Almost conservation laws and global wellposedness

Analysis Seminar

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Abstract

A dynamical reinterpretation of the L^2 mass conservation law for solutions of the KdV equation has led to a general procedure for proving *almost conservation laws* for solutions of nonlinear Hamiltonian PDE. The almost conserved quantities have been used to globalize the available local-in-time wellposedness results for various KdV and NLS type equations and will provide insights into the long-time behavior of solutions. This talk will survey the known local theory, highlighting aspects requiring further investigation. The procedure for constructing almost conserved quantities will be described.

Two families of nonlinear Hamiltonian PDE

- **GNLS**
  Let $u : \mathbb{R}^d \times [0, T] \longmapsto \mathbb{C}$. Form the functional

$$\boldsymbol{H[u, \overline{u}] = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 + F(|u|^2) dx.}$$

  For example, $F(y) = \frac{1}{2}y^2$. Evolve u according to

$$\boldsymbol{iu_t = H_{\overline{u}} \iff iu_t + \Delta u - f(|u|^2)u.}$$

  Important case: $F(y) = \frac{1}{2}y^2,\ f(y) = y$.

- **GKDV**
  Let $u : \mathbb{R} \times [0, T] \longmapsto \mathbb{R}$. Form

$$\boldsymbol{H[u] = \int_{\mathbb{R}^d} \frac{1}{2}u_x^2 + \frac{u^{p+2}}{(p+1)(p+2)} dx.}$$

  Evolve u according to

$$\boldsymbol{u_t = \partial_x H_u \iff u_t + u_{xxx} + u^p u_x = 0.}$$

The case $p = 1$ is the *standard Korteweg-de Vries equation*. Cases $p = 1, 2$ are **completely integrable**.

Research Motivations

Physical Relevance

1. Specific Contexts: NLS and/or KdV arise naturally as model equations in optics, plasma physics, water waves, ...
2. Universal Derivation: Weakly nonlinear and dispersive corrections to standard wave equation lead systematically to NLS and/or KdV.
3. Toy models: GNLS and GKdV are basic examples of *infinite dimensional conservative dynamical systems*.

Research Goal: **Understand Hamiltonian dynamics in infinite dimensions.**

- Approximate dynamics with finite-d models when possible.
- Find "infinite-d phenomena" (turbulence, blow-up, scatter).
- Natural phase spaces consist of **rough functions** $\rightsquigarrow$ **regularity obstructions.**

New techniques from *multilinear harmonic analysis* **are removing this difficulty:** [KPV], [B], [KM]

The intitial value problem for KdV on $\mathbb{R}$ is

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x u^2 = 0$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

($\exists$ other natural boundary value problems for KdV such as $x \in \mathbb{T}$, half-line. [CK] extends some of the theory described below to the inital boundary value setting.)

Solitary Wave Resolution

Integrable machinery has shown, for very nice initial data ϕ, u exists globally (in time) and

$$u(x, t) = \sum_{j=1}^{N} f_{c_j, \theta_j}(x, t) + u_R(x, t)$$

where

$$f_{c,\theta}(x, t) = 3c \operatorname{sech}^2\left(\frac{1}{2}p\sqrt{c}[x - ct + \theta]\right),$$

and u_R is small. **SWR** is not known in the natural phase spaces L^2, H^1 or in any nonintegrable case.

Solutions of KdV_p satisfy

$$\|\boldsymbol{u(t)}\|_{\boldsymbol{L^2}} = \|\boldsymbol{\phi}\|_{\boldsymbol{L^2}}.$$

Problem: Explain how the dynamics of KdV_p exchanges the L^2 mass among the frequencies. How do the components of the system share the energy? Describe motion of "microlocal mass".

Open Question: Let $u(x,t)$ be a global solution of KdV_3 with nice initial data ϕ. Prove that

$$\|\boldsymbol{u(t)}\|_{\boldsymbol{H^s}} \leq \boldsymbol{C}\|\boldsymbol{\phi}\|_{\boldsymbol{H^s}}.$$

(Best known estimates [S] are polynomial-in-time. The integrable cases $KdV_{1,2}$ are known to have this property.)

Wellposedness

Definition: The initial value problem

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x u^2 = 0$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}$$

is $\underline{\textit{wellposed on the time interval } [0, T] \textit{ for data in } H}$ if there is a uniquely defined continuous map $H \ni \phi \longmapsto u \in X$ with $X \subset C([0,T], H)$ and u "solves" KdV. If $T < \infty$, the problem is $\underline{\textit{locally wellposed (LWP)}}$. If we can take $T = \infty$, the problem is $\underline{\textit{globally wellposed (GWP)}}$.

- Rough spaces H require a careful notion of "solves".
- Two notions of continuity in the defintion.
- Typical proofs of LWP have

$$T \sim \|\phi\|_H^{-\alpha}, \ \alpha > 0.$$

- GWP in H follows from LWP in H if $\|u(t)\|_H$ is bounded. The available conserved quantities may provide time independent bounds on the L^2 or H^1 norms which consist of rough functions.

Basic (Nonintegrable) Approaches to Local Wellposedness

- **Classical Energy Method/Compactness**
  Form an approximate sequence. Prove bounds implying compactness. Use Gronwall inequality to prove uniqueness. (Gronwall requires lots of smoothness.) *Euler-Peano construction of ODE solutions.*
- **Fixed Point Argument/Contraction Mapping**
  Rewrite PDE as an integral equation. Prove a contraction estimate in an appropriate space X of functions of space-time. *Picard's approach to ODEs.*

  In what space X should we seek a fixed point?
  - **Maximal Function/Kato Smoothing.**
    Prove estimates for solutions of associated linear homogeneous and inhomogeneous problems. Control nonlinearity using well-matched (e.g. Hölder) linear estimates. Matching conditions identify mixed $L_t^q L_x^p$ space-time norms. Shrink existence interval to shrink contraction constant.
  - **Bourgain Spaces $X_{s,b}$.**
    Detailed study of nonlinearity in spaces related to linear problem. Denominators and calculus smooth nonlinear term. Shrink existence interval to shrink contraction constant.

LWP and GWP results for KdV

Theorem. [Bourgain 1993] *KdV is LWP in L^2.*

Consequently, KdV is GWP in L^2.

Theorem. [Kenig, Ponce, Vega 1996, 2000] *KdV is LWP in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$. (Complex) KdV is illposed in $H^s(\mathbb{R})$ for $s < -\frac{3}{4}$.*

Theorem. [CKSTT 2000] *KdV is GWP in H^s, $s > -\frac{3}{4}$.*

- **Motivated by high/low frequency GWP technique of Bourgain.** Technical issues in KdV context [CST] and $Maxwell - Klein - Gordon$ context [KT] started our collaboration.
- **Proof constructs almost conservation laws.**
  New control of high-to-low frequency transport of "L^2 mass".
- **General Method for globalizing local results.**
  Method improves the GWP theory of various nonlinear Hamiltonian PDE, including 2d cubic NLS, 1d Derivative NLS, periodic and nonperiodic $GKDV$.
- **Almost conservation laws $\rightsquigarrow$ long-time behavior?**
  Regularity of the evolution, weak turbulence, scattering, finite dimensional approximate models, solitary wave resolution....

Motivation for $X_{s,b}$ Spaces

The linear homogeneous ($f = 0$) problem

$$\partial_t u + \partial_x^3 u = f$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}$$

has the explicit solution

$$u(x, t) = S(t)\phi(x) = \int e^{i(x\xi + t\xi^3)}\widehat{\phi}(\xi)d\xi$$

$$= \int\int e^{i(x\xi + t\lambda)}\widehat{\phi}(\xi)\delta(\lambda - \xi^3)d\lambda d\xi.$$

The linear inhomogeneous problem ($f \neq 0$) may be solved via Duhamel's principle as

$$u(x, t) = S(t)\phi + \int_0^t S(t - \tau)f(x, \tau)d\tau$$

$$\sim \mathcal{F}^{-1}\left(\widehat{\phi}(k)\delta(\lambda - k^3)\right) + \mathcal{F}^{-1}\left(\frac{\widehat{f}(k, \lambda)}{(1 + |\lambda - \xi^3|)}\right).$$

Punchline: The space-time Fourier transform of the homogeneous solution lives on the cubic. The space-time Fourier transform of the inhomogeneous solution decays away from the cubic.

$X_{s,b}$ Space Approach to LWP

Define the space $X_{s,b}$ via the norm

$$\|\boldsymbol{u}\|_{X_{s,b}} = \left(\int\int (1+|\xi|)^{2s} (1+|\lambda - \xi^3|)^{2b} |\widehat{u}(\xi,\lambda)|^2 d\xi d\lambda \right)^{\frac{1}{2}}$$

The i.v.p. for KdV is equivalent to **solving** the integral equation

$$u(t) = S(t)\phi + \int_0^t S(t-\tau) \left(\frac{1}{2} \partial_x u^2(\tau) \right) d\tau.$$

Show u exists by proving a contraction estimate in $X_{s,b}$ with $b = \frac{1}{2}+$. Matters reduce to proving a **bilinear estimate**

$$\|\partial_x(uv)\|_{X_{s,b-1}} \leq C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}.$$

Bourgain proved this estimate for $s \geq 0$, with $b = \frac{1}{2}+$. Kenig, Ponce and Vega proved this estimate holds with $b = \frac{1}{2}+$ provided $s > -\frac{3}{4}$ and fails for any b when $s < -\frac{3}{4}$.

Remark: The bilinear estimate smoothes away one derivative in the nonlinearity. We will see "extra" smoothing in a moment.

Ideas in Proving the Bilinear Estimate

We can rewrite the bilinear estimate as

$$\int_* \frac{(1+|k|)^s |k| d(k,\lambda)}{(1+|\lambda-k^3|)^{1-b}} \frac{(1+|k_1|)^{-s} c(k_1,\lambda_1)}{(1+|\lambda_1-k_1^3|)^b}$$

$$\times \frac{(1+|k_2|)^{-s} c(k_2,\lambda_2)}{(1+|\lambda_2-k_2^3|)^b} \leq C\|d\|_{L^2}\|c_1\|_{L^2}\|c_2\|_{L^2}.$$

where $*$ indicates $k = k_1 + k_2,\ \lambda = \lambda_1 + \lambda_2$.

<u>**An arithmetical fact:**</u> The convolution constraints imply

$$\max(|\lambda - k^3|, |\lambda_1 - k_1^3|, |\lambda_2 - k_2^3|) \geq 3kk_1k_2,$$

since $k^3 - k_1^3 - k_2^3 = 3kk_1k_2$.

Assume $c_i(k_i,\lambda_i) = \widehat{\phi}_i(k_i)\delta(\lambda_i - k_i^3)$. We need to show

$$\int d(k_1+k_2, k_1^3+k_2^3) \frac{(1+|k_1+k_2|)^s |k_1+k_2|}{(1+|3k_1k_2(k_1+k_2)|)^{1-b}}$$

$$\times (1+|k_1|)^{-s}\widehat{\phi_1}(k_1)(1+|k_2|)^{-s}\widehat{\phi_1}(k_2)dk_1dk_2$$

is bounded by

$$C\|d\|_{L^2}\left\|\widehat{\phi_1}\right\|_{L^2}\left\|\widehat{\phi_2}\right\|_{L^2}.$$

- Symmetry.
  We can assume $|k_1| \geq |k_2|$ so $|k_1| \gtrsim |k_1 + k_2|$.
- A Change Variables.
  Define $u = k_1 + k_2,\ v = k_1^3 + k_2^3$. Calculate the Jacobian $dudv = 3(k_1^2 - k_2^2)dk_1dk_2$.
- Cauchy-Schwarz.
  Issues collapse to bounding (by $\left\|\widehat{\phi_1}\right\|_{L^2}\left\|\widehat{\phi_2}\right\|_{L^2}$),

$$\left(\int M(k_1, k_2) \times |\widehat{\phi_1}(k_1)|^2 |\widehat{\phi_2}(k_2)|^2 dk_1 dk_2\right)^{\frac{1}{2}}$$

where

$$M = \frac{|k_1 + k_2|^2 (1 + |k_2|)^{-2s}}{(1 + |3(k_1 + k_2)k_1k_2|)^{2(1-b)}(k_1 + k_2)(k_1 - k_2)}.$$

Remark: Suppose $k_1 \sim N$ and $k_2 = k_1 + O(N^{-\frac{1}{2}})$. Then

$$M = N^{2-2s-6(1-b)-1+\frac{1}{2}}.$$

To have M bounded requires

$$s > \frac{3}{4} - 3(1-b) \sim -\frac{3}{4}.$$

Optimality of Kenig-Ponce-Vega Estimate

Let u and v be the same function with $\widehat{u}(k,\lambda) \sim \chi_R(k,\lambda)$ where R is a thin rectangle lying inside

$$\boldsymbol{\left\{(k,\lambda) : |k - N| < O(N^{-\frac{1}{2}}),\ |\lambda - k^3| < O(1)\right\}.}$$

A calculation shows optimality of $-\frac{3}{4}$ for the bilinear estimate.

Bad Denominator Lower Bounds?

Example: In the 2d NLS setting $\lambda - \xi^3$ is replaced by $\lambda + |\xi|^2$ in the definition of $X_{s,b}$. The **<u>bilinear estimate</u>**

$$\|u\overline{v}\|_{X_{s,b-1}} \leq C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}$$

depends upon the **<u>arithmetical fact</u>**,

$$\max(|\lambda+|\xi|^2|, |\lambda_1+|\xi_1|^2|, |\lambda_2-|\xi|^2|) \gtrsim |2(\xi_1+\xi_2)\cdot\xi_2|$$

since $|\xi_1 + \xi_2|^2 - |\xi_1|^2 + |\xi_2|^2 = 2\xi_1 \cdot \xi_2 + 2\xi_2 \cdot \xi_2$.

The **orthogonal interaction** $(\xi_1 + \xi_2) \perp \xi_2$ **turns off** the denominators. New techniques for dealing with this situation were developed in [CDKS]. (The above estimate holds for $s > -\frac{3}{4}$ and this is sharp up to the endpoint.) Independent related work appears in [T]. Similar troubles appear in the KPI and ZS setting.

A robust theory for multilinear estimates in $X_{s,b}$ requires improved techniques for dealing with bad denominator lower bounds.

L^2 Conservation Law Interlude

<u>**Standard Proof:**</u>

$$\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x u^2$$

Multiply by u and reexpress to get,

$$\frac{1}{2}\partial_t u^2 = -\partial_x(u\partial_x^2 u) + \partial_x\left(\frac{1}{2}[\partial_x u]^2\right) - \frac{1}{3}\partial_x u^3.$$

Integrate in x to observe

$$\partial_t \int u^2 \, dx = 0.$$

Note that u is assumed to be $\mathbb{R}$-valued.

Remark: This proof apparently does not reveal <u>how</u> the conserved L^2 mass evolves in frequency space. A dynamical explanation of L^2 mass conservation is a goal of the next few slides.

<u>Fourier Proof:</u> (with notation forecasting later stuff)

$$\int \widehat{u}(\xi)\overline{\widehat{u}}(\xi)d\xi = \int \widehat{u}(\xi)\widehat{\overline{u}}(-\xi)d\xi = \int_{\xi_1+\xi_2=0} \widehat{u}(\xi_1)\widehat{u}(\xi_2),$$

since u is $\mathbb{R}$-valued. Therefore,

$$\partial_t \int \widehat{u}(\xi)\overline{\widehat{u}}(\xi)d\xi = 2\int_{\xi_1+\xi_2=0} \widehat{u_t}(\xi_1)\widehat{u}(\xi_2)$$

$$= 2\int_{\xi_1+\xi_2=0} \left[-(i\xi_1)^3\widehat{u}(\xi_1) - \frac{1}{2}(i\xi_1)\widehat{u^2}(\xi_1)\right]\widehat{u}(\xi_2).$$

Now, symmetrize first term and expand convolution to get

$$= -\int_{\xi_1+\xi_2=0} i(\xi_1^3+\xi_2^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)$$

$$-\int_{\xi_1+\xi_2+\xi_3=0} i(\xi_1+\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)$$

The first term is zero. Upon writing $\xi_1+\xi_2 = -\xi_3$ and symmetrizing, the second term vanishes.

Symmetrization/cancellation argument is apparently more flexible than standard proof. Variants begin to capture the dynamics of L^2 mass in frequency space.

Multilinear Operators

Definitions: A k-multiplier is a function $m : \mathbb{R}^k \longmapsto \mathbb{C}$. A k-multiplier is symmetric if $m(\xi) = m(\sigma(\xi)$ for all $\sigma \in S_k$. The symmetrization of a k-multiplier is

$$[\boldsymbol{m}]_{sym}(\boldsymbol{\xi}) = \frac{\mathbf{1}}{\boldsymbol{n!}} \sum_{\sigma \in S_k} \boldsymbol{m}(\boldsymbol{\sigma}(\boldsymbol{\xi})).$$

A k-linear functional is generated by a k-multiplier via

$$\boldsymbol{\Lambda_k}(\boldsymbol{m}) = \int_{\xi_1+\ldots+\xi_k=0} \boldsymbol{m}(\boldsymbol{\xi}_1, \ldots \boldsymbol{\xi_k})\widehat{\boldsymbol{u}}(\boldsymbol{\xi}_1) \ldots \widehat{\boldsymbol{u}}(\boldsymbol{\xi_k}).$$

Proposition 1. *Suppose u satisfies the KdV equation, and m is a symmetric k-multiplier. Then*

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{\Lambda_k}(\boldsymbol{m}) = \boldsymbol{\Lambda_k}(\boldsymbol{m\alpha_k}) - \boldsymbol{i}\frac{\boldsymbol{k}}{\mathbf{2}}\boldsymbol{\Lambda_{k+1}}\left(\boldsymbol{m}(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi_{k-1}}, [\boldsymbol{\xi_k} + \boldsymbol{\xi_{k+1}}])(\boldsymbol{\xi}\right.$$

where

$$\boldsymbol{\alpha_k} = \boldsymbol{i}(\boldsymbol{\xi}_1^3 + \ldots + \boldsymbol{\xi}_k^3).$$

This follows immediately from the equation and properties of the Fourier transform. The $(k+1)$-multiplier may be symmetrized.

Modified Energies

Let m be an $\mathbb{R}$-valued even 1-multiplier. Define the operator I via

$$\widehat{If}(\xi) = m(\xi)\widehat{f}(\xi).$$

Define the modified energy

$$E_I^2(t) = \|Iu(t)\|^2_{L^2(\mathbb{R}_x)} = \Lambda_2(m(\xi_1)m(\xi_2)).$$

(Last step uses u, m $\mathbb{R}$-valued.) Calculate, using the proposition,

$$\frac{d}{dt}E_I^2(t) = \Lambda_2(m(\xi_1)m(\xi_2)\alpha_2) - i\Lambda_3(m(\xi_1)m(\xi_2+\xi_3)[\xi_2+\xi_3])$$

$$= \Lambda_3(-i[m(\xi_1)m(\xi_2+\xi_3)(\xi_2+\xi_3)]_{sym}).$$

(Note that if $m = 1$, we directly verify L^2 conservation.)

Denote the 3-multiplier above by $M_3(\xi_1, \xi_2, \xi_3)$.

Define

$$E_I^3(t) = E_I^2(t) + \Lambda_3(\sigma_3), \quad \sigma_3 \text{ to be chosen.}$$

Calculate, using the proposition,

$$\frac{d}{dt}E_I^3(t) = \Lambda_3(M_3) + \Lambda(\sigma_3\alpha_3) + \Lambda_4(M_4),$$

where M_4 is given explicitly. Choose

$$\sigma_3 = -\frac{M_3}{\alpha_3}$$

to cancel the Λ_3 terms.

This process may be iterated to formally generate a sequence of modified energies $\{E_I^j(t)\}_{j=2}^{\infty}$, with the property

$$\frac{d}{dt}E_I^j(t) = \Lambda_{j+1}(M_{j+1}).$$

Remarks:

- **The 1-multiplier m has only been assumed to be $\mathbb{R}$-valued and even. There remains a lot of flexibility in choosing m to localize the energy in frequency space.**
- For a particular choice of m, we have shown the first few modified energies are <u>almost conserved</u>, leading to the proof of global wellposedness for $-\frac{3}{4} < s$.

Almost Conserved Quantities

We choose m to depend upon a large parameter N. Let m be a smooth monotone even function satisfying for $s < 0$:

$$\boldsymbol{m(\xi) = 1, \;\; |\xi| < \frac{1}{3}N,}$$

$$\boldsymbol{m(\xi) = \left(\frac{|\xi|}{N}\right)^{s}, \;\; |\xi| > 2N.}$$

The associated operator I commutes with differential operators and barely maps H^s functions into L^2 functions.

We can show, in an appropriate sense, that

$$\boldsymbol{E_I^2(t) \lesssim E_I^4(t),}$$

using Sobolev inequalities. Using the $X_{s,b}$ machinery, we have proven the **main estimate**

$$\boldsymbol{E_I^4(\delta) - E_I^4(0) = \int_0^t \Lambda_5(M_5)(\tau) d\tau}$$

$$\boldsymbol{\leq C N^{-3-\frac{3}{4}+\epsilon} \| I\phi \|_{L^2}^5,}$$

where δ is the lifetime of the local result. Recalling that $E_I^2(t) = \|Iu(t)\|_{L^2}^2$, we see that E_I^2 **is almost conserved.**

Is KdV Globally Wellposed below L^2?

Theorem 1. [C, Staffilani, Takaoka] *The initial value problem for KdV on $\mathbb{R}$ is GWP in $H^s \cap \dot{H}^a$ for certain (related) $s, a < 0$.*

The *homogeneous Sobolev norm* $\dot{H}^a$ is defined using the norm

$$\|\phi\|_{\dot{H}^a(\mathbb{R})} = \left(\int \left| |\xi|^a \widehat{\phi}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}.$$

Note that $L^2 \not\subseteq \dot{H}^a$ so this hypothesis is a bit of a blemish.

The proof adapts the high/low frequency technique of Bourgain (applied to NLS, NLW). We were inspired in part by a similar result of Fonseca, Linares and Ponce for KdV_2. (See also [Keel-Tao], [Takaoka-Tzvetkov], ...)

The idea is to exploit L^2 conservation even though the data $\phi \notin L^2$.

Trick: Evolve the low frequencies according to KdV with L^2 conservation. Evolve the high frequencies "linearly" and hide the high frequency interaction using "extra" smoothing captured in the nonlinear estimate.

Bourgain's High/Low Frequency Technique

Fix $T > 0$. Construct the solution of KdV on $[0, T]$. Decompose the initial data $\phi \in H^s(\mathbb{R}),\ \ -\frac{3}{4} < s \lesssim 0.$

$$\boldsymbol{\phi = \phi_0 + \psi_0,\ \ \phi_0 = \mathbb{P}_N \phi,}$$

$$\widehat{\mathbb{P}_N \boldsymbol{\phi}} = \boldsymbol{\chi_{[-N,N]}\widehat{\phi}.}$$

Later, we **select** $N = N(T)$ to be huge.

Note that $\phi_0 \in L^2$ and

$$\boldsymbol{\|\phi_0\|_{L^2} \sim N^{-s},\ \ \|\psi_0\|_{H^\sigma} \sim N^{\sigma - s};\ \ (\sigma < s).}$$

Low Frequency Evolution
The evolution $\phi_0 \longmapsto u_0(t)$ solving KdV exists globally in time with $\|u_0(t)\|_{L^2} = \|\phi\|_{L^2}$.
High Frequency Evolution
Let $\psi_0 \longmapsto v_0(t)$ evolve so that $u(t) = u_0(t) + v_0(t)$:

$$\boldsymbol{\partial_t v_0 + \partial_x v_0 + \frac{1}{2}\partial_x (v_0^2 + 2u_0 v_0) = 0,}$$

$$\boldsymbol{v_0(0) = \psi_0.}$$

- High frequency evolution is LWP on $[0, \delta]$,

$$\delta = \|\phi_0\|_{L^2}^{-\alpha}.$$

Note: Lifetime depends on (big) L^2 norm of low frequency data.

- Decompose the high frequency evolution.

$$v_0(t) = S(t)\psi_0 + w_0(t).$$

- Extra smoothing shows $w_0 \in L^2$, and is **small**,

$$\sup_{t \in [0,\delta]} \|w_0(t)\|_{L^2} \lesssim N^{-\beta}, \ \beta > 0.$$

Define

$$\begin{aligned} \phi_1 &= u_0(\delta) + w_0(\delta), \\ \psi_1 &= S(\delta)\psi_0, \end{aligned}$$

and repeat.

We can continue with the **same sized** δ until, say

$$\|w_0\|_{L^2} + \|w_1\|_{L^2} + \ldots \gtrsim \|\phi_0\|_{L^2}.$$

Unravelling this gives the result.

Extra Smoothing Bilinear Estimate

If $\widehat{u}$ and $\widehat{v}$ are supported *outside* $|\xi| \geq 1$,

$$\boldsymbol{\|\partial_x uv\|_{X_{0,b-1}} \leq \|u\|_{X_{-\frac{3}{8}+,b}} \|u\|_{X_{-\frac{3}{8}+,b}}.}$$

- LWP tools and this estimate give the L^2 estimate on w_0.
- KPV example shows this is optimal.
- Low frequencies destroy extra smoothing $\rightsquigarrow \dot{H}^a$.

Smoothing Operator I

Recent work **[Keel-Tao]** on simliar NLW problem suggests defining for $s < 0$ and $N \gg 1$ fixed,

$$\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi),$$

where m is smooth and looks like

$$m(\xi) = 1, \ |\xi| < \frac{1}{2}N,$$

$$m(\xi) = N^{-s}|\xi|^{s}, \ |\xi| \geq N.$$

I is the identity on low frequencies and barely smooths H^s tails into L^2 functions. I commutes with differential operators.

Almost L^2 Conservation Property

$$\|Iu(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \int_0^\delta \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau$$

$$= \|Iu(0)\|_{L^2}^2 + 2\int_0^\delta (I(-u_{xxx} - \frac{1}{2}\partial_x u^2), Iu)d\tau$$

$$= \|Iu(0)\|_{L^2}^2 + \int_0^\delta (I(-\partial_x u^2), Iu)d\tau$$

$$= \|Iu(0)\|_{L^2}^2 + \int_0^\delta \int \partial_x \left\{ (I(u))^2 - I(u^2) \right\} Iu \; dx d\tau.$$

By Cauchy-Schwarz,

$$\leq \|Iu(0)\|_{L^2}^2 + \|\partial\{\ \ \}\|_{X_{0,-\frac{1}{2}-}} \|Iu\|_{X_{0,\frac{1}{2}+}}.$$

Theorem 2. [C, Keel, Staffilani, Takaoka, Tao]

$$\|\partial_x \{I(u)I(v) - I(uv)\}\|_{X_{0,-\frac{1}{2}-}}$$

$$\leq CN^{-\frac{3}{4}+} \|Iu\|_{X_{0,\frac{1}{2}+}} \|Iv\|_{X_{0,\frac{1}{2}+}}.$$

(Follows from extra bilinear smoothing estimate and cancellation.)

Global wellposedness of KdV in $H^s,\ s<0$

Our task is to construct the solution on $[0,T]$.

- Rescaling: u solves KdV with data ϕ on $[0,T]$ iff $u_\lambda(x,t) = \frac{1}{\lambda^2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right)$ solves KdV with data ϕ_λ on $[0, \lambda^3 T]$. Choose $\lambda = \lambda(N)$ so that

$$\|I\phi_\lambda\|_{L^2} \sim 1. \text{ (requires } \lambda \sim N^{-\frac{2s}{3+2s}}\text{)}$$

- LWP variant: KdV is wellposed on $[0,\delta]$,

$$\delta \sim \|I\phi\|_{L^2}^{-\alpha},\ \alpha > 0,$$

and

$$\|Iu\|_{X_{0,\frac{1}{2}+}} \leq C\|I\phi\|_{L^2}.$$

The LWP norm doubles after $N^{\frac{3}{4}-}$ steps. We need to have

$$N^{\frac{3}{4}-} > [\lambda(N)]^3 T.$$

Theorem 3. [C, Keel, Staffilani, Takaoka, Tao] *KdV is GWP in H^s, $-\frac{3}{10} < s$.*