the solitons play a determining role in the long time behavior of the KdV evolution $t \mapsto u(t)$:

$$u(t) \sim \sum_i R_{c_i} (x - x_i(t)) + u_R \quad (\text{subcritical}),$$

\curvearrowleft
dispersive.

$$u(t) \sim \sum_i R_{c_i(t)} (x - x_i(t)) + u_R \quad (\text{critical}).$$

This expectation is encouraged by stability results in the neighborhood of solitons.

Theorem [Machl-Merk] (Asymptotic stability for $p=2, 3, 4$).
 Let $p=2, 3, 4$ and let $c_0 > 0$. Let $u_0 \in H^1(\mathbb{R})$, $u_0 \mapsto u(t)$.
 $\exists x_0 > 0$ s.t. if $\|u_0 - R_{c_0}\|_{H^1} < \infty$ then $\exists c_{+\infty} > 0$
 and $t \mapsto x(t)$ s.t.

$$u(t, \cdot + x(t)) \rightarrow R_{c_{+\infty}} \quad \text{in } H^1 \text{ as } t \rightarrow \infty.$$

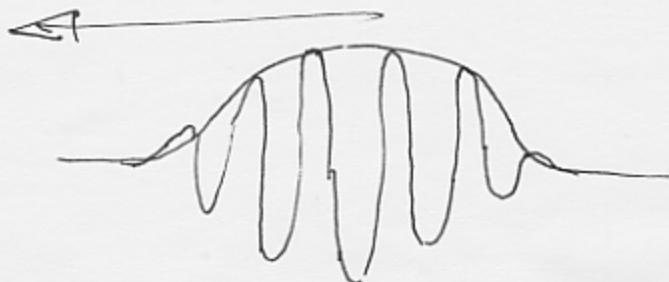
This result relies on a Liouville Theorem:

Theorem [mm] (Liouville property close to R_{c_0} for $p=2, 3, 4$):

Let $p=2, 3, 4$ and $c_0 > 0$. $H^1(\mathbb{R}) \ni u_0 \mapsto u(t) \forall t \in \mathbb{R}$.

(initially close to soliton) $\exists x_0 > 0$ s.t. $\|u_0 - R_{c_0}\|_{H^1} < \infty$ and
 If $\exists t \mapsto x(t)$ s.t. $V(t, x) = u(t, x + x(t))$ satisfies
 (no dispersion) $\forall \delta > 0 \exists A_0 > 0, \forall t \in \mathbb{R} \int_{|x_0| > A_0} V^2(t, x) dx \leq \delta$ (L^2 compactness)
 then $\exists c_1 > 0, x_1 \in \mathbb{R}$ s.t.
 (only solitons) $\forall t \in \mathbb{R}, x \in \mathbb{R} \quad u(t, x) = R_{c_1}(x - x_1 - c_1 t)$.

- Remarks
- ① One might try to drop the (initially close to solution) hypothesis and retain the result. However, for $p=3$ (mkdV) \exists solutions:



called "breathers".

The breathers are not well understood. Are they stable? Do they play a role in the $p=3$ long time behavior? (Ask Randall Pyke)

- ② The proof imposes the decay property hypothesized in the (no dispersion) hypothesis to uniform pointwise exponential decay. We make a weak hypothesis consistent with properties of an asymptotic object and then characterize such objects.

contradiction
 ε is nevertheless
 L^2 -compact.

Assume asym. behavior for ε . fails, where ε is freq. modulated difference.

\exists small-in- H^1 $\varepsilon(s)$ s.t. $\varepsilon(s) \not\rightarrow 0$ in H^1 as $s \rightarrow \infty$.

We use control on mass at right to prove ε satisfies L^2 -compactness, "no dispersion".

reduction to linear leviouville. renormalized time snapshots of ε converge to a nonzero function w that satisfies a linear equation and w is exponentially localized and meets orthogonality conditions.

linear
leviouville

Such a function $w \equiv 0$.

c!

We outline the structure of the argument which pivots off of two statements concerning solutions of a frequency modulated difference equation.

Outline of proof

1. Problem setup / difference equation ε -equation
 - scaling restricts attention to standard ground state. Q
 - derivation of pde for difference of modulated solution $V(t, y)$; and standard ground state. reexpressing
 - orthogonality conditions / choice of modulation parameters.
 $t \mapsto \lambda(t)$, $t \mapsto X(t)$.

Statements for ε -problem (recasting of main [MM] theorems)

Liouville Theorem for ε -equation.

- $\|\varepsilon(\cdot)\|_{H^1}$ small enough small
 - $\forall s, (\varepsilon(s), \varrho) = (\varepsilon(s), Q_Y) = 0$ orthog. condit. \downarrow
 - $\exists s_0 \exists A_0(s_0) > 0$ s.t. $\forall s \in \mathbb{R}$ no dispersion
- $$\|\varepsilon(s)\|_{L^2(|y| > A_0)} \leq s_0$$
- $$\implies \varepsilon \equiv 0 \text{ on } \mathbb{R} \times \mathbb{R}. \quad \downarrow \text{zero}$$

Asymptotic Behavior for ε .

Let $\varepsilon \in C(\mathbb{R}_t, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}_t, H^1(\mathbb{R}))$ solve ε -equation.

$\exists a_2 > 0$ s.t. $\|\varepsilon(\cdot)\|_{H^1} \leq a_2 \implies \varepsilon(s) \rightarrow 0$ in H^1
as $s \rightarrow \infty$.

2. Asymptotic behavior for Σ $\xleftarrow{\text{Section 3 ARMA}}$ Liouville Thm for Σ equation.
- Part C JMPA
- (Via contradiction)
 - introduce quantity to measure Mass at right of soliton.
 - control time variation of mass on right.
- $\Rightarrow L^2$ -compactness, "no dispersion": \mapsto Liouville.

- Prove $\lambda(t)$ converges as $t \rightarrow \infty$.
 - Monotony property in L^2
 - Subcriticality

3. Liouville property close to Q $\xleftrightarrow{\text{Section 4 ARMA}}$ linear Liouville property
- Part A JMPA

- renormalized sequence $w_n = \frac{\Sigma_n}{a_n}$; $a_n = \sup_{s \in \mathbb{R}} \|\Sigma_n(s)\|_{H^1}$.

- L^2 compactness $\Rightarrow \forall s \in \mathbb{R} \quad \forall y \in \mathbb{R}$
 $|w_n(s, y)| \leq c e^{-c_2 |y|}$

$$0 < c \leq \|w_n(s)\|_{L^2}.$$

rescaled Σ 's satisfy
a linear equation at
time infinity.

Compare this discussion
W. [Merk - Vega].

• Σ_n is decomposed into
time decaying + (localized w decay on right)

\uparrow
Monotony property
in L^2 .

4. linear Liouville property (The object w_n does not exist.)
- reduces to study of a quadratic form
explicit calculations for classical linear operators.

1. Problem formulation

Our goal is to show the family of solutions $\{R_c(x - x_0 - ct) \mid c > 0, x_0 \in \mathbb{R}\}$ is asymptotically stable in H^1 :

$$\exists x_0 > 0 \text{ s.t. } \|u_0 - R_c\|_{H^1} \leq x_0 \implies \forall t \geq 0 \ \exists c(t), x(t) \text{ s.t. } u(t, \cdot + x(t)) - R_{c(t)} \xrightarrow[t \rightarrow +\infty]{} 0 \text{ in } H^1.$$

$$c(0) = c_0$$

$$c_\infty = \lim_{t \rightarrow \infty} c(t) \text{ if limit exists.}$$

Scaling invariance ($\forall \lambda_0 > 0 \quad \lambda_0^{\frac{2}{p-1}} u(\lambda_0^3 t, \lambda_0 x)$ is also a solution)

\implies we may restrict $c_0 = 1$.

$$\text{Set } Q(x) = R_1(x) = \left\{ \frac{p+1}{2} \operatorname{sech}^2 \left(\frac{p-1}{2} x \right) \right\}^{\frac{1}{p-1}}, \text{ so}$$

$$Q_{xx} = Q - Q^p.$$

We restrict our attention to data $u_0 \in H^1$ such that

$\|u_0 - Q\|_{H^1} \leq x_0$, chosen later to be small,
and the issue is the long time behavior of the evolution
 $u_0 \mapsto u(t)$.

Orbital stability of subcritical KdVp is known:

$\exists y(t) \text{ s.t. } \forall t$

$$\|u(t) - Q(x - y(t))\|_{H^1} \leq \varepsilon(x_0)$$

where $\varepsilon(x_0) \rightarrow 0$ as $x_0 \rightarrow 0$.

The natural invariances of the equation are the scaling and translation invariances. These are parametrized by $0 < \lambda_0, x_0 \in \mathbb{R}$.

We introduce modulation parameter evolutions $\lambda(t)$, $x(t)$ and consider

$$v(t, y) = \lambda(t)^{\frac{2}{p-1}} u(t, \lambda(t)y + x(t)), \quad \text{How to determine?}$$

$$\varepsilon(t, y) = v(t, y) - Q(y). \quad \text{This choice does not "respect" the way time rescales when } x \mapsto \lambda x.$$

The goal is to show $\varepsilon(t) \xrightarrow{H^1} 0$ as $t \rightarrow \infty$.

[The modulation parameter evolutions are chosen so that the orthogonality conditions hold:

$$\forall t \in \mathbb{R} \quad (\varepsilon(t), Q) = (\varepsilon(t), Q_y) = 0.$$

Why? (explain this choice geometrically, explain impact of this choice in subsequent analysis, is there an optimal choice of orthogonality conditions? How do the orthogonality conditions determine $\lambda(t), x(t)$? ...)

We introduce a respectfully rescaled time variable s :

$$s = \int_0^{t'} \frac{dt}{\lambda^3(t)} \iff \frac{ds}{dt} = \frac{1}{\lambda^3}. \quad \text{as } s \rightarrow \infty \text{ as } t \rightarrow \infty$$

Then $\varepsilon(s, y)$ is our object of concern: $\varepsilon(s) \xrightarrow{H^1} 0$ as $s \rightarrow \infty$.

Exercise Find the equation $\varepsilon(s, y)$ satisfies.

Hint: calculate $v_t, v_y, v_{yy} \Rightarrow$ equation v solves.

$$\frac{d}{dt} = \frac{1}{\lambda^3} \frac{d}{ds} \text{ replacement gives } v_s + \dots = 0 \text{ eg.}$$

$$\text{pass to } \varepsilon \text{ eg using } v = \varepsilon + Q.$$

$\forall s \in \mathbb{R}, y \in \mathbb{R}$ we have the " ε -equation":

$$\left[\begin{aligned} \varepsilon_s &= (L\varepsilon)_y + \frac{\lambda s}{\lambda} \left(\frac{2Q}{P-1} + y Q_y \right) + \left(\frac{x_s}{\lambda} - 1 \right) Q_y \\ &\quad + \frac{\lambda s}{\lambda} \left(\frac{2\varepsilon}{P-1} + y \varepsilon_y \right) + \left(\frac{x_s}{\lambda} - 1 \right) \varepsilon_y - \underbrace{\left[(Q+\varepsilon)^P - (Q^P + P Q^{P-1} \varepsilon) \right]}_{\leq C(\varepsilon^2)}. \end{aligned} \right]$$

where

$$L\varepsilon = L_p \varepsilon = -\varepsilon_{xx} + \varepsilon - P Q^{P-1} \varepsilon.$$