\[ \text{KdV}_p^\pm \]

\[ \begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} &\pm \partial_x \left( \frac{1}{p+1} u^{p+1} \right) = 0 \\
u(0, x) &= \phi(x).
\end{aligned} \]

Q: Does \pm choice result in distinct equations?

1. Equation under \( u \to -u \) transformation:
\[-\partial_x u - \partial_x^3 u \pm (-1)^{p+1} \partial_x \left( \frac{1}{p+1} u^{p+1} \right) = 0\]

\[ \begin{aligned}
\text{even} &\quad \pm \text{ flips to } \mp \\
\text{odd} &\quad \text{equation is invariant under } u \to -u.
\end{aligned} \]

2. Hamiltonian Structure.

\[ H[u] = \int \frac{1}{2} u_x^2 + \frac{1}{(p+2)(p+1)} u^{p+2} \, dx \quad \text{is conserved.} \]

\[ u_t = \partial_x \frac{\delta H}{\delta u} \iff u_t + u_{xxx} + \left( \frac{1}{p+1} u^{p+1} \right)_x = 0. \]

\[ \begin{aligned}
\text{odd} &\quad u \to -u \text{ switches } \mp \text{ to } \pm \\
\text{even} &\quad u \to -u \text{ leaves } H[u] \text{ invariant.}
\end{aligned} \]

If \( p+1 \) is odd, KdV\(_p^\pm\) is \text{ focusing}.

If \( p+1 \) is even, KdV\(_p^\pm\) is neither focusing nor defocusing.
KdV$^\pm_p$ is locally well-posed in a Banach space $\mathcal{A}$ if there exists $T = T(\|u\|_{\mathcal{A}}) > 0$ and a "solution" of KdV$^\pm_p$ s.t.

1. $u \in C([0,T]; \mathcal{A}) \cap X_T = X_T$. 

2. The data-to-solution map from $\mathcal{A} \rightarrow X_T$ is uniformly continuous: $\forall \varepsilon > 0 \exists \delta > 0$ s.t.

$$\|\phi' - \phi_0'\| < \delta \Rightarrow \|u' - v'\|_{X_T} < \varepsilon$$

with $\delta = \delta(\varepsilon, M)$ where $\|\phi^{1/2}\|_{\mathcal{A}} \leq M$.

**Sharp well-posedness problem**

Find the largest space $\mathcal{A}$ of initial data for which KdV$^\pm_p$ is LWP.

We restrict our attention to $\mathcal{A}$ among the $L^2$-based Sobolev spaces $H^s(\mathbb{R})$. Recent work (Vargas-Vega, Tataru, CKS, ...) pushing the envelope outside the $H^s(\mathbb{R})$ scale.

**Scaling Heuristic**

Suppose $u \in \mathcal{A}$ KdV$^\pm_p$. Form $\forall \sigma > 0$

$$u_\sigma(x,t) = \sigma^{-\alpha} u(\sigma x, \sigma^{3-\alpha} t).$$

Choose $\alpha$ such that $\alpha + 3 = \alpha(p+1) + 1$

$$\alpha = \frac{2}{p-1}.$$

Then $u_\sigma$ also $\in \mathcal{A}$ KdV$^\pm_p$. 
\[ \| D_x^s u(t, x) \|_{L^2_x} = \sqrt{\frac{2}{p} + \frac{1}{2}} \| D_x^s u_{-1}(t, x) \|_{L^2_x} \]

Scaling invariant Sobolev index: \( sp = \frac{1}{2} - \frac{3}{p} \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( sp )</th>
<th>( H^s ) well-posedness</th>
<th>( H^s ) ill-posedness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\frac{3}{2})</td>
<td>(-\frac{3}{4} \leq s )</td>
<td>( \text{KdV} ): ( s &lt; -\frac{3}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{1}{2})</td>
<td>( s \leq \frac{1}{2} )</td>
<td>focusing: ( s &lt; \frac{1}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{1}{6})</td>
<td>(-\frac{1}{6} &lt; s )</td>
<td>focusing: ( s &lt; -\frac{1}{6} )</td>
</tr>
<tr>
<td>( p \geq 4 )</td>
<td>( \frac{1}{2} - \frac{3}{p} )</td>
<td>( sp \leq s )</td>
<td>focusing: ( s &lt; sp ).</td>
</tr>
</tbody>
</table>

\[ \text{[cct]} \] results.

We expect the ill-posedness result to extend to \( \text{KdV} \) for \( p \geq 4 \).

We conjecture \( \text{KdV} \) is GWP in \( H^\frac{3}{4} (\mathbb{R}) \) \[ \text{[cct]} \].

The corresponding story for cubic NLS on \( \mathbb{R} \):

\[ \begin{cases} -i \partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \\ u(0, x) = \phi(x) \end{cases} \]

Scaling \( -\frac{1}{2} \)

\( 0 \leq s \) \([ T, CW, GV] \) focusing: \( s < 0 \).
Psuedo-Conformal Transformation

The formulas

$$u(x, t) = \int e^{ix\xi} e^{-it|\xi|^2} \hat{\phi}(\xi) d\xi = \frac{c}{(it)^{\frac{1}{2}}} \int e^{-\frac{i|x-y|^2}{4t}} \phi(y) dy$$

represent solutions of $i\partial_t u + \partial_x^2 u = 0$, $u(0) = \phi$. Replacing $t$ by $\frac{1}{t}$ in the multiplier formula reveals an expression similar to the convolution representation. Following this idea . . .

$$(y, s) := \left(\frac{x}{1 + t}, \frac{1}{1 + t}\right), \quad (x, t) = \left(\frac{y}{s}, \frac{1}{s}\right).$$

$v = pc(u)$, $u = pc^{-1}(u)$ defined by

$$u(x, t) = (1 + t)^{-\frac{1}{2}} e^{-\frac{ix^2}{4(t+1)}} v(y, s),$$

$$v(y, s) = s^{-\frac{1}{2}} e^{-\frac{iy^2}{4s}} u(x, t).$$

Properties of $pc$:

$pc$ is a linear isometry on $L^2(\mathbb{R})$.

$$iv_s + v_{yy} = s^{-\frac{5}{2}} e^{\frac{iy^2}{4s}} (-iu_t + u_{xx}), \quad \forall u.$$
New Results

The following statements are proved in work in progress with Mike Christ and Terry Tao.

**Theorem [CCT]:** Defocussing cubic \( NLS \) is ill-posed in \( H^s(\mathbb{R}) \), \( s < 0 \) (scaling is \( -\frac{1}{2} \)).
Defocussing \( KdV_2 \) is ill-posed in \( H^s(\mathbb{R}) \), \( s < \frac{1}{4} \).
\( \mathbb{R} \)-valued \( KdV_1 \) is ill-posed in \( H^s(\mathbb{R}) \), \( -1 \leq s < -\frac{3}{4} \).

**Theorem [CCT]:** \( KdV_1 \) is LWP in \( H^{-\frac{3}{4}}(\mathbb{R}) \).

We conjecture that \( KdV_1 \) is GWP in \( H^{-\frac{3}{4}}(\mathbb{R}) \). However, the space-time space \( Y_T \) encountered in the local proof at \( -\frac{3}{4} \) differs from the \( X_{s,b} \)-space used for \( -\frac{3}{4} < s \) so a direct adaptation of [CKSTT] does not work.
[BKPV5] ill-posedness results exploit solitary wave solutions

\[ u(x, t) = f(x - vt) \rightarrow \text{KdV}_p^+ \rightarrow \text{ODE}. \quad (\text{Assume } f \text{ decays}) \]

\[ f_p(x - vt) = \left( \frac{p + 2}{2} \right)^{\frac{p}{2}} \text{sech} \left( \frac{p}{2} (x - vt) \right). \]

These can be scaled

\[ f_{p_1, \sigma} (x - \sigma^2 t) = \frac{\sigma^2}{c_p} \text{sech} \left( \frac{p}{2} \sigma (x - \sigma^2 t) \right) \]

Define sequences \( \xi_{\sigma_1}, \xi_{\sigma_2} \) such that \( \sigma_i \rightarrow \infty \)
with

\[ \frac{\sigma_1}{\sigma_2} \rightarrow 1 \]

\[ \sigma_2^2 - \sigma_1^2 \rightarrow \infty. \]

e.g. \( \sigma_1 = n, \sigma_2 = n + 1 \).

\[ \| D_x^{\sigma_2^p} \left( f_p, \sigma_1 (y) - f_p, \sigma_2 (y) \right) \|_{L_x^2} \rightarrow 0, \text{ as } \sigma_i \rightarrow \infty. \]

Fix \( t. \) As \( \sigma_i \rightarrow \infty \)

\[ \| D_x^{\sigma_2^p} \left( f_p, \sigma_1 (x - \sigma_1^2 t) - f_p, \sigma_2 (x - \sigma_2^2 t) \right) \|_{L_x^2} > 1. \]

Hence, there is no uniform continuity in \( H^{\sigma_p}. \)
Defocusing problems do have solitary wave solutions.

\[ U(x,t) = f(x-c(t) \rightarrow u_t + u_{xxx} - u^2 u_x = 0. \]

\[-c f' + f''' - f^2 f' = 0. \]

Multiply by \( xf \) and integrate

\[ \int \left[ -c x (\frac{1}{2} f^2)' + x f f''' - x \left( \frac{4}{3} f f' \right)' \right] dx = 0. \]

\[ \int \left[ \frac{c}{2} f^2 - (xf)' f'' + \frac{4}{3} f f' \right] dx = 0 \]

\( (f+xf') \)

\( + (f')^2 - \kappa (\frac{4}{3} f f')' \)

\[ \Rightarrow \int \left[ \frac{c}{2} f^2 + \frac{3}{2} (f')^2 + \frac{1}{4} f f' \right] dx = 0. \]

If \( c > 0 \) then \( f = 0. \)

A similar argument shows this for \( c < 0. \)
**ODE Approximation Lemma:** Let \( w \in \tilde{H}^7(\mathbb{R}) \) have norm \( O(\varepsilon) \) for small enough \( \varepsilon \). Then \( \exists \ v_1(w) = v_1 \in \tilde{H}^7(\mathbb{R}) \) and an evolution \( v_1 \mapsto v^{(w)} \) solving \( pc(NLS) \) such that

\[
\|v^{(w)} - v^{[w]}\|_{\tilde{H}^5(\mathbb{R})} \lesssim \varepsilon s(1 + |\log s|)^C.
\]

The map \( w \mapsto v^{(w)} \) is Lispchitz from \( \{w \in \tilde{H}^7 : \|w\|_{\tilde{H}^7} \lesssim \varepsilon\} \) to \( L^\infty((0, 1]; \tilde{H}_y^5) \):

\[
\sup_{0 < s \leq 1} \|v^{(w')} - v^{(w)}\|_{\tilde{H}_y^5} \lesssim \|w' - w\|_{\tilde{H}_y^7}.
\]

(The proof uses Duhamel's formula for the difference between the two evolutions.)

**Decoherence Lemma:** If \( a, a' \in [\frac{1}{2}, 2] \) and \( a \neq a' \) then

\[
\limsup_{s \to 0^+} \|v^{[aw]}(s) - v^{[a'w]}\|_{L^2_y} \gtrsim 1.
\]

(Wait a long time and the log pushes the phases apart.)
The initial value problem for defocussing cubic \( NLS \) transforms under \( pc \) into \( pc(NLS) \):

\[
i v_s + v_{yy} = \frac{1}{s} |v|^2 v,
\]

\[
v(1, y) = v_1(y) = e^{iy^2} \phi(y).
\]

\((t \in [0 \rightarrow \infty) \) corresponds to \( s \in (0 \leftarrow 1] \).\)

We ignore the dispersive term \( v_{yy} \) and consider the ODE

\[
i v_t = \frac{1}{s} |v|^2 v.
\]

(This idea was suggested to us by Kenji Nakanishi and goes back to work of Ozawa.) \( \forall \ w : \mathbb{R} \rightarrow \mathbb{C} \), the ODE has explicit solutions

\[
v^{[w]}(s, y) := w(y) e^{-i|w(y)|^2 \log s}.
\]

Note that:

\[
|\partial_y^2 v^{[w]}| \lesssim |\log s|^2 \ll \frac{1}{s}
\]

\( v^{[w]} \) forms a singularity as \( s \rightarrow 0^+ \).
Combining the two lemmas and $pc^{-1}$, we can prove that the $L^2$ global well-posedness of cubic $NLS$ is not uniform in time. (Our argument actually shows the global well-posedness of cubic $NLS$ in $H^5$ is not uniform in time. The approximation via the ODE solution also proves there is no scattering in $L^2$ for the cubic defocussing $NLS$ evolution.

Using the lemmas, $pc^{-1}$, and the dilation and Galilean invariances, we construct global-in-time solutions $u^{(a)}$, $u^{(a')}$ of defocussing cubic NLS satisfying: Let $-5 < s < 0$, $0 < \delta \ll \epsilon < 1$, $T > 0$ be arbitrary,

$$\|u^{(a)}(0)\|_{H^s_x}, \|u^{(a')}(0)\|_{H^s_x} \lesssim \epsilon,$$

$$\|u^{(a)}(0) - u^{(a')}(0)\|_{H^s_x} \lesssim \delta,$$

$$\sup_{0 \leq t < T} \|u^{(a)}(t) - u^{(a')}(t)\|_{H^s_x} \gtrsim \epsilon.$$