Reading Course Notes   March 19 - 2009   Yuri

A Lie group is a group where each element depends differentiably on a set of real parameters, that is $g = g(\theta^a)$ where $\theta^a \in \mathbb{R}$ for $1 \le a \le n$ → a generic element of the group

Also θ is chosen s.t. $g(0) = 1$ → (identity element) $\forall g$.

A (linear) Representation of group (call it R) assigns $g \mapsto D_R(g)$ for each group element where $D_R(g)$ is a linear operator acting on the basis of R

E.g. if the basis is $\mathbb{R}^n$ $D_R(g)$ is an $n \times n$ matrix → $D_R(g) = (D_R(g))^i_j$, $1 \le i,j \le n$

D_R satisfies (i) $D_R(\underset{\text{in group}}{1}) = \underset{\text{identity in basis}}{1}$

(ii) $D_R(g_1) D_R(g_2) = D_R(g_1 g_2)$ $\forall g_1, g_2$

→ preserves group structure

Given an element of the basis say $\varphi = (\varphi^1 \ldots \varphi^n)$ then g will induce a transformation

$\varphi^i \mapsto (D_R(g))^i_j \varphi^j$

So that an abstract group is given a more physical interpretation.

② Ex. if the lie group is $SO(3)$ and the basis is $\mathbb{R}^3$ then $g \in SO(3)$ induces a rotation in $\mathbb{R}^3$

R is reducible if $\exists S \subset \text{Basis}(R)$ s.t. $\forall s \in S$ $D_R(g)s \in S$ $\forall g$

S is an invariant subspace for the group

if Basis(R) contains no invariant subspace R is irreducible

R is completely reducible if $\text{Basis}(R) = A_1 \oplus \dots \oplus A_k$ where $A_j \subset \text{Basis}(R)$ is invariant $\forall j$

Lie Algebra (Does not depend on R)

Given Lie group & some representation consider the neighborhood of the identity corresponding to $|\theta^a| \ll 1$ $\forall a$

Then we can taylor expand: $D_R(g(\theta)) \simeq 1 + \theta_a \left.\frac{\partial D_R(g(\theta))}{\partial \theta_a}\right|_{\theta_a=0}$

Let $T^a_R = -i \left.\frac{\partial D_R(g(\theta))}{\partial \theta_a}\right|_{\theta_a=0}$ so that $\simeq 1 + i\theta_a T^a_R$

(we insert the i by convention, makes hermitian, unitary cases easier).

T^a_R is called the generator of the group in R

Now there is a theorem, which ~~gives~~ tells us that for generic θ, we can then write

$D_R(g(\theta)) = e^{i\theta_a T^a_R}$. I donot yet know how to prove this though!

If we accept the theorem for know, we can also observe that if T^a_R is a Hermitian operator $D_R(g(\theta))$ will be unitary (something we proved in class)

Now let us find the Lie Algebra

given g_1, g_2 we know $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$

but how do parameters θ relate?

writing $D_R(g_1) = e^{i\alpha_a T^a_R}$ $\quad D_R(g_2) = e^{i\beta_a T^a_R}$

$\& \; D_R(g_1 g_2) = e^{i\delta_a T^a_R}$

what is the relationship between $\alpha_a, \beta_a, \delta_a$?

if $[T^a_R, T^b_R] = 0 \;\forall a,b$ we have $\delta_a = \alpha_a + \beta_a$

In general this is not the case.

Lets suppose again that $|\alpha_a|, |\beta_a| \ll 1$

then $e^{i\alpha_a T^a_R} e^{i\beta_a T^a_R} = e^{i\delta_a T^a_R}$

$\Rightarrow i\delta_a T^a_R \simeq \ln\left[\left(1 + i\alpha_a T^a_R + \frac{1}{2}(i\alpha_a T^a_R)^2\right)\left(1 + i\beta_a T^a_R + \frac{1}{2}(i\beta_a T^a_R)^2\right)\right] + O((\alpha_a+\beta_a)^3)$ $\quad$ & ln both sides

$\Rightarrow i\delta_a T^a_R \simeq \ln\Big(1 + i\beta_a T^a_R + \frac{1}{2}(i\beta_a T^a_R)^2 + i\alpha_a T^a_R - \alpha_a\beta_b T^a_R T^b_R + \frac{1}{2}(i\alpha_a T^a_R)^2 + \dots\Big)$

$= \ln\left(1 + i(\alpha_a+\beta_a)T^a_R - \frac{1}{2}(\beta_a T^a_R)^2 - \frac{1}{2}(\alpha_a T^a_R)^2 - \alpha_a\beta_b T^a_R T^b_R\right)$

Now recall $\ln(1+x) \simeq x - \frac{1}{2}x^2$ $\quad$ for $x = i(\alpha_a+\beta_a)T^a_R - \frac{1}{2}(\beta_a T^a_R)^2 - \frac{1}{2}(\alpha_a T^a_R)^2 - \dots$

So we have

$$i\delta_a T^a_R \simeq i(\alpha_a+\beta_a)T^a_R - \frac{1}{2}((\beta_a T^a_R)^2 + (\alpha_a T^a_R)^2) - \alpha_a\beta_b T^a_R T^b_R$$
$$+ \frac{1}{2}((\alpha_a+\beta_a)T^a_R)^2 + \ldots$$

$$\simeq i(\alpha_a+\beta_a)T^a_R - \frac{1}{2}((\beta_a T^a_R)^2 + (\alpha_a T^a_R)^2) - \alpha_a\beta_b T^a_R T^b_R$$
$$+ \frac{1}{2}\left[(\alpha_a T^a_R)^2 + (\beta_a T^a_R)^2 + \alpha_a\beta_b T^a_R T^b_R + \beta_a\alpha_b T^a_R T^b_R\right]$$

$$= i(\alpha_a+\beta_a)T^a_R - \frac{1}{2}\alpha_a\beta_b(T^a_R T^b_R - T^b_R T^a_R)$$

$$\Rightarrow \underline{\alpha_a\beta_b[T^a_R, T^b_R] = -2i(\delta_a - \alpha_a - \beta_a)T^a_R} \quad \circledast$$

Observe that the LHS is linear in both α_a & β_a

Now $\delta_a = \delta_a(\alpha, \beta) \Rightarrow -2(\delta_a - \alpha_a - \beta_a)$ must be linear in both α_a & β_a

meaning $\exists$ constant $f^{ab}{}_c$ s.t

$$\alpha_a\beta_b f^{ab}{}_c = -2(\delta_c - \alpha_c - \beta_c)$$

we can then write

$$[T^a_R, T^b_R] = i f^{ab}{}_c T^c_R$$

this relationship defines our Lie Algebra & $f^{ab}{}_c$ are the structure constants of the group.

2 Things we should note here:

(i) We only need to consider small α, β to get the Algebra.

(ii) the constants $f^{ab}{}_c$ are <u>independent</u> of R -> the equation for $[T^a_R, T^b_R]$ holds $\forall$ R

To argue this: suppose contrary that $f^{ab}{}_c = (f^{ab}{}_c)_R$ then we would have $\delta^a = (\delta^a)_R$ but δ_a reflects the group operation $g_1 \cdot g_2$ meaning that given g_1, g_2 ~~α, β~~ we set $\delta_a(\alpha, \beta)$ by the group operation alone which cannot depend on R.

<u>Applications to Lorentz group / Spin for spinors</u>

A Lie group that, when represented in the basis $(Y_1 - Y_m, X_1 .. X_n) \in \mathbb{R}^{m+n}$ does not, for any elements change the number $(Y_1^2 + .. + Y_m^2) - (X_1^2 + .. + X_n^2)$ is called $O(n, m)$

$\therefore$ Lorentz group $= O(3, 1)$ & leaves invariant $t^2 - (x^2 + y^2 + z^2)$ = space-time interval in physics.

If $\eta_{\mu\nu}$ = ~~Min~~ Minkowski metric

& we put $\Lambda_R = \Lambda$ an ~~ma~~ element of Lorentz group in $\mathbb{R}^4$ basis (drop the R since we use this rep)

this translates then to $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$

where $\Lambda = \Lambda^\rho{}_\sigma \to 4\times 4$ matrix

those Λ w/ $\det \Lambda = +1$ are ~~called~~ in a subgroup called $SO(3,1)$ (we only care about it)

Find generators If we go back to our defn of generators, we need to ~~think~~ look only at infinitesimal Lorentz transformations

so ~~we~~ consider then $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \omega_{\mu\nu}$

where $|\omega_{\mu\nu}| \ll 1$

observe then:
$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$
$$= \eta_{\mu\nu} (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma)$$
$$= \eta_{\rho\sigma} + \eta_{\rho\nu}\omega^\nu{}_\sigma + \eta_{\mu\sigma}\omega^\mu{}_\rho + O(\omega^2)$$
$$\Rightarrow \eta_{\rho\sigma} = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho}$$
$$\Rightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$

so $\omega_{\mu\nu}$ has 6 independent components

These correspond to 6 generators

we call then $T^a{}_R \to J^{\mu\nu}$ the generators of Lorentz group & search for explicit forms.

Now Walid's class told us the Lie algebra for the group

Namely if we call $J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$ ← "rotation"

$K^i = J^{i0}$ ← "boost"

we have $[J^i, J^j] = i\varepsilon^{ijk}J^k$ — i, j are spatial (1, 2, 3) indices.

$[J^i, K^j] = i\varepsilon^{ijk}K^k$

$[K^i, K^j] = -i\varepsilon^{ijk}J^k$

Note that we separated the $J^{\mu\nu}$ into 2 separate 3 component things.

Knowing the Algebra holds & representations Apply it to 4-component solutions of the Dirac equation. Get $J^{\mu\nu}$

So we have $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

let's give it a Lorentz transformation and see how ψ transforms

so we send $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$\partial_\mu \mapsto \partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$

$\psi \mapsto \psi'(x') = S(\Lambda)\psi(x)$

where $S(\Lambda)$ is to be discovered...

so $(i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0 = (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)S(\Lambda)\psi(x) = 0$

$\Rightarrow i S^{-1}(\Lambda) \gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S(\Lambda) \psi(x) - m \psi(x) = 0$

but — $m\psi(x) = i\gamma^\mu \partial_\mu \psi(x)$ (where we started!)

$\Rightarrow \underline{S^{-1}(\Lambda) \gamma^\mu \Lambda_\mu{}^\nu S(\Lambda) = \gamma^\nu}$ (1)

Now we assume $S(\Lambda)$ can also be written w/ some generator namely we say: $S(\Lambda) = e^{\frac{-i}{4}\omega_{\mu\nu}\beta^{\mu\nu}}$

where $\beta^{\mu\nu} = (\beta^{\mu\nu})_{\rho\sigma}$ = 4x4 matrices w/ $\beta^{\mu\nu} = -\beta^{\nu\mu}$

$\therefore S(\Lambda) \simeq 1 - \frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu}$

& $S(\Lambda)^{-1} \simeq (1 - \frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu})^{-1} \simeq 1 + \frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu} + O(\omega^2)$

where $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu$ as before.

then use (1) to say $S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda) - \gamma^\nu = 0$

& expand LHS up to 2nd order

$0 = (1 + \frac{i}{4}\beta_{\alpha\beta}\omega^{\alpha\beta})\gamma^\mu(\delta_\mu{}^\nu + \omega_\mu{}^\nu)(1 - \frac{i}{4}\beta_{\alpha\beta}\omega^{\alpha\beta}) - \gamma^\nu$

$= \cancel{\gamma^\mu \delta_\mu{}^\nu} + \gamma^\mu \omega_\mu{}^\nu + \frac{i}{4}\beta_{\alpha\beta}\omega^{\alpha\beta}\gamma^\mu\delta_\mu{}^\nu - \frac{i}{4}\gamma^\mu \delta_\mu{}^\nu \beta_{\alpha\beta}\omega^{\alpha\beta} \cancel{-\gamma^\nu}$

$\Rightarrow \frac{i}{4}(\beta_{\alpha\beta}\omega^{\alpha\beta}\gamma^\nu - \gamma^\nu\beta_{\alpha\beta}\omega^{\alpha\beta}) = -\gamma_\mu\omega^{\mu\nu} = \gamma_\mu\omega^{\nu\mu}$

So lower the ν's get

$\frac{i}{4}(\beta_{\alpha\beta}\omega^{\alpha\beta}\gamma_\nu - \gamma_\nu\beta_{\alpha\beta}\omega^{\alpha\beta}) = \gamma_\mu\omega_\nu{}^\mu = g_{\nu\rho}\gamma_\mu\omega^{\rho\mu}$ (2)

Now ω Antisymm

$\Rightarrow g_{\nu\rho}\gamma_\mu \omega^{\rho\mu} = \frac{1}{2}\omega^{\rho\mu}(\eta_{\nu\rho}\gamma_\mu - \eta_{\nu\mu}\gamma_\rho)$

Change some names on LHS of (2) get

$$\omega^{\rho\mu}\frac{i}{4}(\beta_{\rho\mu}\gamma_\nu - \gamma_\nu\beta_{\rho\mu}) = \frac{1}{2}\omega^{\rho\mu}(\eta_{\nu\rho}\gamma_\mu - \eta_{\nu\mu}\gamma_\rho)$$

Now ~~to arbitrary antisymm~~ drop ω since its pretty arbitrary to get

$$[\gamma_\nu, \beta_{\rho\mu}] = 2i(\eta_{\nu\rho}\gamma_\mu - \eta_{\nu\mu}\gamma_\rho)$$

Now we observe that for β we can put something easily made from Dirac matrices namely $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ then Check

$$[\gamma_\nu, \frac{i}{2}[\gamma_\rho, \gamma_\mu]] = 2i(\eta_{\nu\rho}\gamma_\mu - \eta_{\nu\mu}\gamma_\rho)$$

well.. just expand

$$[\gamma_\nu, \frac{i}{2}[\gamma_\rho, \gamma_\mu]] = \frac{i}{2}(\gamma_\nu(\gamma_\rho\gamma_\mu - \gamma_\mu\gamma_\rho) - (\gamma_\rho\gamma_\mu - \gamma_\mu\gamma_\rho)\gamma_\nu)$$

$$= \frac{i}{2}(\gamma_\nu\gamma_\rho\gamma_\mu - \gamma_\nu\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\rho\gamma_\nu) \quad (3)$$

Now $\gamma_\nu\gamma_\rho = 2\eta_{\nu\rho} - \gamma_\rho\gamma_\nu$

$\gamma_\nu\gamma_\mu = 2\eta_{\nu\mu} - \gamma_\mu\gamma_\nu$

so $(3) = \frac{i}{2}(\eta_{\nu\rho}\gamma_\mu - \gamma_\rho\gamma_\nu\gamma_\mu - \eta_{\nu\mu}\gamma_\rho + \gamma_\mu\gamma_\nu\gamma_\rho - \eta_{\mu\nu}\gamma_\rho + \gamma_\rho\gamma_\nu\gamma_\mu$
$+ \eta_{\rho\nu}\gamma_\mu - \gamma_\mu\gamma_\nu\gamma_\rho)$

we finally get

$$\frac{i}{2}\left(\cancel{4g}\, 4\eta_{\nu\rho}\gamma_\mu - 4\eta_{\nu\mu}\gamma_\rho\right) = 2i\left(\eta_{\nu\rho}\gamma_\mu - \eta_{\nu\mu}\gamma_\rho\right)$$

As claimed

So now ~~we~~ we see that $\psi'(x') = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}}\psi(x)$

Let's also compare with our generator

From Lie Algebra stuff we have that:

$\psi'(x) = e^{-\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}}\psi(x)$ where $J_{\mu\nu}$ depend on ~~represent~~ representation

here we consider small ω & Dirac repr.

$$\Rightarrow \psi'(x) \simeq \left(1 - \frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\right)\psi(x) \quad (4)$$

while we just found $\psi'(x') = \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x)$

but.. we can also Teylor expand the above

$$\psi'(x') = \psi'(x) + \frac{\partial\psi'(x)}{\partial x^\mu}(x' - x)^\mu$$

when $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$

$$\Rightarrow \psi'(x') = \psi'(x) + \omega^{\mu\nu}x_\nu\partial_\mu\psi'(x)$$

but.. $\omega^{\mu\nu}x_\nu\partial_\mu = \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)$

$$\Rightarrow \psi'(x) + \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi'(x) = \psi(x) - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x)$$

$$\Rightarrow \psi(x) = \psi'(x) + \frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\psi(x)$$

(from eqn (4))

So $-\frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi'(x) = \frac{i}{2}\omega^{\mu\nu}(J_{\mu\nu} - \frac{1}{2}\sigma_{\mu\nu})\psi(x)$

Now $\psi'(x) = \psi(x) + \frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\psi(x)$ but we only work to 1st order in ω so replace $\psi'(x)$ by $\psi(x)$ above to get.

$\frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi(x) = \frac{i}{2}\omega^{\mu\nu}(J_{\mu\nu} - \frac{1}{2}\sigma_{\mu\nu})\psi(x)$

Now this holds $\forall\omega$ & $\forall\psi$ in the relevant sets

So $J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}$

Observe that:

$S^i = \frac{1}{2}\epsilon^{ijk}\sigma^{jk} = \frac{\sigma^i}{2}$ ← Pauli if we use the Dirac-Pauli stuff for γ^μ matrizes

ie we get bak the $\vec{S} = \begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$ stuff

$\therefore$ ψ carry intrinsic spin when we consider Lorentz transformation.