Jared Stang
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Differential Representation of Maxwell's Equations. (in Minkowski space)

Claim:

$$\boxed{\begin{array}{lll} dF = 0 & \Leftrightarrow \quad \partial_{[\sigma} F_{\mu\nu]} = 0 & (*) \\ d{}^*F = {}^*J & \Leftrightarrow \quad \partial_\mu F^{\mu\nu} = J^\nu & (**) \end{array}}$$

Defs:

$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ – Faraday 2-form.

$*$ – Hodge star operator $\left({}^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}\right)$
← potato notes

${}^*F$ – Maxwell 2-form

$J = \frac{J^\alpha}{6} \varepsilon_{\alpha\beta\mu\nu}\, dx^\beta \wedge dx^\mu \wedge dx^\nu$ – current 3-form.

Remark: Source free equations $(*)$ are ones we get for "free", by definition of Faraday tensor.

Equations with a source $(**)$ are derived from action principle.

Justification:

$(*)$ $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$.

By definition,

$$dF = \frac{1}{2} \partial_\alpha F_{\mu\nu}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu \overset{\text{set}}{=} 0.$$

Now, $dx^\alpha \wedge dx^\mu \wedge dx^\nu$ is completely antisymmetric. A symmetric tensor contracted with an antisymmetric tensor gives zero.

$$(\eta^{\mu\nu}F_{\mu\nu} = \eta^{\nu\mu}F_{\nu\mu} = -\eta^{\mu\nu}F_{\mu\nu} \Rightarrow \eta^{\mu\nu}F_{\mu\nu} = 0)$$

Therefore, the antisymmetric part of $\partial_\alpha F_{\mu\nu}$ must vanish.

ie. $\partial_{[\alpha} F_{\mu\nu]} = 0$, as desired.

(In vacuum first $\rightarrow J = 0$)

(**) $\cdot\ {}^*F = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\,dx^\mu \wedge dx^\nu$

$$\Rightarrow d^*F = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\,\partial_\gamma F^{\alpha\beta}\,dx^\gamma \wedge dx^\mu \wedge dx^\nu \overset{\text{set}}{=} 0.$$

Now $dx^\gamma \wedge dx^\mu \wedge dx^\nu = 0$ unless $\gamma \neq \mu, \nu$.
$\Rightarrow \gamma = \alpha, \beta$ (Minkowski space ... only 4 choices for each index. $\varepsilon_{\mu\nu\alpha\beta} = 0$ unless $\mu \neq \nu \neq \alpha \neq \beta$).

$$\Rightarrow 0 = d^*F = \sum_{\gamma=\alpha,\beta} \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\,\partial_\gamma F^{\alpha\beta}\,dx^\gamma \wedge dx^\mu \wedge dx^\nu$$

$$= \sum_\alpha{}' \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha F^{\alpha\beta}\,dx^\alpha \wedge dx^\mu \wedge dx^\nu + \sum_\beta{}' \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\,\partial_\beta F^{\alpha\beta}\,dx^\beta \wedge dx^\mu \wedge dx^\nu$$

Now, $\varepsilon_{\mu\nu\alpha\beta}\partial_\beta F^{\alpha\beta} = \varepsilon_{\mu\nu\beta\alpha}\partial_\beta F^{\beta\alpha}$, so renaming the dummy indices in the second sum above ($\alpha \rightarrow \beta$, $\beta \rightarrow \alpha$) gives:

$$0 = \frac{1}{2}\sum_\alpha{}' \varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha F^{\alpha\beta}\,dx^\alpha \wedge dx^\mu \wedge dx^\nu$$

permuted dx^δ's

$$= \frac{1}{2}\sum_\alpha{}' \varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha F^{\alpha\beta}\,dx^\mu \wedge dx^\nu \wedge dx^\alpha \qquad (\#)$$

This is just a linear combination of basis wedge products:

$\frac{1}{2}\sum_{\alpha} \varepsilon_{\mu\nu\alpha\beta}\, \partial_\alpha F^{\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha$

not a form, just a coefficient.

$= a_0\, dx^1 \wedge dx^2 \wedge dx^3 + a_1\, dx^0 \wedge dx^2 \wedge dx^3$
$\quad + a_2\, dx^0 \wedge dx^1 \wedge dx^3 \quad + a_3\, dx^0 \wedge dx^1 \wedge dx^2,$

where the subscript i in each coefficient a_i refers to the value of the index β for all terms making up the coefficient. (Ex. When $\beta=0$, μ, ν, and α will assume values from $\{1,2,3\}$ for non-trivial terms in the sum, as $\varepsilon_{\mu\nu\alpha\beta}=0$ if $\mu=\beta$, $\nu=\beta$, OR $\alpha=\beta$. The dx^i's can be permuted then to $dx^1 \wedge dx^2 \wedge dx^3$, with the resulting coefficient a_0).

As a linear combination of independent elements set to zero, we have that each coefficient $\underline{a_\beta = 0}$.

What are the a_β's?

$\longrightarrow$ picked up from permuting dx^i's.

$$a_\beta = \sum_\alpha \tfrac{1}{2}\, \varepsilon^{\mu\nu\alpha}\, \varepsilon_{\mu\nu\alpha\beta}\, \partial_\alpha F^{\alpha\beta}$$
$$= \partial_\alpha F^{\alpha\beta} \quad \Rightarrow \quad \partial_\alpha F^{\alpha\beta} = 0.$$

Essentially,
Negatives picked up from permuting the dx^i's are cancelled by the $\varepsilon_{\mu\nu\alpha\beta}$, leaving the divergence $\partial_\alpha F^{\alpha\beta}$.

To see this concretely, consider $\beta=0$.

$$\frac{1}{2}\sum_\alpha \varepsilon_{\mu\nu\alpha 0}\, \partial_\alpha F^{\alpha 0}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha$$

$$= \frac{1}{2}\Big(\varepsilon_{1230}\partial_3 F^{30} dx^1\wedge dx^2 \wedge dx^3 + \varepsilon_{2130}\partial_3 F^{30} dx^2 \wedge dx^1 \wedge dx^3$$

$$+ \varepsilon_{1320}\partial_2 F^{20} dx^1 \wedge dx^3 \wedge dx^2 + \varepsilon_{3120}\partial_2 F^{20} dx^3 \wedge dx^1 \wedge dx^2$$

$$+ \varepsilon_{2310}\partial_1 F^{10} dx^2 \wedge dx^3 \wedge dx^1 + \varepsilon_{3210}\partial_1 F^{10} dx^3 \wedge dx^2 \wedge dx^1 \Big)$$

$$(\#\#) \quad = \frac{1}{2}\left(-\partial_3 F^{30} - \partial_3 F^{30} - \partial_2 F^{20} - \partial_2 F^{20} - \partial_1 F^{10} - \partial_1 F^{10}\right) dx^1 \wedge dx^2 \wedge dx^3$$

$$= -\partial_\alpha F^{\alpha 0}\, dx^1 \wedge dx^2 \wedge dx^3 .$$

Note: $F^{\mu\nu}$ antisymmetric $\Rightarrow F^{\beta\beta} = 0$, so it doesn't matter that the term with $\alpha = \beta$ disappears due to $\varepsilon_{\mu\nu\alpha\beta}$.

To add a source term, we consider the current 3-form, which is just the Hodge dual of the four current. In component form, it is:

$${}^*J_{\beta\mu\nu} = J^\alpha \varepsilon_{\alpha\beta\mu\nu} .$$

Expanded in the wedge product basis,

$${}^*J = \frac{1}{6} J^\beta \varepsilon_{\beta\alpha\mu\nu}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu .$$

We equate this with $d{}^*F$, and get, at $(\#)$,

$$\cancel{d{}^*F - {}^*J = \frac{1}{4}\partial_\alpha F^{\alpha\beta} - \frac{1}{6}J^\beta}$$

$$0 = \sum_\alpha \varepsilon_{\mu\nu\alpha\beta}\left(\frac{1}{2}\partial_\alpha F^{\alpha\beta} + \frac{1}{6}J^\beta\right) dx^\mu \wedge dx^\nu \wedge dx^\alpha$$

The J^β's essentially just go along for the ride in the above manipulations, and then at (##), we get 6 factors of $\frac{1}{6}J^0$, giving us the desired result:

$$\partial_\alpha F^{\alpha\beta} + J^\beta = 0$$

(modulo a negative times J^β which I can't seem to figure out).