McMaster Talk 25 January 2002.

(joint w. M. Christ + T. Tao.)

$$KdV_p^{\pm} \quad \begin{cases} \partial_t u + \partial_x^3 u \pm \partial_x\left(\frac{1}{p+1} u^{p+1}\right) = 0 & u : [0,\infty) \times \mathbb{R} \to \mathbb{R} \\ u(0,x) = \phi(x). \end{cases}$$

Q: Does $\pm$ choice result in distinct equations?

① equation under $u \mapsto -u$ transformation:

$$-\partial_t u - \partial_x^3 u \pm (-1)^{p+1} \partial_x\left(\frac{1}{p+1} u^{p+1}\right) = 0$$

$$p+1 \begin{cases} \text{even} & \pm \text{ flips to } \mp \text{ under } u \mapsto -u. \\ \text{odd} & \text{equation is invariant under } u \mapsto -u. \end{cases}$$

② Hamiltonian Structure.

$$H[u] = \int \tfrac{1}{2} u_x^2 \mp \frac{1}{(p+2)(p+1)} u^{p+2} \, dx. \quad \text{is conserved.}$$

$$u_t = \partial_x \frac{\delta H}{\delta u} \iff u_t + u_{xxx} + \left(\frac{1}{p+1} u^{p+1}\right)_x = 0.$$

$$p+2 \begin{cases} \text{odd} & u \mapsto -u \text{ switches } \mp \text{ to } \pm. \\ \text{even} & u \mapsto -u \text{ leaves } H[u] \text{ invariant.} \end{cases}$$

If $p+1$ is odd, $KdV_p^{\pm}$ is $\begin{matrix} + \text{ focusing} \\ - \text{ defocusing} \end{matrix}$.

If $p+1$ is even, $KdV_p^{\pm}$ is neither focusing nor defocusing.

$KdV_p^{\pm}$ is <u>locally well-posed in a Banach space $\mathcal{A}$</u> if $\exists\ T = T(\|\phi\|_{\mathcal{A}}) > 0$ and a "solution" of $KdV_p^{\pm}$ s.t.

(1) $u \in C([0,T]; \mathcal{A}) \cap \mathcal{Y}_T = X_T$.

(2) The data-to-solution map from $\mathcal{A} \mapsto X_T$ is uniformly continuous: $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$$\|\phi^1 - \phi^2\|_{\mathcal{A}} < \delta \implies \|u^1 - u^2\|_{X_T} < \varepsilon$$

with $\delta = \delta(\varepsilon, M)$ where $\|\phi^{1,2}\|_{\mathcal{A}} \le M$.

<u>Sharp well-posedness problem</u>

Find the largest space $\mathcal{A}$ of initial data for which $KdV_p^{\pm}$ is LWP.

We restrict our attention to $\mathcal{A}$ among the L^2-based Sobolev spaces $H^s(\mathbb{R})$. $\exists$ recent work (Vargas-Vega, Tataru, CKS, ...) pushing the envelope outside the $H^s(\mathbb{R})$ scale.

<u>Scaling Heuristic</u>

Suppose u ⓢ $KdV_p^{\pm}$. Form $\forall\ \sigma > 0$

$$u_\sigma(x,t) = \sigma^{\alpha} u(\sigma x, \sigma^3 t).$$

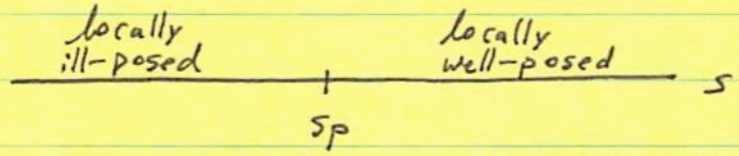
Choose α such that $\alpha + 3 = \alpha(p+1) + 1$

$$\alpha = \frac{2}{p}.$$

Then u_σ also ⓢ $KdV_p^{\pm}$.

$$\| D_x^s u(t,x) \|_{L^2_x} = \sigma^{\frac{2}{p}+s-\frac{1}{2}} \| D_x^s u_{\sigma=1}(t,x) \|_{L^2_x}$$

Scaling invariant Sobolev index : $\boxed{s_p = \frac{1}{2} - \frac{2}{p}.}$



(heuristic).

<u>Known Results</u> [KPV, BKPSV, G]

p	Scaling	H^s well-posedness	H^s ill-posedness
1	$-\frac{3}{2}$	$-\frac{3}{4} \le s$	~~C~~-KdV: $s < -\frac{3}{4}$
2	$-\frac{1}{2}$	$\frac{1}{4} \le s$	~~focusing~~: $s < \frac{1}{4}$
3	$-\frac{1}{6}$	$-\frac{1}{6} < s$	$s < -\frac{1}{6}$
$p \ge 4$	$\frac{1}{2} - \frac{2}{p}$	$s_p \le s$	focusing : $s < s_p$.

[CCT] results.

We expect the ill-posedness result to extend to $KdV^{\text{defocusing}}_{p \ge 4}$.

We conjecture KdV is GWP in $H^{\frac{3}{4}}(\mathbb{R})$ [CKSTT].

using techniques I described here 3~~4~~6 days ago

$\exists$ corresponding story for <u>cubic NLS</u> on $\mathbb{R}$:

$$\begin{cases} -i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0 , & u : (0,\infty) \times \mathbb{R} \to \mathbb{C} \\ u(0,x) = \phi(x) . & \end{cases}$$

$+$ focus
$-$ defocus

Scaling	H^s well-posedness	H^s ill-psedness
$-\frac{1}{2}$	$0 \le s$ [T, CW, GV]	~~focusing~~ : $s < 0$.

This, in some sense, is our principal result

Psuedo-Conformal Transformation

The formulas

$$u(x,t) = \int e^{ix\xi} e^{-it|\xi|^2} \widehat{\phi}(\xi) d\xi = \frac{c}{(it)^{\frac{1}{2}}} \int e^{-i\frac{|x-y|^2}{4t}} \phi(y) dy$$

represent solutions of $i\partial_t u + \partial_x^2 u = 0,\ u(0) = \phi$. Replacing t bt $\frac{1}{t}$ in the multiplier formula reveals an expression similar to the convolution representation. Following this idea....

$$(y,s) := (\frac{x}{1+t}, \frac{1}{1+t}),\ (x,t) = (\frac{y}{s}, \frac{1^{-\varsigma}}{s}).$$

$v = pc(u),\ u = pc^{-1}(u)$ defined by

$$u(x,t) = (1+t)^{-\frac{1}{2}} e^{-\frac{ix^2}{4(t+1)}}\ v(y,s),$$

$$v(y,s) = s^{-\frac{1}{2}} e^{-\frac{iy^2}{4s}}\ u(x,t).$$

Properties of pc:

$$pc \text{ is a linear isometry on } L^2(\mathbb{R}).$$

$$iv_s + v_{yy} = s^{-\frac{5}{2}}\ e^{\frac{iy^2}{4s}}(-iu_t + u_{xx}),\ \forall u.$$

New Results

The following statements are proved in work in progress with **Mike Christ** and **Terry Tao**.

Theorem [CCT]: Defocussing cubic NLS is ill-posed in $H^s(\mathbb{R}),\ s < 0$ (scaling is $-\frac{1}{2}$).
Defocussing KdV_2 is ill-posed in $H^s(\mathbb{R}),\ s < \frac{1}{4}$.
$\mathbb{R}$-valued KdV_1 is ill-posed in $H^s(\mathbb{R}),\ -1 \leq s < -\frac{3}{4}$.

Theorem [CCT]: KdV_1 is LWP in $H^{-\frac{3}{4}}(\mathbb{R})$.

We conjecture that KdV_1 is GWP in $H^{-\frac{3}{4}}(\mathbb{R})$. However, the space-time space Y_T encountered in the local proof at $-\frac{3}{4}$ differs from the $X_{s,b}$-space used for $-\frac{3}{4} < s$ so a direct adaptation of **[CKSTT]** does not work.

[BKPVS] ill-posedness results exploit <u>solitary wave solutions</u>

$$u(x,t) = f(x-t) \longrightarrow KdV_p^+ \longrightarrow \text{ODE}. \quad (\text{Assume } f \text{ decays})$$

$\Longrightarrow$

$$f_p(x-t) = \left(\frac{p+2}{2}\right)^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}(x-t)\right).$$

These can be scaled

$$f_{p,\sigma}(x-\sigma^2 t) = \sigma^{\frac{2}{p}} c_p \operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}\sigma(x-\sigma^2 t)\right)$$

Define sequences $\{\sigma_1\}$, $\{\sigma_2\}$ such that $\sigma_i \longrightarrow \infty$ with

$$\frac{\sigma_1}{\sigma_2} \longrightarrow 1$$

$$\sigma_2^2 - \sigma_1^2 \longrightarrow \infty.$$

e.g. $\sigma_1 = n$, $\sigma_2 = n+1$.

$$\| D_x^{s_p} \left(f_{p,\sigma_1}(y) - f_{p,\sigma_2}(y)\right) \|_{L^2_x} \longrightarrow 0, \text{ as } \sigma_i \longrightarrow \infty.$$

Fix t. As $\sigma_i \longrightarrow \infty$

$$\| D_x^{s_p} \left(f_{p,\sigma_1}(x-\sigma_1^2 t) - f_{p,\sigma_2}(x-\sigma_2^2 t)\right) \|_{L^2_x} \gtrsim 1.$$

Hence, there is no uniform continuity in $\dot{H}^{s_p}$.

Defocussing problems d/n have solitary wave solutions.

$$u(x,t) = f(x-ct) \longrightarrow u_t + u_{xxx} - u^2 u_x = 0.$$

$$-cf' + f''' - f^2 f' = 0.$$

Multiply by xf and integrate

$$\int \left[-cx\left(\tfrac{1}{2}f^2\right)' + xff''' - x\left(\tfrac{1}{4}f^4\right)' \right] dx = 0.$$

$$\Longrightarrow \int \left[\frac{c}{2} f^2 - \underbrace{(xf)'}_{(f+xf')} f'' + \frac{1}{4} f^4 \right] dx = 0$$

$$+ (f')^2 - x\left(\tfrac{1}{2}(f')^2\right)'$$

$$\rightsquigarrow \int \left[\frac{c}{2} f^2 + \frac{3}{2}(f')^2 + \frac{1}{4} f^4 \right] dx = 0.$$

If $c \geq 0$ then $f = 0$.

A similar argument shows this for $c < 0$.

ODE Approximation Lemma: Let $w \in \tilde{H}^7(\mathbb{R})$ have norm $O(\epsilon)$ for small enough ϵ. Then $\exists\ v_1(w) = v_1 \in \tilde{H}^7(\mathbb{R})$ and an evolution $v_1 \longmapsto v^{\langle w \rangle}$ solving $pc(NLS)$ such that

$$\|\boldsymbol{v}^{\langle w \rangle} - \boldsymbol{v}^{[w]}\|_{\tilde{H}^5(\mathbb{R})} \lesssim \boldsymbol{\epsilon s}(\mathbf{1} + |\log \boldsymbol{s}|)^{\boldsymbol{C}}.$$

The map $w \longmapsto v^{\langle w \rangle}$ is Lispchitz from $\{w \in \tilde{H}^7 : \|w\|_{\tilde{H}^7} \lesssim \epsilon\}$ to $L^\infty((0,1]; \tilde{H}^5_y)$:

$$\sup_{0<s\leq 1} \|\boldsymbol{v}^{\langle w' \rangle} - \boldsymbol{v}^{\langle w \rangle}\|_{\tilde{H}^5_y} \lesssim \|\boldsymbol{w}' - \boldsymbol{w}\|_{\tilde{H}^7_y}.$$

(The proof uses Duhamel's formula for the difference between the two evolutions.) energy estimates

Decoherence Lemma: If $a, a' \in [\frac{1}{2}, 2]$ and $a \neq a'$ then

$$\limsup_{s\to 0+} \|\boldsymbol{v}^{[aw]}(\boldsymbol{s}) - \boldsymbol{v}^{[a'w]}\|_{L^2_y} \gtrsim \mathbf{1}.$$

(Wait a long time and the log pushes the phases apart.)

The initial value problem for defocussing cubic NLS transforms under pc into $pc(NLS)$:

$$iv_s + v_{yy} = \frac{1}{s}|v|^2 v,$$

$$v(1, y) = v_1(y) = e^{iy^2}\phi(y).$$

$(t \in [0 \longrightarrow \infty)$ corresponds to $s \in (0 \longleftarrow 1].)$

We ignore the dispersive term v_{yy} and consider the ODE

$$iv_t = \frac{1}{s}|v|^2 v.$$

(This idea was suggested to us by Kenji Nakanishi and goes back to work of Ozawa.) $\forall\ w : \mathbb{R} \longmapsto \mathbb{C}$, the ODE has explicit solutions

$$v^{[w]}(s, y) := w(y)e^{-i|w(y)|^2 \log s}.$$

Note that:

$$|\partial_y^2 v^{[w]}| \lesssim |\log s|^2 \ll \frac{1}{s}$$

$v^{[w]}$ forms a singularity as$s \longrightarrow 0^+$.

Combining the two lemmas and pc^{-1}, we can prove that the L^2 global well-posedness of cubic NLS is **not uniform in time.** (Our argument actually shows the global well-posedness of cubic NLS in H^5 is not uniform in time. The approximation via the ODE solution also proves there is no scattering in L^2 for the cubic defocussing NLS evolution.

Using the lemmas, pc^{-1}, and the dilation and Galilean invariances, we construct global-in-time solutions $u^{\langle a \rangle}$, $u^{\langle a' \rangle}$ of defocussing cubic NLS satisfying: Let $-5 < s < 0$, $0 < \delta \ll \epsilon < 1$, $T > 0$ be arbitrary,

$$\|u^{\langle a \rangle}(0)\|_{H_x^s}, \|u^{\langle a' \rangle}(0)\|_{H_x^s} \lesssim \epsilon,$$

$$\|u^{\langle a \rangle}(0) - u^{\langle a' \rangle}(0)\|_{H_x^s} \lesssim \delta,$$

$$\sup_{0 \leq t < T} \|u^{\langle a \rangle}(t) - u^{\langle a' \rangle}(t)\|_{H_x^s} \gtrsim \epsilon.$$