Navier Stokes Equations

1. Reformulations, structural properties, Hodge decomposition

The fluid velocity $\underline{u}(t,x)$ of a viscous, homogeneous, incompressible fluid that fills the entire space $\mathbb{R}^d$ evolves according to the Navier - Stokes equations

$$(1) \qquad \begin{cases} \rho\, \partial_t \underline{u} = \mu \Delta \underline{u} - \rho (\underline{u} \cdot \underline{\nabla}) \underline{u} - \underline{\nabla} p + \underline{f} \\ \underline{\nabla} \cdot \underline{u} = 0. \end{cases}$$

The condition $\underline{\nabla} \cdot \underline{u} = 0$ express <u>incompressibility</u> of the fluid. ρ is the (constant) <u>density</u> of the fluid. μ is the <u>viscosity coefficient</u>. p is the (unknown) <u>pressure</u>. Without loss of generality, we may assume $\rho = 1$ and will redenote $\mu = \nu$. (We could also assume $\nu = 1$ but will retain ν for various reasons.) f is <u>external force</u>.

In coordinate notation, we denote $\underline{u} = (u^1, \dots, u^d)$ and can reexpress (1) as

$$(2) \qquad \begin{cases} \partial_t u^\ell + u^j \partial_j u^\ell = \nu \Delta u^\ell - \partial_\ell p + f^\ell \\ \partial_\ell u^\ell = 0. \end{cases}$$

The <u>tensor product</u> of two vectors is a matrix

$$(\underline{v} \otimes \underline{w})_{\ell k} = v^\ell w^k.$$

We calculate

$$\underline{\nabla} \cdot (\underline{v} \otimes \underline{w}) = \partial_\ell (v^\ell w^k) = (\partial_\ell v^\ell) w^k + v^\ell \partial_\ell w^k.$$

$$= (\underline{\nabla} \cdot \underline{v})\, \underline{w} + (\underline{v} \cdot \underline{\nabla})\, \underline{w}.$$

For <u>divergence free</u> $\underline{u}$ (so $\underline{\nabla} \cdot \underline{u} = \partial_k u^k = 0$)

$$\underline{\nabla} \cdot (\underline{u} \otimes \underline{u}) = (\underline{u} \cdot \underline{\nabla})\, \underline{u} = u^j \partial_j u^k.$$