General Relativity APM 426 U. Toronto Mathematics Dept. Spring 2002

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WE 69 MWF 3-4

Overview: General Relativity (GR) is the theory of space, time and gravitation formulated by Einstein in 1915. It has a reputation of being very difficult:

- GR goes against some deeply ingrained intuitive notions
- GR is expressed using sophisticated mathematics.

Einstein's equations for general relativity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \underbrace{8\pi G}_{\substack{\text{constant chosen}\\\text{by comparing with}\\\text{Newtonian gravity}}} T_{\mu\nu}$$

Ricci curvature tensor

Ricci Scalar

(pseudo-riemannian) spacetime metric

stress-energy-momentum tensor

These equations are a relativistically invariant generalization of the Poisson equation for the Newtonian gravitational potential equation

$$\Delta \Phi = 4\pi G \rho$$

Laplacian

Newtonian potential

Mass density

A basic goal of this course is to understand the einstein equations and the notion of "relativistically invariant".

I. Force-Free particle dynamics

A. Variational characterization of dynamics; Hamiltonian dynamics

B. Classical Free particle

- simplest case; Cartesian coordinates
- general coordinates; Riemannian manifolds

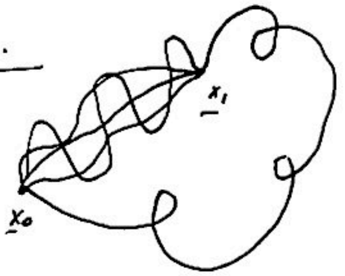
C. Geodesics

- geodesic equation
- geodesic deviation; curvature

D. Relativistic Free particle

- spacetime structure; spacetime interval
- simultaneity / colocation / Lorentz transformations
- causal structure
- proper time; proper length
- Lagrangian dynamics of relativistic free particle; 4-momentum.

A. Variational characterization of dynamics; Hamiltonian dynamics.



$$\mathcal{A} = \{ w : [t_0, t_1] \longrightarrow \mathbb{R}^3 \mid w(t_0) = \underline{x}_0 \,,\ w(t_1) = \underline{x}_1 \}$$

"admissible class" "competitors" "paths"

This section based on PDE by L.C. Evans.

Action Functional: $\mathcal{A} \longrightarrow \mathbb{R}$

$$I[w] = \int_{t_0}^{t_1} L(\dot{w}(t), w(t), t)\, dt \qquad \text{where} \quad L : \underset{\dot{w}}{\mathbb{R}^3} \times \underset{w}{\mathbb{R}^3} \times \underset{t}{\mathbb{R}} \longmapsto \mathbb{R} \quad \text{is given.}$$

<u>Hamilton's Principle</u> Mechanical systems posses a Lagrangian L such that the actual motion $t \longmapsto y(t)$ satisfies

$$I[y] = \min_{y \in \mathcal{A}} I[w].$$

System of n 2nd order ODE.

Note: $L \rightarrow L + c$ d/n affect dynamics.

calculus of variations $\Longrightarrow$ y ⓢ Euler-Lagrange equation(s)

(eqs. of motion) $$-\frac{d}{dt}\left(L_{\dot{y}}(\dot{y}(t), y(t), t)\right) + L_y(\dot{y}(t), y(t), t) = 0.$$ (Lagrangian dynamics)

Example $L(\dot{w}, w, t) = \frac{1}{2} m |\dot{w}|^2 - V(y)$.

$$- \frac{d}{dt} m \dot{w} - \nabla_y V = 0$$

$$m \ddot{w} = - \nabla_y V$$ (Newton's law for a particle in the potential V.)

Remark: We will study the case $V = 0$ of this example in various generalized settings. The resulting force-free particle dynamics will be a key device in our tool box in this class. Before turning to this, we recall Hamiltonian dynamics.

Hamiltonian Dynamics Suppose we are given a Lagrangian → Lagrangian dynamics

)

Define

$$p(t) = L_{\dot{y}}(\dot{y}(t), y(t), t).$$

Form $$H(p(t), y(t), t) = \sup_{\dot{y}} \left(p(t) \cdot \dot{y} - L(\dot{y}, y(t), t) \right).$$

(H is the Legendre transformation of L.)

Then, the Lagrangian dynamical equations may be reexpress

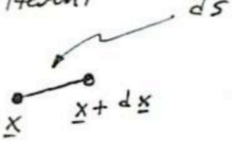
Hamilton's equations $$\begin{cases} \dot{p} = - H_y \\ \dot{y} = H_p . \end{cases}$$ [system of $2n$ 1st order ODE.

Remarks:

- Lagrangian + Hamiltonian descriptions are canonical.
- Hamilton's eqs. are special among general 1st order ODE systems.
  $\Longrightarrow$ phase space volume conservation.

B. Classical Free Particle

We will describe the motion in 3d space in 3 different systems of coordinates:



(taken from H. Stephani GR book; Chapter 1) & §3.2.

Cartesian Coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 .$$

Spherical Coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

General Coordinates

$$ds^2 = \underbrace{g_{\alpha\beta}}_{\text{metric tensor}} (x^\nu)\, dx^\alpha dx^\beta \quad ; \quad \alpha, \beta, \nu = 1, 2, 3$$

symmetric $g_{\alpha\beta} = g_{\beta\alpha}$

invertible with inverse $g^{\alpha\beta}$

$$g^{\alpha\beta} g_{\beta\nu} = \delta^\alpha_\nu$$

These expressions explain how our coordinates are related to length. A deflection of dx, dy or dz changes length of $(\underline{x} + d\underline{x}) - \underline{x}$ by a pythagorean rule. But, a change in $d\theta$ has a different impact.

I will do Cartesian and general cases and assign the discussion of spherical Coordinates as an exercise.

Lagrangian derivation of eqs. of motion for force-free classical particle (general coordinates) $t \mapsto x^\kappa(t)$ is a particle trajectory.

$$L = \frac{m}{2} |v|^2 = \frac{m}{2} g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = \frac{m}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta .$$

velocity

(implicitly, we have used

$$|v|^2 = \left| \frac{ds}{dt} \right|^2 = \frac{ds^2}{dt^2} = g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} .)$$

Variation

$\rightsquigarrow$ C.o.V. / Hamilton's principle $\rightsquigarrow$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L}{\partial x^\nu} = 0$$

$$\frac{\partial L}{\partial \dot{x}^\nu} = 2 \frac{m}{2} g_{\alpha\nu} \dot{x}^\alpha$$

$$\frac{\partial L}{\partial x^\nu} = \frac{m}{2} g_{\alpha\beta,\nu} \dot{x}^\alpha \dot{x}^\beta .$$

$,\nu$ notation

~~chain rule~~

~~$\frac{m}{2} g_{\alpha\nu,\beta} \dot{x}^\alpha \dot{x}^\beta$~~ + ~~$\frac{m}{2} g_{\alpha\nu} \ddot{x}^\alpha - \frac{m}{2} g_{\alpha\beta,\nu} \dot{x}^\alpha \dot{x}^\beta = 0$~~

~~$\frac{m}{2} \frac{1}{2} (g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha}) \dot{x}^\alpha \dot{x}^\beta$~~

~~[illegible]~~ ~~$+ g_{\alpha\nu} \ddot{x}^\alpha$ [illegible] $\frac{1}{2} [g_{\alpha\nu,\beta}$ [illegible]~~

✓ $$\cancel{m} g_{\alpha\nu,\beta} \dot{x}^\alpha \dot{x}^\beta + \cancel{m} g_{\alpha\nu} \ddot{x}^\alpha - \frac{\cancel{m}}{2} g_{\alpha\beta,\nu} \dot{x}^\alpha \dot{x}^\beta = 0$$

$\frac{1}{2}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha}) \dot{x}^\alpha \dot{x}^\beta$

$$g_{\alpha\nu} \ddot{x}^\alpha + \frac{1}{2} (g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \dot{x}^\alpha \dot{x}^\beta = 0$$

Apply $g^{\mu\nu}$ on left

Christoffel symbol.

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0}$$

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha} .$$

Example Cartesian Coordinates

$$g_{\mu\nu} = \delta_{\mu\nu} \Longrightarrow g_{\mu\nu,\alpha} = 0 \quad \forall\ \alpha, \mu, \nu.$$

$\longrightarrow$ $\ddot{x}^{\mu} = 0$ so, in these coordinates, we have that

$x^{\mu}(t) = a^{\mu} t + b^{\mu}$ and we find straight line motion

as the solution of the force free particle dynamics equation.

Exercise 1.1 Find the equations of motion for force-free particle motion in 3d spherical coordinates. Find the Christoffel coefficients for ~~the~~ ~~2d~~ ~~sphere~~ ~~sitting~~ ~~inside~~ ~~R^3~~. $\mathbb{R}^3$ using spherical coordinates. Calculate $\left[\Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\rho\beta}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\mu\beta}\right]$.

(from Stephani)

C. Geodesic Equation

path followed by force-free particle in $\mathbb{R}^3$ $\Longleftrightarrow$ straight line $\Longleftrightarrow$ shortest curve between 2 points

geodesic curve

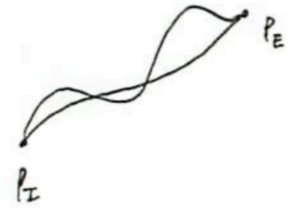
$\uparrow$

based here ... we generalize.

path followed by force-free particle moving on a surface sitting inside $\mathbb{R}^3$

or in a variable metric generalized $\mathbb{R}^3$ $\Longleftrightarrow$ geodesic.

e.g. 2-sphere $ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$



paths from P_I to P_E in our space.

A <u>geodesic curve</u> is the shortest curve between 2 points.

$$S \doteq \int_{P_I}^{P_E} ds = \text{extremum.}$$

Let $\lambda \in [\lambda_I, \lambda_E]$ parametrize the paths uniformly in the sense that $\underline{X}(\lambda_{I,E}) = P_{I,E}$. The parametrization allows us to write

$$S = \int_{\lambda_I}^{\lambda_E} \frac{ds}{d\lambda}\, d\lambda = \int_{\lambda_I}^{\lambda_E} \underbrace{\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}}_{\equiv \sqrt{F} = L \ \longrightarrow \ E-L \text{ eq.}}\, d\lambda = \text{extremum.}$$

(crank) and write $\frac{dx^\alpha}{d\lambda} = x'^\alpha$.

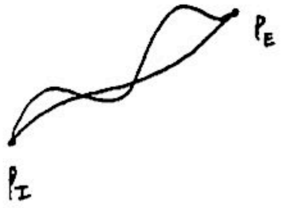
$$0 = \frac{1}{2F\sqrt{F}} \left[-\frac{dF}{d\lambda} g_{\alpha\gamma} x'^\alpha + 2F \frac{d}{d\lambda}\left(g_{\alpha\gamma} x'^\alpha\right) - F g_{\alpha\beta,\gamma} x'^\alpha x'^\beta \right]$$

analogous to setting initial velocity

Suppose we <u>restrict</u> the parametrization λ s.t. $\lambda \sim s$ (arc length) in the setting of the minimizer. The geodesic segment connecting P_I to P_E has arc length $\lambda_E - \lambda_I$; and the λ parametrization traces the curve with <u>unit</u> speed. Thus $F = 1$ for the geodesic. Thus

$$\frac{d}{d\lambda}\left(g_{\alpha\gamma} x'^\alpha\right) - \frac{1}{2} g_{\alpha\beta,\gamma} x'^\alpha x'^\beta = 0.$$

$\longrightarrow$ The geodesic solves the force-free particle motion equation.



paths from P_I to P_E in our space.

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$$S = \int_{\lambda_I}^{\lambda_E} \frac{ds}{d\lambda}\, d\lambda = \int_{\lambda_I}^{\lambda_E} \underbrace{\sqrt{g_{\alpha\beta} \frac{d\underline{x}^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}}\, d\lambda = \text{extremum.}$$

$$\equiv \sqrt{F} = L \longrightarrow E-L \text{ eq.}$$

(crank) and write $\frac{dx^\alpha}{d\lambda} = X'^\alpha$.

$$0 = \frac{1}{2F\sqrt{F}} \left[-\frac{dF}{d\lambda} g_{\alpha\gamma} X'^\alpha + 2F \frac{d}{d\lambda}\left(g_{\alpha\gamma} x'^\alpha\right) - F g_{\alpha\beta,\gamma} X'^\alpha X'^\beta \right.$$

→ analogous to setting initial velocity

Suppose we <u>restrict</u> the parametrization λ s.t. $\lambda \sim s$ (arc length) in the setting of the minimizer. The geodesic segment connecting P_I to P_E has arc length $\lambda_E - \lambda_I$, and the λ parametrization traces the curve with <u>unit</u> speed. Thus $F = 1$ for the geodesic. Thus

$$\frac{d}{d\lambda}\left(g_{\alpha\gamma} x'^\alpha\right) - \frac{1}{2} g_{\alpha\beta,\gamma} X'^\alpha X'^\beta = 0.$$

⟶ The geodesic solves the force-free particle motion equation.

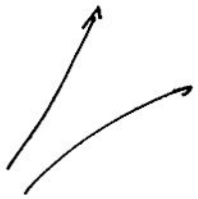
A force-free particle moves on a geodesic

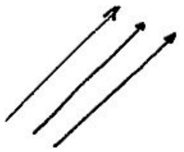
$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

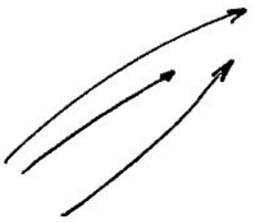
of the space or surface where the dynamics are constrained.

Note: The form of the geodesic/force-free motion equation is heavily dependent on the choice of coordinates. Witness the form in Cartesian coordinates vs. spherical coordinates in 3d.

Suppose the geodesic flows look like

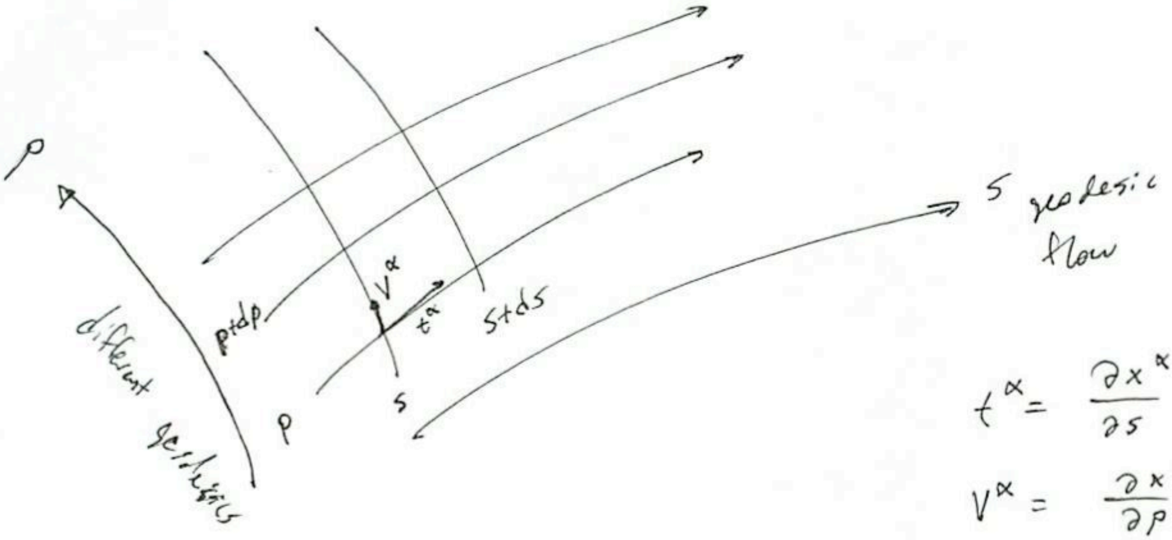






We may therefore hope to infer structural properties of the space containing the dynamics from the flow. We set our sights lower and hope to answer:

Q: How can we determine from the Christoffel symbols (or at least from the geodesic equation) whether the space is actually $\mathbb{R}^3$ (or flat Euclidean)?



$$t^\alpha = \frac{\partial x^\alpha}{\partial s}$$

$$V^\alpha = \frac{\partial x^\alpha}{\partial \rho}$$

Mixed partials

$$\frac{\partial t^\alpha}{\partial \rho} = \frac{\partial V^\alpha}{\partial s}.$$

Consider

$$\frac{D}{Ds} t^\mu = \frac{d}{ds} t^\mu + \Gamma^\mu_{\alpha\beta} t^\alpha \frac{dx^\beta}{ds}.$$

$s \longleftrightarrow t^\alpha$

$$\frac{D}{Ds} W^\mu = \frac{d}{ds} W^\mu + \Gamma^\mu_{\alpha\beta} t^\alpha W^\beta.$$

$$\frac{D}{Ds}(\cdot)^\mu = \frac{d}{ds}(\cdot)^\mu + \Gamma^\mu_{\alpha\beta} t^\alpha (\cdot)^\beta$$ operator

$\rho \longleftrightarrow V^\alpha$

$$\frac{D}{D\rho} = \frac{d}{d\rho}(\cdot) + \Gamma^\mu V^\alpha (\cdot)$$ operator

Calculate: $\frac{D}{D\rho} t^\alpha$, $\frac{D}{Ds} V^\alpha$.

$\frac{D}{Ds}$, $\frac{D}{D\rho}$.

We choose to look at $\frac{D^2 V^\alpha}{Ds^2}$ as a tool to understand

The surface or space we are in.

$$\frac{D^2 V^\alpha}{Ds^2} = \frac{D}{Ds}\left(\frac{D}{D\rho} t^\alpha\right)$$

A calculation shows that

$$\frac{D^2 v^\alpha}{Ds^2} = \frac{D}{Ds}\left(\frac{D}{Dp} t^\alpha\right)$$

$$= \cdots = t^\beta t^\mu v^\nu \underbrace{\left(\Gamma^\alpha_{\mu\nu,\beta} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\alpha_{\rho\beta}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\rho\nu}\Gamma^\rho_{\mu\beta}\right)}_{R^\alpha{}_{\mu\nu\beta}} .$$

<u>Note</u>: $\Gamma^\mu_{\alpha\beta} = g^{\text{[illegible]}} \frac{1}{2}\left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right) \underset{\uparrow}{=} 0.$

if $g_{\mu\nu} = \delta_{\mu\nu}$.

Euclidean / Cartesian

D. Relativistic Free Particle

(from Landau-Lifschitz Classical Th. of Fields)

A *system of reference* is a coordinate system serving to indicate the *location* of a particle as well as a system of clocks fixed in this system to indicate *time*.

systems of reference in which a free-force motion proceeds w. constant velocity are called *inertial*.

∃ many inertial systems of reference.

Experiments have validated the *principle of relativity*:

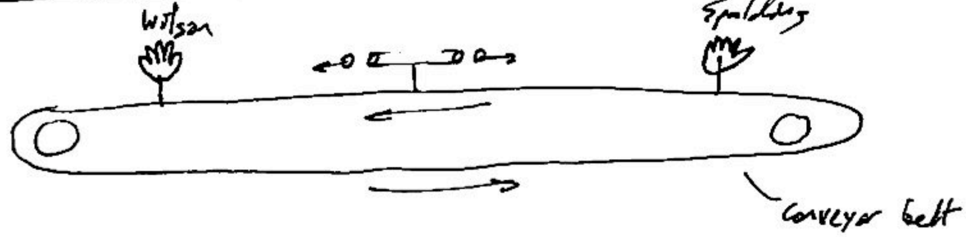
- laws of nature are identical in all inertial systems of reference
- eqs. expressing laws of nature are invariant under coordinate transformations between inertial reference systems.

Experiments have shown that natural processes are governed by interactions with a maximum signal velocity.

$$\text{light speed} = c = 2.999 \times 10^{10}\ \text{cm}/\text{s}. \qquad (\text{huge})$$

principle of relativity ⟹ light speed is same in all inertial frames.

Thought Experiment



⟹ simultaneity fails.

Q: What is the structure of spacetime? Are events causally linked?

This is rather unsettling...

Light speed invariance w.r.t. inertial frame

• P_2: light is received.

• P_1: light is released.

observed in K system	observed in K' system
$P_1: (t_1, x_1, y_1, z_1)$	$P_1: (t_1', x_1', y_1', z_1')$
$P_2: (t_2, x_2, y_2, z_2)$	$P_2: (t_2', x_2', y_2', z_2')$
light transit time → $c(t_2 - t_1)$ space units separate P_1, P_2 in K frame	$C(t_2' - t_1')$ space units between P_1, P_2 in K' frame.
$(X_2-X_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2 = 0$	$(x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2 - c^2(t_2'-t_1')^2 = 0$

(P_1, P_2 are events connected by light.)

Definition

$$I = \left\{ -c^2(t_2-t_1)^2 + \left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \right] \right\}^{\frac{1}{2}}$$

is called the <u>spacetime interval</u> between events P_1, P_2.

Fact: The interval between two events is the same in all inertial frames.

☆ Sign troubles in lecture lead me to redefine the spacetime interval as

$$I^2 = \left\{ -c^2(t_2-t_1)^2 + \left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \right] \right\}.$$

Simultaneity Question

Given events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) in an inertial frame, does $\exists$ another inertial frame in which the events are simultaneous:

$$t_1' = t_2' \quad ?$$

Write t_{12} for $t_1 - t_2$ and l_{12}^2 for $(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$.

$$s_{12}^2 = -c^2 t_{12}^2 + l_{12}^2 .$$

In any new frame K', t_{12}' and l_{12}' must satisfy

$$-c^2 t_{12}^2 + l_{12}^2 = -c^2 (t_{12}')^2 + (l_{12}')^2 .$$

We want $(t_{12}')^2 = 0$ so, necessarily

$$-c^2 t_{12}^2 + l_{12}^2 > 0 . \qquad \text{(spacelike)}$$

Colocation Question

In K', can we force $(l_{12}')^2 = 0$: colocation.

We must have

$$-c^2 t_{12}^2 + l_{12}^2 < 0 \qquad \text{(timelike)}.$$

We will use Lorentz transformations to show these necessary conditions suffice for bringing events into simultaneity or colocation via inertial frame changes

Our intuition suggests spacetime = $\{\text{events}\}$ has a <u>causal structure:</u>

Given events p, q. $\exists$ 3 alternatives:

1. It is possible, in principle, for an observer to go from q to p. (q is <u>to the past</u> of p)

2. p is to the past of q

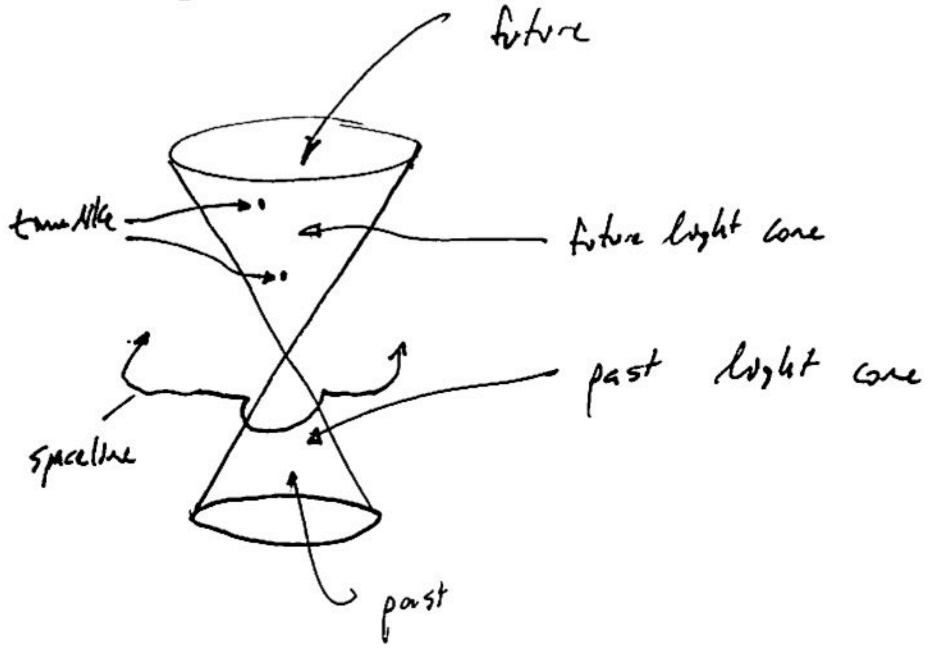
3. It is impossible, even in principle, for an observer to go from p to q or from q to p. (p and q are simultaneous.)

Subdivide 3. further:

(i) events in 3) on $\partial\{\text{points to future of } p\}$ (future light cone)

(ii) " " past " (past light cone)

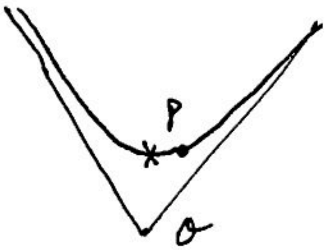
(iii) " " not in either light cone.



(from Wald, if I recall correctly.)

Colocation Picture

O, P timelike

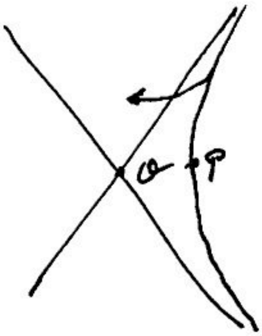


The changing of inertial frames sweeps P along the upper sheet of a hyperboloid of 2 sheets above O.

Choosing the frame pushing P closest to O colocates events P, O.

Simultaneity picture

O, P spacelike



changing frames sweeps of a hyperboloid of 1 sheet.

Choosing frame → P nearest O simultanizes P, O.

Causality is rescued!

Temporal ordering of events is the same in all frames.

spacelike events are not time ordered.

Rotation Matrices

Let R be an $n \times n$ matrix. Suppose that

$$(R)^T (Id) (R) = Id.$$

$\Rightarrow$ $R^T = R^{-1}$ and, if we have a real inner product

$$(u, v) = (u, R^{-1}Rv) = (u, RR^{-1}v) = (R^T u, R^{-1} v)$$
$$= (R^{-1}u, R^{-1}v)$$

So R and R^{-1} preserve inner products and lengths.

Such matrices are said to be <u>orthogonal.</u>

Orthogonal matrices preserve lengths.

Lorentz Transformations

The matrices Λ which satisfy

$$\eta = \Lambda^T \eta \Lambda$$

~~define changes of inertial frame preserve the spacetime interval.~~

define linear changes of variable $x' = \Lambda x$ (x 4-vector)

which preserve the spacetime interval.

[illegible] Such matrices are called <u>Lorentz transformations</u>

The general Lorentz transformation depends on 6 parameters:

3 for the **boosts** $\quad x t$, $y t$, $z t$

3 for the **rotations** $\quad x y$, $x z$, $y z$.

rotations (in space)

$\theta \in [0, 2\pi]$ ↑ rotation parameter

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Boosts (spacetime rotations)

$\phi \in (-\infty, \infty)$

$$\Lambda = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lorentz Group $\cup$ $\{$translations$\}$ = Poincaré group.

Boosts:

$$t' = t\cosh\phi - x\sinh\phi$$
$$x' = -t\sinh\phi + x\cosh\phi$$

$x' = 0$ is moving w. speed

$$V = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi.$$

We can replace ϕ by $\tanh^{-1}(V)$, and obtain

$$t' = \gamma(t - Vx) \qquad \text{where} \quad \gamma = (1-V^2)^{-1/2}.$$
$$x' = \gamma(x - Vt)$$

(Assuming $c = 1$, here.)

Lorentz Transformations d/n necessarily commute.

Addenda

(t, x, y, z) set $c = 1$.

$P_1 \quad (0, 0, 0, 0)$

spacelike events in this frame. K

$P_2 \quad (\frac{1}{2}, 2, 0, 0)$

Change coordinates using a Lorentz transformation into a new frame K' so that P_1 and P_2 are simultaneous.

We execute a Lorentz boost in xt. These are parametrized by the velocity of the frame which we take along x.

$$t' = \gamma(t - vx) \qquad \text{with} \quad \gamma = \gamma(v).$$
$$x' = \gamma(x - vt)$$

P_1 in K' w. speed v has coordinates

$$(\gamma(t_1 - vx_1), \gamma(x_1 - vt_1), 0, 0) = (\underset{\uparrow}{0}, 0, 0, 0)$$

simultaneous

P_2 in K' w. speed v has coordinates

$$(\gamma(\tfrac{1}{2} - v \cdot 2), \gamma(2 - v\tfrac{1}{2}), 0, 0) \underset{\substack{\uparrow \\ \text{if } v = \frac{1}{4}}}{=} (\overset{\downarrow}{0}, \gamma(\tfrac{1}{4})(2 - \tfrac{1}{8}), 0, 0)$$

Lorentz transformation example

S, S' inertial frames.

at $t = t' = 0$, spatial origins ($\underline{x} = 0$, $\underline{x}' = 0$) coincide.

Suppose S' moves at speed V along $\underset{\substack{\shortparallel \\ x}}{x_1}$ axis when (x, y, z)
as measured by S.

$$\text{Set} \quad \beta = \frac{V}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

4 equations relate the S' and S coordinates

$$\begin{cases} y' = y, \quad z' = z \\ ct' = \gamma(ct - \beta x) = \dfrac{ct - \beta x}{\sqrt{1-\beta^2}} \\ x' = \gamma(x - \beta ct) = \dfrac{x - \beta ct}{\sqrt{1-\beta^2}}. \end{cases}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$x^{\alpha'} = L^{\alpha'}{}_{\beta} \, x^{\beta}$$

$$x^{\beta} = L^{\beta}{}_{\alpha'} \, x^{\alpha'}$$

inverse matrix.

Proper time

Suppose you and Randy Johnson synchronize your watches and stick a similarly synchronized clock into a baseball. Then you play catch with the baseball. After a few (disturbingly fast) tosses, you observe the baseball's clock and find:

$$\boxed{\text{The clock runs slow on moving objects.}}$$

Observe a clock in motion relative to us. At a certain instant, we may consider the clock to be in an inertial frame with a (relative) veclocity, v.

During a short moment dt (in our frame) the clock travels a distance $\{ dx^2 + dy^2 + dz^2 \}^{1/2}$.

In the clock's frame, the corresponding moment has a dt' and $dx' = dy' = dz' = 0$.

By invariance of spacetime interval, we have

$$(dt')^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}$$

$$\Longrightarrow \quad dt' = dt \left\{ 1 - \frac{dx^2 + dy^2 + dz^2}{c^2 \, dt^2} \right\}^{\frac{1}{2}}$$

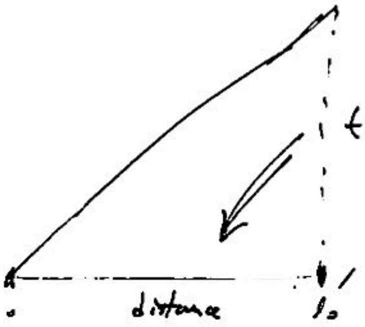
$$= dt \left(1 - \frac{|v|^2}{c^2} \right)^{1/2}.$$

$dt' < dt$ .... hmmm.

Definition: The time read by a clock moving w. a given object is called the proper time for this object.

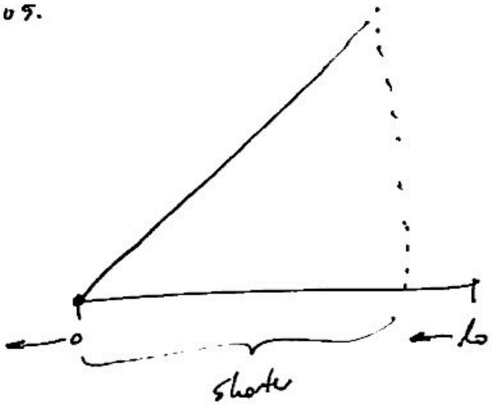
Proper length

Suppose a rod of length l_0' in our frame.



What does it mean that rod has length l_0?
A flash at 0 reaches l_0 in a particular amount of time.

Suppose the rod is moving at high speed to left relative to us.



$$l_0' = l_0\sqrt{1-\frac{|v|^2}{c^2}}.$$

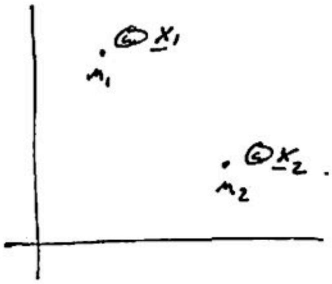
A moving rod is "Lorentz contracted" in length along propagation direction. Proper length ≥ any other measured length.

Addenda

<u>Remark</u>: Newtonian Gravity violates special relativity.

NG is based on a force law:

$$F = G\frac{m_1 m_2}{|x_1 - x_2|^2}$$ is force between



small motion of mass m_2 is instantly felt at m_1. But no signals can travel instantaneously or at any speed faster than light speed.

Lagrangian Derivation of relativistic force-free particle dynamics

Let P_I be an event to the past of P_E. Consider a particle travelling along a world-line connecting P_I to P_E.

We introduce the quantity

$$\boxed{S = \alpha \int_{P_I}^{P_E} ds} = \alpha \text{ (spacetime transit interval).}$$

Note that:

- S is invariant under Lorentz transformations of spacetime.
- S is a scalar.
- $\int_{P_I}^{P_E} ds < 0$ so $S > 0$ if we require $\alpha < 0$.

free parameter to be chosen later.

Also, if $\alpha > 0$ we could make S smaller by making wild motion. We seek action minimizers.

☆ We will demand this length be negative, which is a choice for $ds \to -\sqrt{|ds^2|}$

Thus, S is a candidate Lagrangian for relativistic particle dynamics. Experiments reveal it to provide a good theory of fast particles.

We recast S in our reference frame. Suppose $P_I = (t_I, \underline{x}_I)$, $P_E = (t_E, \underline{x}_E)$ and we write

$$- \quad S = \int_{t_I}^{t_E} L \, dt .$$

Then, proper time notions demand that

$$\boxed{L = + \alpha c \sqrt{1 - \frac{|V|^2}{c^2}}}$$

where V is instantaneous velocity.

We call L the (special) relativistic lagrangian.

We choose α by relying on a correspondence principle:

Relativistic mechanics should be well approximated by classical mechanics in the (so called nonrelativistic) limit $c \to \infty$.

Since

$$L_{\text{classical}} = \frac{1}{2} \overset{\text{rest mass}}{m} v^2,$$

we can expand using Taylor's formula

$$L = + \underset{\substack{\text{constant b/a}\\ \text{affect Lagrangian}\\ \text{dynamics}}}{\alpha c} - \frac{\alpha v^2}{2c} + o\Big(\alpha \underset{\text{small}}{\frac{v^2}{c^2}}\Big) \quad \text{as} \quad c \to \infty.$$

We obtain a nice correspondence if we choose

$$\boxed{\alpha = -\overset{\text{rest mass}}{m} c}$$

The action of a free particle with mass m is

$$S = -mc \underset{\text{"proper frame"}}{\int_{P_I}^{P_E} ds} = \underset{\text{"Lab frame"}}{-mc^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}}\, dt}$$

with lagrangian (Lab frame)

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

As in classical Hamiltonian dynamics, we define the momentum

$$\underline{p} = \frac{\partial L}{\partial \underline{v}}$$

which, for the free particle with rest mass m, reads

$$\underline{p} = \frac{m\,\underline{v}}{\sqrt{1 - \frac{|v|^2}{c^2}}} .$$

Note: As $c \to \infty$, $\underline{p} \to m\,\underline{v}$ the classical momentum.

As $v \to c$, momentum of a particle w. mass $m > 0$ goes $\nearrow$.

The energy $\mathcal{E}$ of the particle is defined to be

$$\mathcal{E} = \underline{p} \cdot \underline{v} - L .$$

(why no su/ $\dot{v}$?

See next page)

Note: $v << c;\ c \to \infty \rightsquigarrow$ ~~$\mathcal{E} = mc^2 + \frac{1}{2} m|v|^2$~~.

We can be more accurate.

$$\mathcal{E} = \frac{m\,\underline{v}}{\sqrt{}} \cdot \underline{v} + mc^2 \sqrt{}\ \frac{\sqrt{}}{\sqrt{}}$$

$$= \frac{m|\cancel{v}|^2 + mc^2 - \cancel{mc^2}\left(\frac{m|v|^2}{\cancel{mc^2}}\right)}{\sqrt{}}$$

so

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{|v|^2}{c^2}}} . \sim mc^2 \left[1 - \tfrac{1}{2}(1+x)^{-3/2}\Big|_{x=0} \overset{x = \frac{-v^2}{c^2}}{x} \right] = mc^2 + \tfrac{1}{2} m v^2 + \dots$$

Thus, for $v = 0$, the energy of a free particle is $\mathcal{E} = mc^2$.

rest energy

A side calculation

$L = L(x, v)$

$\mathcal{E} = \underline{p} \cdot \underline{v} - L$ Why is this interesting ?

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) v + \cancel{\underline{p}\,\frac{dv}{dt}} - \frac{\partial L}{\partial x}\underbrace{\frac{dx}{dt}}_{v} - \cancel{\frac{\partial L}{\partial v}\frac{dv}{dt}}$$

$$= \underbrace{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x}\right]}_{\uparrow} v = 0$$

E–L equation.

Remark : The dynamics are unaffected by adding a constant to L and are similarly unaffected by adding a constant to $\mathcal{E}$. Q: Why is the "rest mass" so important? In what way does this choice of constant provide significant insight?

So, how does the particle move? What is the equation of motion?

Remark: The Lagrangian formulation of particle dynamics requires a parametrization for the world-line connecting the initial event P_I to terminal event P_E. In lab frame, we naturally use t. In the "particle frame", we might use arc length along world line or spacetime interval.

We carry out the derivation in an arbitrary parametrization and discover more interesting structure.

For $w \in \tilde{\mathcal{A}} = \{ w : [\lambda(t_0), \lambda(t_1)] \longrightarrow \mathbb{R}^4 , w(\lambda(t_0)) = P_I , w(\lambda(t_1)) = P_E \}$ where $\lambda : [t_0, t_1]$ is an "arbitrary" parametrization of the world line,

we form

$$I[w] = -mc \int_{\lambda(t_0)}^{\lambda(t_1)} \frac{ds}{d\lambda}\, d\lambda \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

$$= -mc \int_{\lambda_0}^{\lambda_1} \sqrt{\left| \eta_{\mu\nu} \frac{dw^\mu}{d\lambda} \frac{dw^\nu}{d\lambda} \right|}\, d\lambda .$$

We minimize $I[\cdot]$ over $\tilde{\mathcal{A}}$. Let $h : [\lambda(t_0), \lambda(t_1)] \xrightarrow{\text{smooth}} \mathbb{R}^4$ w. $h(\lambda(t_i)) = 0$

$$\lim_{\varepsilon \to 0} \frac{I[w + \varepsilon h] - I[w]}{\varepsilon} = \left\langle \frac{\delta I}{\delta w}, h \right\rangle .$$

We calculate

(from Stephani)

$$= -mc\frac{1}{\cancel{\varepsilon}}\int_{\lambda(t_0)}^{\lambda(t_1)} \frac{-\eta_{\mu\nu}\left[\cancel{\varepsilon}\frac{dh^\mu}{d\lambda}\frac{dw^\nu}{d\lambda} + \cancel{\varepsilon}\frac{dw^\mu}{d\lambda}\frac{dh^\nu}{d\lambda}\right]}{2\sqrt{-\eta_{\mu\nu}\frac{dw^\mu}{d\lambda}\frac{dw^\nu}{d\lambda}}}\, d\lambda .$$

$\eta_{\mu\nu} = \eta_{\nu\mu}$ so terms on numerator contribute the same

$$= -mc\int_{\lambda(t_0)}^{\lambda(t_1)} \frac{-\eta_{\mu\nu}\left[\frac{dw^\mu}{d\lambda}\right]\frac{dh^\nu}{d\lambda}}{\sqrt{-\eta_{\mu\nu}\frac{dw^\mu}{d\lambda}\frac{dw^\nu}{d\lambda}}}\, d\lambda$$

$\frac{dw^\mu}{d\lambda}\frac{d\lambda}{ds} \longrightarrow$ hmmm...

$$= +mc\int_{\lambda(t_0)}^{\lambda(t_1)} \eta_{\mu\nu}\frac{dw^\mu}{ds}\frac{dh^\nu}{d\lambda}\, d\lambda$$

change variable of integration $\lambda \longmapsto s$.

$$= mc\int_{s(t_0)}^{s(t_1)} \eta_{\mu\nu}\frac{dw^\mu}{ds}\frac{dh^\nu}{ds}\, ds$$

IBP

$$= -mc\int_{s(t_0)}^{s(t_1)} \eta_{\mu\nu}\frac{d}{ds}\left(\frac{dw^\mu}{ds}\right) h^\nu\, ds \quad + \text{ zero } \partial\text{-term.}$$

$$\Longrightarrow \quad \frac{\delta I}{\delta w} = -mc\frac{d}{ds}\left(\frac{dw^\mu}{ds}\right).$$

A relativistic force-free particle moves according to

$$\frac{d}{ds}\left(\frac{dw^\mu}{ds}\right) = 0 \quad \text{so} \quad \frac{dw^\mu}{ds} = v^\mu \text{ constant}$$

4-velocity

Since $ds^2 = -\eta_{\mu\nu}\, dw^\mu dw^\nu$, we see that

$\uparrow$ time like choice to make $ds^2 > 0$

$$\eta_{\mu\nu} U^\mu U^\nu = \eta_{\mu\nu} \frac{dw^\mu}{ds} \frac{dw^\nu}{ds} = -1$$

Thus, the 4-velocity is normalized

$$\boxed{\eta_{\mu\nu} U^\mu U^\nu = -1}$$

Exercise 1.2 Find the 4-velocity of a particle moving with constant velocity v.

The 4-momentum of a particle w. rest mass m is

$$p^\mu = m U^\mu \qquad \text{(also constant in force-free setting.)}$$

where U^μ is the particle's 4-velocity.

Relativistic Charged Particle

properties of a particle w.r.t. interaction with em field are characterized by a number — the <u>charge</u> e.

(charge may be <0, $=0$, >0.)

Properties of the em field are characterized by a 4-vector, the <u>4-potential</u> A_i, whose components are functions of spacetime.

The relativistic free particle Lagrangian is supplemented

$$S = \underset{\text{integration along world line}}{\int_a^b} \Big(\underset{\text{free particle}}{-mc\,ds} + \frac{\overset{\text{particle characteristic charge}}{e}}{c} \underset{\text{field}}{A_i}\, dx^i \Big)$$

$$A_i = (\overset{\text{scalar potential}}{\phi}, \underset{\text{vector potential}}{\underline{A}})$$

$$= \int_a^b \Big(-mc \frac{ds}{\cancel{dt}} + \frac{e}{c} \underline{A} \cdot d\underline{x} - e\phi\, dt \Big)$$

$$= \int_a^b \Big[-mc \frac{ds}{dt} + \frac{e}{c} \underline{A} \cdot \frac{d\underline{x}}{dt} - e\phi \Big] dt$$

$$= \int_a^b \Big[\underbrace{-mc\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c} \underline{A} \cdot \underline{v} - e\phi}_{\text{Lagrangian for a charge in an em field.}} \Big] dt$$

$$E-L \iff \frac{d}{dt}\Big(\frac{\partial L}{\partial \underline{v}}\Big) = \frac{\partial L}{\partial \underline{x}} .$$

Recall relativistic free particle momentum $\underline{p} = \frac{m\underline{v}}{\sqrt{1-\frac{v^2}{c^2}}}$,

E-L unravels to read

per unit charge

$$\frac{d\underline{p}}{dt} = -\underbrace{\frac{e}{c}\frac{\partial \underline{A}}{\partial t} - e\nabla\phi}_{\text{independent of velocity}} + \underbrace{\frac{e}{c}\underline{v}\times \text{curl}\,\underline{A}}_{\text{velocity dependent in direction } \perp \text{ to } \underline{v} \text{ and } \underline{A}.} .$$

Changes momentum

motivate thinking of RHS pieces as "forces"

"force" per unit charge.

Velocity independent force per unit charge

$$-\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \nabla\phi := \underline{E}$$

electric field intensity.

velocity dependent force per unit charge

$$\frac{1}{c}\,\underline{v}\times\underbrace{\text{curl}\,\underline{A}}_{\overset{\shortparallel}{\underline{H}}}$$

magnetic field intensity.

$$\frac{d\underline{p}}{dt} = e\,\underline{E} + \frac{e}{c}\,\underline{v}\times\underline{H}.$$

II. Manifolds, Tensors

A. Manifolds

- intrinsic rather than extrinsic
- definition; examples
- maps between manifolds; pullback of function

B. Vectors

- can't "add" arrows on manifold.
- algebraic definition
- dimension of tangent space; coordinate basis
- transformation property
- tangents to curves
- 1 parameter groups of diffeos; odes
- tangent vector field; commutator.

C. Covectors / 1-forms / dual vectors

- dual vectors; dual space; dual basis
- df as a covector.
- double dual vector space.

D. Tensors

- definition
- contraction, outer product, (symmetrization), [antisymmetrization]
- metric tensor; raising/lowering indices
- Notational remarks
- Maxwell's equations in tensor notation.

II. Manifolds, Tensors

A. Manifolds

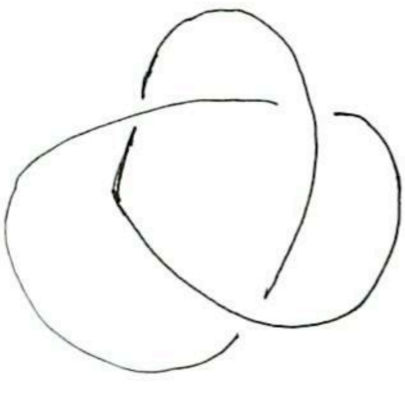
Locally, we perceive the surface of the earth as $\mathbb{R}^2$.

But its not.

We want to introduce a notion which permits us to describe sets which a locally like $\mathbb{R}^n$ but which may have different global properties.

A natural approach (which we do not follow): We understand that the surface of the earth sits in $\mathbb{R}^3$ as the surface S^2. One approach would be to study surfaces in $\mathbb{R}^3$ and then extend the study to higher dimensions.

Example



Trefoil knot is an embedding of

$$S^1 \longrightarrow \mathbb{R}^3.$$

No projection of the trefoil will avoid a self intersection.

Whitney Thm: Any smooth, connected, closed manifold M of dimension n can be smoothly embedded in $\mathbb{R}^{2n+1}$.

Instead, we develop an "internal" abstract notion of manifold. Note that we do not have a natural higher dimensional Euclidean space which contains our spacetime.

Definitions

An open ball in $\mathbb{R}^n$ of radius r centered around point $y = (y^1, \ldots, y^n)$ is the set of points

$$B(y, r) = \{ x \in \mathbb{R}^n : |x - y| < r \},$$

where

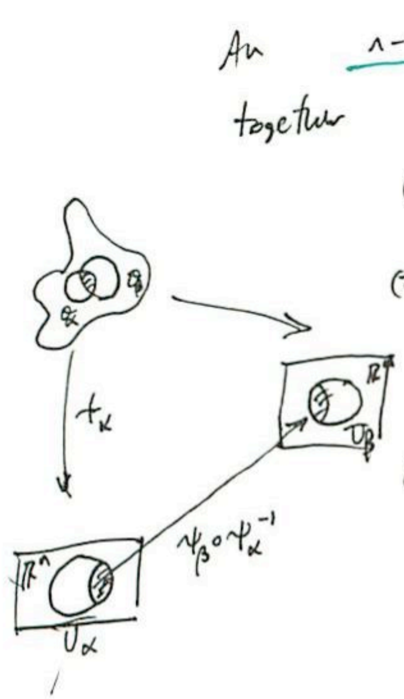
$$|x - y| = \left\{ \sum_{\mu=1}^{n} |x^\mu - y^\mu|^2 \right\}^{\frac{1}{2}}.$$

An open set in $\mathbb{R}^n$ is any set which may be expressed as a union of open balls.

A manifold is a set made of pieces that "look like" open subsets of $\mathbb{R}^n$ "sewn together" smoothly.

An n-dimensional C^∞ real manifold M is a set together with a collection of subsets $\{O_\alpha\}$ satisfying:

(1) Every $p \in M$ lies in at least one O_α; $\{O_\alpha\}$ covers M.



(2) $\forall\ \alpha\ \exists$ one-to-one onto map $\psi_\alpha : O_\alpha \rightarrow U_\alpha$ where $U_\alpha \subset \mathbb{R}^n$ is an open set.

(3) If any two ~~st~~ O_α and O_β overlap, $O_\alpha \cap O_\beta \neq \phi$, we can consider the map $\psi_\beta \circ \psi_\alpha^{-1}$ which takes points in $\psi_\alpha [O_\alpha \cap O_\beta] \subset U_\alpha \subset \mathbb{R}^n$ to points in $\psi_\beta [O_\alpha \cap O_\beta] \subset U_\beta \subset \mathbb{R}^n$. We require these subsets to be open and this map be C^∞.

Remarks The maps ψ_α are called charts. To avoid trivialities, we require that the chart family $\{\psi_\alpha\}$ be maximal: all coordinate systems compatible with conditions (2), (3) are included.

The definition above involved the smoothness (C^∞) of the manifold in the requirement that

$$\psi_\beta \circ \psi_\alpha^{-1} : \overset{\subset \mathbb{R}^n}{\psi_\alpha[\mathcal{O}_\alpha \cap \mathcal{O}_\beta]} \longrightarrow \overset{\subset \mathbb{R}^n}{\psi_\beta[\mathcal{O}_\alpha \cap \mathcal{O}_\beta]}$$

is a C^∞ map. We may relax or adjust the smoothness of the manifold by relaxing or adjusting this requirement.

To obtain a complex manifold, replace $\mathbb{R}^n$ by $\mathbb{C}^n$ in the definition.

Examples

$\mathbb{R}^n$ with single chart $\mathcal{O} = \mathbb{R}^n$, ψ = identity map.

~~$S^2 = \{(x^1, x^2, x_3)$~~

$$S^2 = \{(x^1, x^2, x^3) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

$$\mathcal{O}_i^{\pm} = \{(x^1, x^2, x^3) \in S^2 : \pm x^i > 0\}.$$

The six "spherical caps" cover S^2.

The caps are homeomorphic images of disks in $\mathbb{R}^2$ lying below them. The overlap functions can be verified to be C^∞.

$S^{n-1} = \{(x^1, \dots, x^n) : (x^1)^2 + \dots + (x^{n-1})^2 = 1\}$ is also a manifold.

A simple procedure allows us to form a new manifold from two given manifolds M, M'. We can make the product space $M \times M' = \{(p, p') : p \in M, p' \in M'\}$ into an $(n+n')$-dimensional manifold:

If $\psi_\alpha : O_\alpha \longrightarrow U_\alpha$, $\psi_\beta' : O_\beta' \longrightarrow U_\beta'$,

we can define a chart

$$\psi_{\alpha\beta} : O_{\alpha\beta} \longrightarrow U_{\alpha\beta} \subset \mathbb{R}^{n+n'}$$

by taking

$$O_{\alpha\beta} = O_\alpha \times O_\beta'$$

$$U_{\alpha\beta} = U_\alpha \times U_\beta' .$$

Most manifolds considered in GR are expressible as products of $\mathbb{R}^n$ and S^m.

Maps between Manifolds

M	M'	are manifolds with corresponding
$\{\psi_\alpha\}$	$\{\psi_\beta'\}$	family of chart maps.

A map $f: M \longrightarrow M'$ is said to be C^∞ if $\forall\ \alpha, \beta$, the map

$$\psi_\beta' \circ f \circ \psi_\alpha^{-1} : \underset{\mathbb{R}^n}{\underset{\cap}{U_\alpha}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\cap}{U_\beta'}} \quad \text{is } C^\infty .$$

If $f: M \to M'$ is C^∞, one-to-one, onto and has C^∞ inverse map, then f is called a diffeomorphism and M and M' are said to be diffeomorphic.

e.g. $f: M \to \mathbb{R}$ is a function on M.

~~next page~~

$p \in M$

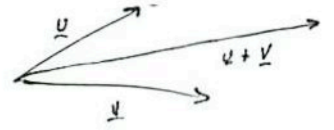
$\phi: M \to M'$

$g: M' \to \mathbb{R}$

$\underbrace{\phi^* g}_{\text{function on } M}(p) = \phi(g(p)).$

B. Vectors

The natural vector space structure on $\mathbb{R}^n$ allows us to "add" two points in $\mathbb{R}^n$ and obtain another point in $\mathbb{R}^n$:



On manifolds, this vector space structure is not so clear. For example, how do you "add" two points on the sphere S^2 to obtain a third point on the sphere? Infinitesimally, this seems ok...

"Infinitesimal displacements" or tangent vectors ressurect (some of) the vector structure on $\mathbb{R}^n$ and allow us to develop calculus on manifolds.

We wish to define tangent vectors "intrinsically"; that is, without reference to a particular embedding of the manifold.

In $\mathbb{R}^n$, $\exists$ one-to-one correspondence between vectors and directional derivatives:

$$(V_0^1, \dots, V^n) \longleftrightarrow \sum_{\mu=1}^{n} V^\mu \frac{\partial}{\partial x^\mu} .$$

Directional derivatives are characterized by their linearity and "Leibnitz rule" behavior when acting on functions.

Let $\mathcal{F} = \{ f : m \longrightarrow \mathbb{R} \mid f \in C^\infty \} = C^\infty(m; \mathbb{R})$.

<u>Definition</u> A <u>tangent vector</u> V at a point $p \in M$ is a map $V : \mathcal{F} \longrightarrow \mathbb{R}$ which satisfies:

(1) (linearity) $V(af + bg) = a\,V(f) + b\,V(g)$

$\forall\, f, g \in \mathcal{F}\,;\; a, b \in \mathbb{R}\,;$

(2) (Leibnitz rule) $V(fg) = f(p)\,V(g) + V(f)\,g(p)$.

<u>Note</u>: If $h \in \mathcal{F}$ is constant then $V(h) = 0$.

Why?

$$V(h^2) = h(p)\,V(h) + V(h)\,h(p) = 2c\,V(h). \quad \text{(using (2))}.$$

$$V(h^2) = V(ch) = c\,V(h) \quad \text{(using (1))}$$

$$\Longrightarrow \quad c\,V(h) = 2c\,V(h) \quad \Longrightarrow \quad V(h) = 0 .$$

The linearity property (1) implies that the collection of tangent vectors at the point p is a vector space. We define the addition law naturally:

$$(V_1 + V_2)(f) = V_1(f) + V_2(f).$$

The scalar multiplication law:

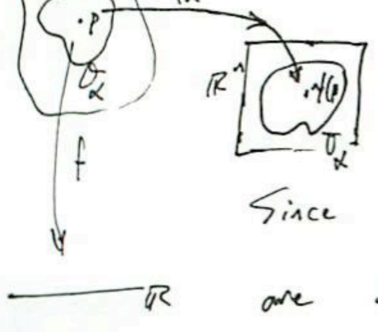
tangent space

$$(aV)(f) = V(af).$$

Theorem Let M be an n-dimensional manifold. Let $p \in M$ and V_p denote the tangent space at p. Then $\dim V_p = n$.

proof: We first construct n linearly independent tangent vectors $X_\mu \in V_p$. Then we prove that $\{X_\mu\}_{\mu=1}^{n}$ spans V_p.

Given $p \in M$, $\exists$ $O_\alpha \ni p$ and $\psi_\alpha : O_\alpha \longrightarrow U_\alpha \subset \mathbb{R}^n$, with ψ_α a bijection. We consider $\forall$ $f \in \mathcal{F} = C^\infty(M;\mathbb{R})$. We may then define the function $f \circ \psi_\alpha^{-1} : U_\alpha \longrightarrow \mathbb{R}$. We may then define the tangent vector X^μ at p by the formula



$$X_\mu(f) = \frac{\partial}{\partial x^\mu}\left(f \circ \psi_\alpha^{-1}\right)\Big|_{x = \psi_\alpha(p)}.$$

very clear definition. A procedure for generating a set of #'s from f at $p \in M$.

Since ψ_α is a bijection of O_α onto U_α, $\{X^\mu\}_{\mu=1}^{n}$ are linearly independent. Therefore $\dim V_p \geq n$.

We now show $\{\mathbb{X}_\mu\}_{\mu=1}^{n}$ spans V_p, after some preparatory work.

Let $F \in C^\infty(\mathbb{R}^n;\mathbb{R})$. $\forall\ \underline{a} = (a_1,\dots,a_n) \in \mathbb{R}^n$ $\exists$ $H_\mu \in C^\infty(\mathbb{R}^n;\mathbb{R})$ such that

$$F(x) = F(a) + \sum_{\mu=1}^{n} (x^\mu - a^\mu)\, H_\mu(x). \qquad (\text{MVT})$$

We also know that

$$H_\mu(a) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=a}. = \mathbb{X}^\mu(F)$$

for this choice.

Take $F = f\circ\psi_\alpha^{-1}$ and $a = \psi_\alpha(p)$. $\forall\ q \in \mathcal{O}_\alpha$,

$$\underset{F(\psi(q))}{\underset{\shortparallel}{f(q)}} = \underset{F(\psi(p))}{\underset{\shortparallel}{f(p)}} + \sum_{\mu=1}^{n} \left[x^\mu\circ\psi(q) - x^\mu\circ\psi(p)\right] H_\mu(\psi(q)).$$

Let $v \in V_p$. We wish to show $v \in \text{span}\{\mathbb{X}_\mu\}$.

Apply v to f using the formula above. Using Leibnitz rule,

$$v(f) = \overset{\nearrow 0}{v[f(p)]} + \sum_{\mu=1}^{n} \overset{\nearrow 0}{\left[x^\mu\circ\psi(q) - x^\mu\circ\psi(p)\right]}\Big|_{q=p} v\left(H_\mu(\psi(q))\right)$$

$$+ \sum_{\mu=1}^{n} v\left(\left[x^\mu\circ\psi(q) - \overset{\nearrow 0}{x^\mu\circ\psi(p)}\right]\right) H_\mu(\psi(p))$$

$$= \sum_{\mu=1}^{n} v\left(x^\mu\circ\psi\right) H_\mu(\psi(p))$$

$$= \sum_{\mu=1}^{n} v^\mu\, \mathbb{X}_\mu(F),$$ which completes the proof.

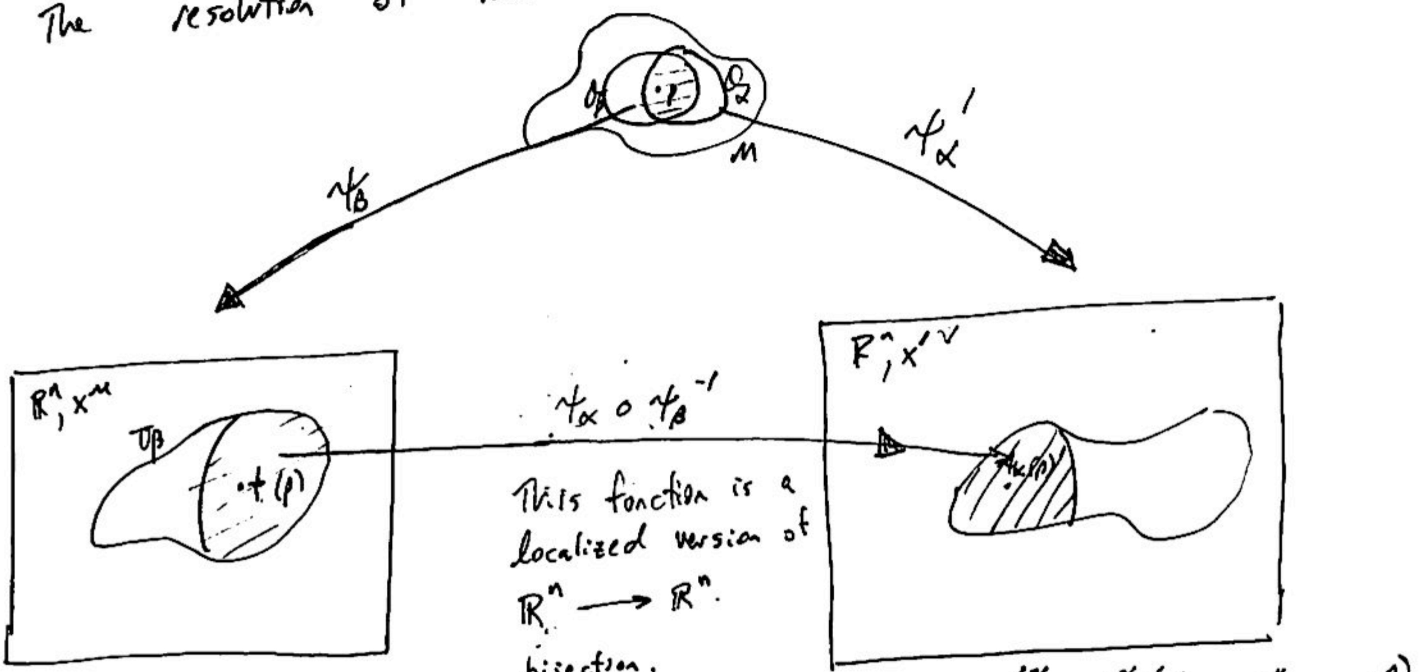
The basis $\{\mathbb{X}_\mu\}$ of V_p constructed above is called a coordinate basis. Often $\mathbb{X}_\mu$ is simply called $\frac{\partial}{\partial x^\mu}$.

Two natural (related) issues emerge.

- The basis $\{\mathbb{X}_\mu\}$ depends upon the coordinates x^μ inside $U_\alpha \subset \mathbb{R}^n$. If we change coordinates on $\mathbb{R}^n$ and hence in U_α, what happens to the basis of V_p?
- The point p may also lie in $\mathcal{O}_\beta$ which maps under ψ_β to U_β, an open subset of a different $\mathbb{R}^n$. With some coordinates in U_β, we can define a different basis of V_p. How do these two bases relate to each other?

$\longrightarrow$ $\boxed{\mathbb{R}^n \to \mathbb{R}^n \text{ calculation next page}}$

The resolution of the second issue takes care of the first.



This function is a localized version of $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ bisection.

$$x'^\nu = x'^\nu(x^1, \dots, x^\mu, \dots, x^n)$$

$$\mathbb{X}_\mu = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})$$

$$\frac{\partial}{\partial x^\mu} = \sum_{\nu=1}^{n} \frac{\partial (x')^\nu}{\partial x^\mu} \frac{\partial}{\partial (x')^\nu} \quad \text{(chain rule).}$$

$f(x'^\nu)$ given & $x'^\nu = x'^\nu(x^\mu)$. $\frac{\partial}{\partial x^\mu} f = \cdots$

$\mathbb{R}^n \longrightarrow \mathbb{R}^n$ vector transformation property.

Suppose V is a vector on $\mathbb{R}^n$ at p.
Then V acts on function $f: \mathbb{R}^n \to \mathbb{R}$ to produce a number. This action does not depend upon the coordinates used to represent the calculations.

Vf is intrinsic.

Coordinate representation

x'^{ν} coordinates on $\mathbb{R}^n$. $\frac{\partial}{\partial x'^{\nu}}$ basis for V_p.

$$Vf = \left(V'^{\nu} \frac{\partial}{\partial x'^{\nu}}\right)\Big|_p f .$$

So V'^{ν} are components of V in x'^{ν} coordinates

We introduce new coordinates x^{μ}.

If $f = f(x^{\mu})$ then we can calculate $x^{\mu} = x^{\mu}(x'^{\nu})$

$$\frac{\partial}{\partial x'^{\nu}} f = \frac{\partial f}{\partial x^{\mu}} \underbrace{\frac{\partial x^{\mu}}{\partial x'^{\nu}}}_{\substack{\text{Jacobian} \\ \text{matrix}}}$$

Multiply by V'^{ν} (and sum).

$$Vf = V'^{\nu} \frac{\partial}{\partial x'^{\nu}} f = V'^{\nu} \frac{\partial f}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

$$= V^{\mu} \frac{\partial f}{\partial x^{\mu}}$$

$$\boxed{V^{\mu} = V'^{\nu} \frac{\partial x^{\mu}}{\partial x'^{\nu}}}$$

Hence, we can represent the tangent vector $\mathbb{X}_\mu$ in the new basis as

$$\mathbb{X}_\mu = \sum_{\nu'=1}^{n} \frac{\partial x'^{\nu}}{\partial x^{\mu}}\Big|_{\psi_\beta(p)} \frac{\partial}{\partial x'^{\nu}} = \sum_{\nu'=1}^{n} \frac{\partial x'^{\nu}}{\partial x^{\mu}} \mathbb{X}'_{\nu} .$$

Given $v \in V_p$ in the basis $\{\mathbb{X}_\mu\}$:

$$v = \sum_{\mu=1}^{n} v^\mu \mathbb{X}_\mu ,$$

we may ask for v in terms of the basis $\{\mathbb{X}'_{\nu}\}$:

$$v'^{\nu} = \sum_{\mu=1}^{n} v^\mu \frac{(\partial x')^\nu}{\partial x^\mu} .$$

<u>vector transformation law</u>

and then

$$v = \sum_{\nu} (v')^\nu \mathbb{X}'_\nu .$$

~~Summary: If $x'^\nu = x'^\nu(x^\mu)$ and~~ ~~$\mathbb{R}^n \to \mathbb{R}^n$~~

~~$f(x'^\nu)$ is considered, we can calculate~~

~~$\frac{\partial}{\partial x^\mu} f(x'^\nu) = \underbrace{\frac{\partial f}{\partial x'^\nu}}_{\nabla'} \underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{\text{Jacobian}}$~~

~~Basis element of a vector~~

~~$\left(\frac{\partial}{\partial x^\mu}\right) f(x'^\nu) = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)$~~

28 Jan. 2002

Summary

Manifold $\quad M \qquad \mathcal{O}_\alpha \subset M \qquad \psi_\alpha : \mathcal{O}_\alpha \xrightarrow[\text{bijection}]{} U_\alpha , \quad U_\alpha \subset \mathbb{R}^n$

$$M \subset \bigcup_\alpha \mathcal{O}_\alpha$$

$$\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \phi$$

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \xrightarrow[\text{smooth.}]{} \psi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$$

---

tangent vectors at $p \in M$

$$\mathcal{F} = C^\infty(M;\mathbb{R})$$

$V : \mathcal{F} \longrightarrow \mathbb{R}$ s.t. $\quad V$ is linear + satisfies Leibnitz Rule.
depends on p.

$V_p = \{$ tangent vectors at $p \}$

V_p is a vector space of dimension n.

$p \in M \Rightarrow \exists\ \mathcal{O}_\alpha \ni p$ and $\psi_\alpha : \mathcal{O}_\alpha \longmapsto U_\alpha \subset \mathbb{R}^n$.
The chart ψ_α leads us to coordinate basis of V_p

$$V_p = \text{span}\ \{ X_\mu \} . \qquad \left(\text{sometimes } X_\mu \text{ written } \frac{\partial}{\partial x^\mu}\right)$$

$X_\mu : \mathcal{F} \longmapsto \mathbb{R}$ by formula

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}\left(f \circ \psi_\alpha^{-1}\right)\Big|_{\psi_\alpha(p)} .$$

Vector transformation law a.k.a. Chain Rule

$$V = \sum_{\mu=1}^{n} V^\mu X_\mu = \sum_{\nu=1}^{n} V'^\nu X'_\nu \qquad \text{where}$$

$$V'^\nu = \sum_{\mu=1}^{n} V^\mu \frac{\partial x'^\nu}{\partial x^\mu} .$$

<u>smooth curve C on M</u> $\quad I \subset \mathbb{R}$ is an interval

$$C : I \longrightarrow M.$$

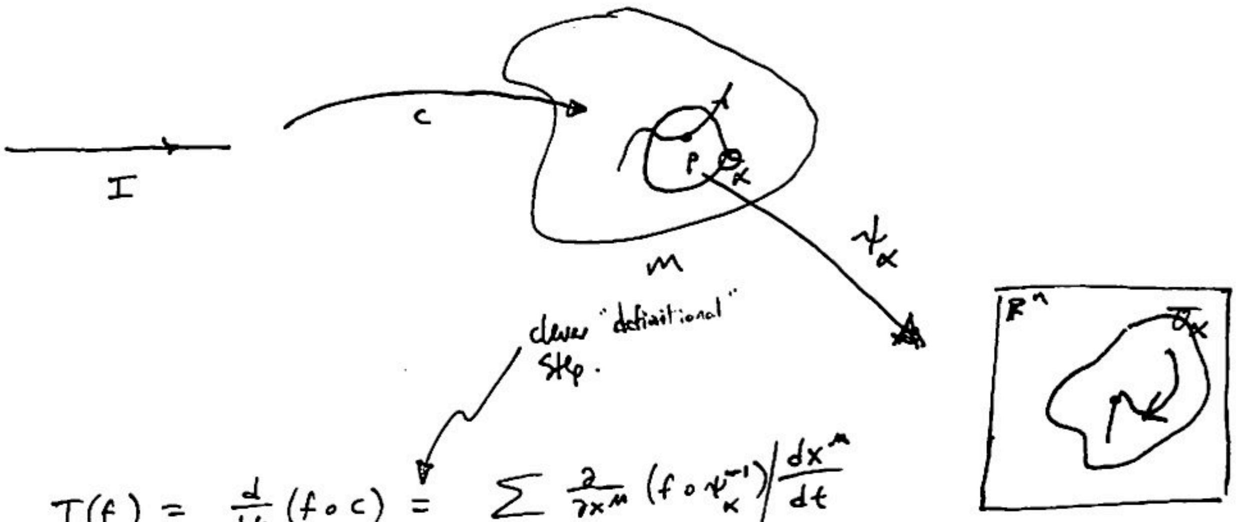
$\forall\, p \in M$ lying on C we can associate a tangent vector $T \in V_p$:

How? $\quad \forall\, f \in \mathcal{F} = C^\infty(M; \mathbb{R})$

$T(f) =$ derivative of $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ evaluated at p.

$$= \frac{d}{dt}(f \circ c)\Big|_{c^{-1}(p)} .$$

We can understand the coordinate basis in this way. Given a chart ψ defined on $\mathcal{O} \ni p$, consider the curves on M defined by varying x^μ and keeping the other coordinates fixed. The curves generated are associated as above with X_μ.



clever "definitional" step.

$$T(f) = \frac{d}{dt}(f \circ c) = \sum_\mu \frac{\partial}{\partial x^\mu}(f \circ \psi_\alpha^{-1})\Big|_{\psi_\alpha(p)} \frac{dx^\mu}{dt}$$

$$= \sum_\mu \frac{dx^\mu}{dt}\, X_\mu(f)$$

$$= \sum_\mu T^\mu X_\mu(f) \quad ; \quad T^\mu = \frac{dx^\mu}{dt} .$$

(Hawking – Ellis)

Addenda "push forward" $r > 1$.

Suppose $\phi : M \xrightarrow{C^r} M'$ and $\lambda(t)$ is a curve passing through $p \in M$. Then $\phi(\lambda(t))$ passes through $\phi(p) \in M'$. The tangent vector to $\phi(\lambda(t))$ at p will be denoted by

$$\phi_* \left(\left(\frac{\partial}{\partial t} \right)_\lambda \right) \Big|_{\phi(p)} .$$

This may be regarded as the image, under the map ϕ, of the vector $\left(\frac{\partial}{\partial t} \right)_\lambda \Big|_p$. Clearly, ϕ_* is a linear map of $T_p(M) \longrightarrow T_{\phi(p)}(M')$.

relation w. pull back of functions. Let X be a vector at p. Let f be a function at $\phi(P)$.

$$X(\phi^* f)\Big|_p = (\phi_* X)(f)\Big|_{\phi(P)} .$$

Definition: A C^r map $\phi : M \to M'$ is an immersion if $\forall\, p \in M \ \exists$ nhd $\mathcal{U} \subset M$ with $p \in \mathcal{U}$ s.t. $\phi^{-1}\big|_{\phi(\mathcal{U})}$ is a C^r map. (This implies $n \le n'$.)

By implicit function theorem ϕ will be an immersion $\Longleftrightarrow$ ϕ_* injective at every $p \in M$.

$$\Downarrow$$

$$\phi_* : T_p \xrightarrow[\substack{\text{isomorphism}\\ \text{of vector}\\ \text{spaces.}}]{} \phi_*(T_p) \subset T_{\phi(p)}$$

$\phi(M)$ is an "immersed submanifold" in M.

An immersion is an imbedding if it is a homeomorphism onto its image in the induced topology.

Remark The manifold structure does not provide a natural way to compare vectors $v \in V_p$ with vectors $w \in V_q$ for $p \neq q$, $p, q \in M$. This comparison is possible after we impose additional structure on the manifold, namely a notion of "parallel transport" along curves connecting $p, q \in M$.

tangent field on M

This is an assignment of V_p $\forall\, p \in M$:

$$v : M \longrightarrow V_p .$$

Despite the remark above, $\exists$ natural notion of what it means for v to vary smoothly from point to point in M.

If $f \in C^\infty(M; \mathbb{R})$ then $\forall\, p \in M$, $v|_p(f) \in \mathbb{R}$ which means $v|_p(f) : M \longrightarrow \mathbb{R}$ (by varying p).

Tangent field v is said to be smooth if $\forall\, f \in \mathfrak{F}$, $v(f) : M \rightarrow \mathbb{R}$ is also smooth.

Exercise Verify that the coordinate basis fields are smooth.

A *one-parameter group of diffeomorphism ϕ_t* is a C^∞ map $\mathbb{R}\times M \longmapsto M$
s.t. $\forall$ fixed $t \in \mathbb{R}$, $\phi_t : M \longrightarrow M$ is a diffeo. and
$\forall\ t, s \in \mathbb{R}$ $\quad \phi_t \circ \phi_s = \phi_{t+s}$. (Hence $\phi_{t=0}$ is identity.)

We can associate to ϕ_t a vector field:

$\forall\ p \in M$, $\phi_t(p) : \mathbb{R} \longrightarrow M$ is a curve called the *orbit of ϕ_t passing through p* at $t = 0$. Define $V|_p$ to be the tangent to this curve at $t = 0$.

V can be thought of as the *infinitesimal generator* of ϕ_t.

Conversely, given a smooth vector field $V : M \longrightarrow \overset{TM}{\cancel{\mathbb{R}}}$, we can associate a one-parameter group of diffeomorphism ϕ_t $\quad \mathbb{R}_t \times M \longrightarrow \mathbb{R}$.

How? $\quad \forall\ p \in M \quad \exists\ V|_p$, a tangent vector.

$p \in O_\alpha$ with chart ψ_α. We construct

$V = \sum_{\mu=1}^{n} V^\mu \frac{\partial}{\partial x^\mu}$. $\quad$ a solution of the ODE system on $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx^\mu}{dt} = V^\mu(x^1, \cdots, x^n) \\ \psi_\alpha(x^1, \dots, x^n) = p. \end{cases} \qquad \text{ODE i.v.p}$$

$\| \ ?$

$V^\mu(\psi(p))$

This may always be solved for short time.

Hence ϕ_t may be defined for $t \in I$

but perhaps may not be extendible to

$t \in \mathbb{R}$.

(Lie $\underset{\text{[illegible]}}{\overset{\text{add}}{\text{derivative}}}$)

Given vector fields V, W, we may define a new vector field called the *commutator of V and W* denoted $[V, W]$ via the formula

$$[V, W](f) = V[W(f)] - W[V(f)].$$

Exercise (a) Verify that the commutator defined above satisfies the requirements ~~that~~ for it to be called a vector field.

(b) Let X, Y, Z be smooth vector fields on a manifold $\mathcal{M}$. Verify that their commutator satisfies the *Jacobi identity*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Dual Vectors a.k.a. covectors

C.

Let V be a finite dimensional vector space over the real numbers. Consider the collection $V^* = \{ f : V \to \mathbb{R} , f \text{ linear} \}$. We can define addition and scalar multiplication in V^* in an obvious way and V^* becomes a vector space.

V^* is the dual vector space to V. Elements of V^* are called dual vectors and are also called covectors.

If $v_1, \dots, v_n$ are the basis vectors in V, we can define elements $v^{1*}, \dots, v^{n*}$ in the dual vector space V^* by

$$v^{\mu *}(v_\nu) = \delta^\mu{}_\nu ,$$

where

$$\delta^\mu{}_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

The action of $v^{\mu *}$ is then determined on all $v \in V$ by linearity. It follows that $\{ v^{1*}, \dots, v^{n*} \}$ is a basis of V^* called the dual basis to the basis $\{ v_1, \dots, v_n \}$ of V.

$$\Longrightarrow \quad \dim V^* = \dim V$$

$$v_\mu \longleftrightarrow v^{\mu *} \quad \text{gives an isomorphism} \quad V \longleftrightarrow V^*.$$

Note: Isomorphism depends upon the given basis.

Manifold Setting

Example Given $f: M \to \mathbb{R}$ we can define a covector field θ_f on M. We do so via formula: $\forall$ v.f. X on M,

$$\theta_f[X] = Xf.$$

We denote θ_f by the name df.

In coordinates we write

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

and have $(dx^\mu)\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$ so that

$$(df)(X) = \left(\frac{\partial f}{\partial x^\mu} dx^\mu\right)\left(V^\alpha \frac{\partial}{\partial x^\alpha}\right)$$

$$= \underset{\underset{a_\mu}{\shortparallel}}{\frac{\partial f}{\partial x^\mu}} V^\mu = Xf.$$

Q: Suppose $x'^\nu = x'^\nu(x^\mu)$. is a change of variables.

Similarly ; $x^\mu = x^\mu(x'^\nu)$.

We represent dx^μ in terms of dx'^ν.

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

$$a_\mu \frac{\partial x^\mu}{\partial x'^\nu} = a'_\nu.$$

covector transformation

so $$df = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu.$$

The Double Dual Vector Space

V^* is a finite dimensional vector space. We can define its dual space V^{**}. An element $v^{**} \in V^{**}$ is a linear map on V^* into $\mathbb{R}$.

Claim V^{**} is naturally isomorphic to V.

$\forall$ $v \in V$ we associate the map in V^{**} whose value on the dual vector $w^* \in V^*$ is given by $w^*(v)$. We obtain a one-one linear map of V onto V^*. Hence, taking double dual gives nothing new: $V^{**} \longleftrightarrow V$.

D. Tensors

Let V be a finite dimensional vector space. Let V^* be the dual vector space of V. A tensor T of type (k, l) over V is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \longrightarrow \mathbb{R} .$$

A (0,1) tensor is a dual vector. A tensor of type (1,0) is an element of V^{**} which we identify with V. A (1,1) tensor T is a multilinear map $V^* \times V \longrightarrow \mathbb{R}$. If we fix a $v \in V$ and form $T(\cdot, v)$ then we obtain a linear map on V^*, that is an element of $V^{**} \longleftrightarrow V$.

The obvious rules of addition & scalar multiplication endows the collection $\mathcal{T}(k,l) = \{$ tensors of type $(k,l) \}$ with a vector space structure.

What is the dimension of $\mathcal{T}(k,l)$?

Action of $T \in \mathcal{T}(k,l)$ specified on basis vectors in V^* and V. Hence

$$\dim \mathcal{T}(k,l) = n^{k+l}.$$

Operations on Tensors

Contraction w.r.t. ith dual + jth vector slots is a map

$$C : \mathcal{T}(k, \ell) \longrightarrow \mathcal{T}(k-1, \ell-1)$$

defined

$$CT = \sum_{\sigma=1}^{n} T(\dots, \underset{i}{v^{\sigma *}}, \dots, .j\dots, \underset{j}{v_\sigma}, \dots)$$

where $\{v_\sigma\}$ is a basis of V, $\{v^{\sigma *}\}$ is its dual basis and these vectors are inserted into the ith & jth slots of T.

Outer product

T of type (k, ℓ). T' of type (k', ℓ').

The outerproduct of T and T', denoted $T \otimes T'$, is defined as follows:

Given $(k+k')$ dual vectors $(v^{1*}, \dots, v^{k+k *})$ and $(\ell+\ell')$ vectors $(w_1, \dots, w_{\ell+\ell'})$, we define $T \otimes T'$ acting on these vectors to be

$$T(v^{1*}, \dots, v^{k*}; w_1, \dots, w_\ell) \times T(v^{k+1*}, \dots, v^{k+k' *}; w_{\ell+1}, \dots, w_{\ell+\ell'}),$$

1 February 2002

V finite-d vector space. $\dim V = n$.

$\{v_\mu\}$ basis of V.

$V^* = \{\text{linear maps } f : V \longrightarrow \mathbb{R}\}$ dual vector space.

Natural basis of V^* called the dual basis $\{v^{\mu *}\}$ defined

$$v^{\mu *}(v_\nu) = \delta^\mu{}_\nu .$$

Tensor of type (k, l) acting on vector space V (and its dual V^*).

$$T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \longrightarrow \mathbb{R} \quad \text{linear in each slot.}$$

Contraction is an operation which takes a (k, l) tensor and creates a $(k-1, l-1)$ tensor. Contraction w.r.t. ith dual-vector & j-th vector slot defined by

$$CT = \sum_{\sigma=1}^{n} T(\cdots, \overset{i}{v^{\sigma *}}, \cdots ; \cdots, \overset{j}{v_\sigma}, \cdots)$$

where v_σ (v_σ^*) are basis (dual basis) vectors.

Q Does contraction depend upon the basis $\{V_\sigma\}$ of V ?

Answer No. Here is why ?

Suppose $\{W_\nu\}$ is a different basis of V. $\{W_\nu\}$ induces another basis $\{w^{\nu *}\}$ of V^*, via $w^{\nu *}(w_\mu) = \delta^\nu_{\ \mu}$.

Represent old basis of V in terms of new basis.

$$V_\mu = \sum_{\beta=1}^{n} \Lambda^\beta_{\ \mu} \, w_\beta .$$

Represent old basis of V^* in terms of new basis

$$V^{\sigma *} = \sum_{\nu=1}^{n} K^\sigma_{\ \nu} \, w^{\nu *} .$$

How do the matrices $\Lambda^\nu_{\ \sigma}$ and $K^\sigma_{\ \nu}$ relate to each other ?

$$\delta^\sigma_{\ \mu} = V^{\sigma *}(V_\mu) = \left(\sum_{\nu=1}^{n} K^\sigma_{\ \nu} \, w^{\nu *}\right)\left(\sum_{\beta=1}^{n} \Lambda^\beta_{\ \mu} \, w_\beta\right)$$

$$= \sum_{\nu=1}^{n} \sum_{\beta=1}^{n} K^\sigma_{\ \nu} \Lambda^\beta_{\ \mu} \, w^{\nu *}(w_\beta)$$

$$= \sum_{\nu=1}^{n} \sum_{\beta=1}^{n} K^\sigma_{\ \nu} \Lambda^\beta_{\ \mu} \, \delta^\nu_{\ \beta}$$

$$= \sum_{\nu=1}^{n} K^\sigma_{\ \nu} \Lambda^\nu_{\ \mu}$$

$$\Longrightarrow \quad \boxed{K^\sigma_{\ \nu} = (\Lambda^\nu_{\ \mu})^{-1}.}$$

$T \in \mathcal{T}(2;1)$, $CT \in \mathcal{T}(1;0)$.

$$CT(\cdot) = \sum_{\sigma=1}^{n} T(\cdot\,, v^{\sigma *}; v_{\sigma})$$

$$= \sum_{\sigma=1}^{n} T\left(\cdot\,, \sum_{\gamma=1}^{n} K^{\sigma}_{\gamma} w^{\gamma *}; \sum_{\beta=1}^{n} \Lambda^{\beta}_{\cancel{\sigma}} w_{\beta}\right)$$

$$= \sum_{\sigma=1}^{n} \sum_{\gamma=1}^{n} \sum_{\beta=1}^{n} K^{\sigma}_{\gamma} \Lambda^{\beta}_{\sigma} \; T(\cdot\,, w^{\gamma *}; w_{\beta})$$

$$= \sum_{\gamma=1}^{n} \sum_{\beta=1}^{n} \underbrace{\sum_{\sigma=1}^{n} K^{\sigma}_{\gamma} \Lambda^{\beta}_{\sigma}}_{\delta_{\gamma}{}^{\beta}} \; T(\cdot\,, w^{\gamma *}; w_{\beta})$$

$$= \sum_{\gamma=1}^{n} T(\cdot\,, w^{\gamma *}; w_{\gamma}) \qquad \checkmark$$

$\longrightarrow$ Manifold Setting.

We now consider the vector space V to be $V_p = \{$ tangent vectors at $p \in M \}$. Vectors in V_p are also called contravariant vectors and the dual space V_p^* of dual vectors also have two other names: covariant vectors and one forms.

$\exists$ a coordinate basis of V_p consisting of $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ defined as before:

$$\frac{\partial}{\partial x^i} (f)\Big|_p = \frac{\partial}{\partial x^i} (f \circ \psi^{-1}) \Big|_{\psi(p)} .$$

The associated dual basis of V_p^* is usually denoted $dx^1, \dots, dx^n$. That is, dx^μ is the symbol for the linear map

$$dx^\mu \left(\sum_\nu a^\nu \frac{\partial}{\partial x^\nu} \right) = a^\nu \quad \text{i.e.} \quad dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu{}_\nu .$$

$V \in V_p$ has components V^μ w.r.t. basis $\{\frac{\partial}{\partial x^\mu}\}$.

W.r.t. a different basis $\{\frac{\partial}{\partial x'^\nu}\}$, V has components V'^ν given by vector transformation law

$$V'^\nu = \sum_{\mu=1}^{n} V^\mu \frac{\partial x'^\nu}{\partial x^\mu} . \qquad (\text{contravariance})$$

If $\omega \in V_p^*$ has components ω_μ w.r.t. $\{dx^\mu\}$. then w.r.t. $\{dx'^\nu\}$ ω has components given by

$$\omega'_\nu = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} . \qquad (\text{covariance})$$

Given $T \in \mathcal{T}(k, l)$, we can now understand how T transforms under a change of basis:

$$T'^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'} = \sum_{\mu_1 \cdots \nu_l = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x'^{\nu_l'}} .$$

Tensor transformation law .

We will rarely use this at the coordinate level.

A tensor field is an assignment of a tensor over V_p $\forall\, p \in M$. A covariant vector field ω is said to be smooth if $\forall$ smooth contravariant (aka tangent) vector field V, the function $\omega(V)$ on M is a smooth function.

A tensor field is smooth if its action on any smooth contravariant / covariant vector fields defines a smooth function on M.

Physics is encoded in tensor fields defined on spacetime.

Metric Tensor

A metric or metric tensor on a manifold M is a symmetric nondegenerate tensor field of type $(0,2)$.

So a metric takes two vectors in V_p and gives a number and it does so at all points $p \in M$.

Symmetric: $\forall\, v_1, v_2 \in V_p$ $\quad g(v_1, v_2) = g(v_2, v_1)$.

nondegenerate: Suppose we fix $v_1 \in V_p$ and consider the $(0,1)$ tensor defined by $g(\cdot, v_1)$. The $(0,2)$ tensor g is nondegenerate at p if $g(\cdot, v_1) = 0$ only when $v_1 \equiv 0$.

Given a coordinate basis for V_p, we obtain a dual basis $\{dx^\mu\}$ of V_p^* and may expand the metric tensor in this basis as

$$g = \sum_{\mu,\nu} g_{\mu\nu}\, dx^\mu dx^\nu .$$

Now we see the analogy with infinitesimals emerging. The formula above conveys the metric as "infinitesimal square distance".

~~Suppose X and Y are vectors The metric~~

Suppose X, Y are vector fields on M. The metric $g_{\alpha\beta}$ assigns a magnitude of X at p via the formula

$$\left(\left| g_{\alpha\beta}(p) X_p^\alpha X_p^\beta \right| \right)^{\frac{1}{2}} .$$

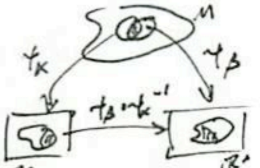
Similarly, the cos angle

$$\frac{g_{\alpha\beta}(X^\alpha Y^\beta)}{|g_{\alpha\beta} X^\alpha X^\beta|^{\frac{1}{2}}\ |g_{\alpha\beta} Y^\alpha Y^\beta|^{\frac{1}{2}}} = \cos(X, Y)$$

provided magnitudes of X, Y are nonzero and we have X is orthogonal to Y if $g_{\alpha\beta} X^\alpha Y^\beta = 0$.

SUMMARY

4 Feb. 2002



M C^∞ real manifold:

$\exists$ $O_\alpha \subset M$, $\bigcup_\alpha O_\alpha = M$, smooth bijection $\psi_\alpha : O_\alpha \longrightarrow U_\alpha \subset \mathbb{R}^n$

respecting overlap $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \psi_\beta(U_\alpha \cap U_\beta)$

subset of $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth map.

Functions on M:

$f \in \mathcal{F} = C^\infty(M, \mathbb{R})$, $f: M \to \mathbb{R}$ smooth if $\forall \alpha$ $f \circ \psi_\alpha^{-1} : U_\alpha \to \mathbb{R}$ smooth.

Tangent vectors on M at p:

$V_p = \{ V : \mathcal{F} \to \mathbb{R}$ s.t. V is linear + satisfies Leibnitz rule. $\}$

coordinate basis for V_p $\{ \frac{\partial}{\partial x^\mu} \}$ defined by

$$\frac{\partial}{\partial x^\mu} f \Big|_p = \frac{\partial}{\partial x^\mu} (f \circ \psi_\alpha^{-1}) \Big|_{\psi(p)}$$

smooth vector field: smooth assignment of $v \in V_p$ $\forall p \in M$. "smooth" $V : \mathcal{F} \longrightarrow \mathcal{F}$.

Covectors on M at p:

$V_p^* = \{ v^* : V_p \longrightarrow \mathbb{R}$ linear. $\}$ $V_p \simeq V_p^*$.

dual basis to a given coordinate basis $\{ \frac{\partial}{\partial x^\mu} \}$ of V_p is $\{ dx^\mu \}$.

$$dx^\nu \left(\frac{\partial}{\partial x^\mu} \right) = \delta^\nu{}_\mu .$$

smooth covector field: assignment of $v^* \in V_p$ $\forall p \in M$ with "smoothness"

$$v^* : \{ \text{smooth}\overset{\text{any}}{\ } \text{vector field} \} \longrightarrow \mathcal{F} .$$

Tensor of type (k, l) at p:

$$\mathcal{T}_p(k,l) = \{ T : \underbrace{V_p^* \times \cdots \times V_p^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \longrightarrow \mathbb{R} \quad \text{linear in each slot} \}$$

basis of $\mathcal{T}(k,l)$ $\{ \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l} \}$

e.g. $T : V \longrightarrow \mathbb{R}$

$\longrightarrow$ $T(\frac{\partial}{\partial x^\mu}) = T_\mu$ so we write $T = T_\mu dx^\mu$.

tensor components

$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$.

$T : V^* \longrightarrow \mathbb{R}$

$T(dx^\nu) = T^\nu$ so write $T = T^\nu \frac{\partial}{\partial x^\nu}$.

smooth tensor field assignment of a tensor $T \in \mathcal{T}_p(k,l)$ $\forall p \in M$ "smoothly" in the sense that T: smooth (co)vector fields $\to \mathcal{F}$.

Transformation Laws

vectors

$v \in V_p$ expressed w.r.t. coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$ as $v = v^\mu \frac{\partial}{\partial x^\mu}$.

New basis $\{\frac{\partial}{\partial x'^\nu}\}$, $v = v'^\nu \frac{\partial}{\partial x'^\nu}$ where

$$v'^\nu = v^\mu \underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{\text{Jacobian matrix of Transformation}} .$$

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, bend left] & \\ \boxed{\mathbb{R}^n} \arrow[rr] & & \boxed{\mathbb{R}^n} \\ x & & x' \end{tikzcd}$$

covectors

$w \in V_p^*$ w.r.t. $\{dx^\mu\}$ as $w = w_\nu \, dx^\nu$.

New basis $\{dx'^\beta\}$, $w = w_\beta' \, dx'^\beta$ where

$$w_\beta' = w_\nu \underbrace{\frac{\partial x^\nu}{\partial x'^\beta}}_{\text{Jacobian matrix of transformation.}}$$

tensors

$T \in \mathcal{T}(k,l)$ expressed w.r.t coordinate (& dual) bases $\{\frac{\partial}{\partial x^\mu}, dx^\nu\}$

as
$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l} .$$

New basis $\{\frac{\partial}{\partial x'^\alpha}, dx'^\beta\}$

$$T = T^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l} \frac{\partial}{\partial x'^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial x'^{\alpha_n}} \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_l}$$

where

$$T^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l} = T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_l} \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial x'^{\alpha_k}}{\partial x^{\mu_n}} \otimes \frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}} \otimes \cdots \otimes \frac{\partial x^{\nu_l}}{\partial x'^{\beta_l}} .$$

Some books define tensors as collections of numbers which transform according to the tensor transformation rule.

Example 1 A symmetric nondegenerate tensor of type (0,2) is called a metric or metric tensor.

$g \in \mathcal{T}(0,2)$ means $g: V_p \times V_p \longrightarrow \mathbb{R}$ linearly at each $p \in M$.

Suppose $v, w \in V_p$. g is symmetric at p if $g(v,w) = g(w,v)$. $g \in \mathcal{T}(0,2)$ is a symmetric tensor field if g is symmetric at p $\forall p \in M$.

one can also Symmetrize tensors by summing over various permutations of the indices. ( )

↙ can also antisymmetrize [ ]

Suppose we fix $v \in V_p$ and consider the (0,1) tensor at p defined by $g(\cdot, v)$. The (0,2) tensor g is nondegenerate at p if $g(\cdot, v) = 0$ only when $v = 0$. Suppose v is a smooth tensor field. g is a nondegenerate tensor field if $g(\cdot, v) = 0$ only for $v = 0$.

Given a coordinate basis of V_p, $\exists$ dual basis $\{dx^\mu\}$ of V_p^*. In this basis, we may express the metric tensor as

$$g = \sum_{\mu,\nu} g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu dx^\nu.$$

Metric Structure

A metric tensor g defined on a manifold M gives M a lot more structure. We can consider $g: V_p \times V_p \longrightarrow \mathbb{R}$ to define an inner product on V_p and with it we can define what it means for two vectors in V_p to be orthogonal. We can also define the norm of a vector in V_p using this inner product.

Another tensor is the (1,1)-tensor known as the Kronecker delta, δ^μ_ν. The nondegeneracy hypothesis on the metric tensor implies the existence of a (2,0)-tensor h with the property

$$g_{\mu\alpha} h^{\alpha\beta} = \delta_\mu^\beta .$$

The (2,0)-tensor h is called the inverse metric and is also denoted by g but with upper indices:

$$(g_{\mu\nu})^{-1} = (g^{\mu\nu}).$$

The metric and its inverse allow us to raise and lower indices on tensors.

Given $T \in \mathcal{T}(k,l)$ with components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$,

we can raise and lower indices:

$$T^{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_k}{}_{\mu_j}{}_{\nu_1\cdots\nu_l} = g_{\mu_j\gamma}\, T^{\mu_1\cdots\mu_{j-1}\gamma\mu_{j+1}\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

$$g^{\mu\alpha} T_{\alpha\beta}{}^{\gamma\delta} = T^{\mu}{}_{\beta}{}^{\gamma\delta} .$$

Vectors $V^\mu \frac{\partial}{\partial x^\mu}$ may be turned into covectors $V_\mu dx^\mu$ by the operation

$$V_\mu = g_{\mu\nu} V^\nu.$$

Also,

$$V^\nu = g^{\nu\mu} V_\mu .$$

Example   Maxwell equations

4-current

Suppose we are given a vector field J^{α} on Minkowski spacetime. We work in a global Cartesian chart so the flat metric may be written

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

We also have

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}.$$

4-potential

We impose PDE conditions on a covector field A_{μ} in terms of the given vector field J^{α} as follows:

$$\text{Define} \quad F_{\mu\nu} = A_{[\mu,\nu]} = A_{\mu,\nu} - A_{\nu,\mu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}.$$

$\longrightarrow$ $F_{\mu\nu} = - F_{\nu\mu}$ so 6 unknowns.

Faraday or E-M tensor

We demand A_{μ} satisfy certain conditions when we require

$$\partial_{\alpha} F^{\alpha\beta} = 4\pi J^{\beta} \quad \text{and} \quad \partial_{[\alpha} F_{\beta\gamma]} = 0$$

4 equations   4 equations

8 equations in 6 unknowns so this seems overdetermined.

__Exercise__ Suppose we give the components of the Faraday tensor some names:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}, \qquad \begin{array}{l} \underline{E} = (E_1, E_2, E_3) \\ \underline{B} = (B_1, B_2, B_3) \end{array}$$

Suppose we give components of the 4-current some names:

$$J^\alpha = (\rho, J_1, J_2, J_3) \quad \text{and} \quad \underline{J} = (J_1, J_2, J_3).$$

Verify that the differential conditions

$$\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta \quad \text{and} \quad \partial_{[\alpha} F_{\beta\gamma]} = 0$$

unravel to read, using $\underline{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$,

$$\underline{\nabla} \times \underline{E} - \partial_t \underline{B} = 4\pi \underline{J} \qquad \underline{\nabla} \cdot \underline{E} = 4\pi\rho$$

$$\underline{\nabla} \times \underline{\beta} + \partial_t \underline{E} = 0 \qquad \underline{\nabla} \cdot \underline{B} = 0.$$

__Remark:__

$$\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta.$$

Apply ∂_β:

$$\underset{\overset{\|}{0}}{\partial_\beta \partial_\alpha F^{\alpha\beta}} = 4\pi \partial_\beta J^\beta$$

$$\Longrightarrow \partial_\beta J^\beta = 0 \quad \Longleftrightarrow \quad \partial_t \rho + \underline{\nabla} \cdot \underline{J} = 0$$

(Conservation of charge.)

Maxwell's Equations (Example) (in Minkowski spacetime)

(skip until I.36)

$$\begin{cases} \underline{\nabla} \times \underline{E} - \partial_t \underline{B} = 4\pi \underline{J} \\ \underline{\nabla} \times \underline{B} + \partial_t \underline{E} = 0 \end{cases} \qquad \begin{matrix} \underline{\nabla} \cdot \underline{E} = 4\pi \rho \\ \underline{\nabla} \cdot \underline{B} = 0 \end{matrix}$$

$$\varepsilon^{ijk} = \varepsilon^{i}{}_{j}{}^{k} = \cdots = \begin{cases} +1 & ijk = \sigma(123) \quad \sigma \text{ even permutation} \\ -1 & \qquad\qquad\qquad\quad \text{odd} \\ 0 & \text{otherwise.} \end{cases}$$

$j, i = 1, 2$ or 3.

properties of curl $\Longrightarrow$ Maxwell's eq. can be rewritten

$$\begin{cases} \varepsilon^{ij}{}_{k} \partial_j E^k - \partial_0 B^i = 4\pi J^i \\ \varepsilon^{ij}{}_{k} \partial_j B^k + \partial_0 E^i = 0 \end{cases} \qquad \begin{matrix} \partial_i E^i = 4\pi J_0 \\ \partial_i B^i = 0. \end{matrix} \qquad (J_0 := \rho.)$$

current 4-vector $\quad J^\mu = (J^0, J^1, J^2, J^3) = (\rho, \underline{J}).$

Faraday or EM Field Tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} = -F_{\nu\mu}$$

antisymmetric $(0,2)$-tensor.

Lorentz Metric

$$\underset{(0,2)}{\eta_{\mu\nu}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underset{(2,0)}{\eta^{\mu\nu}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we can use the metric to raise and lower indices.

$$F^{\alpha\beta} = \eta^{\alpha\mu} \eta^{\beta\nu} F_{\mu\nu}$$

Jim:
(tighten this discussion)

$$F^{\alpha\alpha} = \eta^{\alpha\mu}\eta^{\alpha\nu}F_{\mu\nu} = (\pm 1)\,\delta^{\alpha\mu}\delta^{\alpha\nu}F_{\mu\nu} = \pm F_{\alpha\alpha} = 0.$$

$$F^{0i} = \eta^{0\mu}\eta^{i\nu}F_{\mu\nu} = (-1)\,\delta^{0\mu}\,(1)\,\delta^{i\nu}F_{\mu\nu} = -F_{0i}$$

$$F^{ij} = \eta^{i\mu}\eta^{j\nu}F_{\mu\nu} = \delta^{i\mu}\delta^{j\nu}F_{\mu\nu} = F_{ij}\,.$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$\partial_\alpha F^{\overset{\beta\alpha}{\cancel{\alpha\beta}}} = 4\pi J^\beta$$

$$\left.\begin{aligned}
&\underline{\beta=0}: \quad \partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = 4\pi J^0 \\
&\underline{\beta=1}: \quad -\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2 = 4\pi J^1 \\
&\underline{\beta=2}: \quad -\partial_0 E^2 - \partial_1 B^3 + \partial_3 B^1 = 4\pi J^2 \\
&\underline{\beta=3}: \quad -\partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1 = 4\pi J^3
\end{aligned}\right\} \text{encodes} \quad \begin{aligned} &\nabla\times B - \partial_t E = 4\pi J \\ &\nabla\cdot E = 4\pi\rho\,. \end{aligned}$$

Similarly

$$\underset{\substack{\uparrow \\ \text{antisymmetrization}}}{\partial_{[\alpha}}F_{\beta\gamma]} = 0,$$ encodes the other Maxwell equations

Check $\alpha = 1,\ \beta = 2,\ \gamma = 3.$

1 2 3

123 e   213 o   312 e
132 o   231 e   321 o

$$\partial_1 F_{23} - \partial_1 F_{32} + \partial_2 F_{31} - \partial_2 F_{13} + \partial_3 F_{12} - \partial_3 F_{21} = 0$$

Antisymmetry of $F_{\mu\nu}$ =>

$$2\left[\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}\right] = 2\left[\partial_1(+B_1) + \partial_2(B_2) + \partial_3 B_3\right]$$

$$= \nabla \cdot \underline{B} = 0.$$

Check $\alpha = 0,\ \beta = 1,\ \gamma = 2$

0 1 2

012 e   102 o   201 e
021 o   120 e   210 o

$$2\left[\partial_0 F_{12} + \partial_1 F_{02} + \partial_2 F_{01}\right] = 0$$

$$2\left[\partial_0 B_3 + \partial_1(-E_2) + \partial_2(E_1)\right] = 0$$

$$\partial_0 B_3 + \partial_2 E_1 - \partial_1 E_2 = 0$$

a component of the $\nabla \times E$ equation . . .

etc.

~~For $i = 1$, look at 2nd Maxwell eq.~~

~~$$\varepsilon^{1jk} \partial_j E_k - \partial_0 B^1 = \partial_2 E_3 - \partial_3 E_2 - \partial_0 B^1$$~~

~~$$= \partial_2 F_{30} - \partial_3 F_{02} - \partial_0 F_{23}$$~~

~~$$= \partial_{[2} F_{30]}$$~~

~~2 3 0 permutes:~~

~~$2\,3\,0_e$ $3\,0\,2_e$ $0\,2\,3_e$~~

~~$2\,0\,3_o$ $3\,2\,0_o$ $0\,3\,2_o$~~

Conservation of charge:

$$\partial_\mu F^{\overset{\nu\mu}{\cancel{\mu\nu}}} = 4\pi J^\nu .$$

Apply ∂_ν to this equation

$$\partial_\nu \partial_\mu F^{\mu\nu} = 4\pi \partial_\nu J^\nu .$$

The left side is equal to ~~$\partial_\mu \partial_\nu F^{\nu\mu}$~~ its negative due to antisymmetry of $F^{\mu\nu}$

so

$$\partial_\nu J^\nu = 0 .$$

$$\boxed{\partial_t \rho + \nabla \cdot J = 0.}$$

Abstract Index Notation Remarks

We have seen that a tensor $T \in \mathfrak{T}(k,l)$ may be characterized using the basis elements $\{ \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l} \}$ which are available in any $O_\alpha \subset M$ using the chart $\psi_\alpha : O_\alpha \to U_\alpha \subset M$. The Tensor is then characterized with components $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_k}$

$$\circledast \quad \sum_{\mu_i, \nu_i} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_k} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes dx^{\nu_l} .$$

We compress this sometimes by just writing

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_k}$$

for the tensor.

We have seen that tensors transform under coordinate transformations in a rather complicated way but, at the end of the story, we obtain new components

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_k'}$$

in which a modification (with primes) of formula $\circledast$ above again defines the tensor. We will <u>abstractly</u> denote tensors with

$$T^{a_1 \cdots a_k}{}_{b_1 \cdots b_k}$$

~~to denote~~ with reference to "any basis".

Antisymmetric $(0,l)$ tensors are called differential $\boxed{l\text{-forms.}}$

Exterior derivative

Antisymmetric $(0,r)$-tensors are called r-forms. We define a map d which takes r-forms to $(r+1)$-forms.

A zero-form field is a function. d acts on f to produce a 1-form, that is, a covector field. How does df act on vectors?

$\forall$ v.f. $\mathbb{X}$,

$$df(\mathbb{X}) = \mathbb{X} f.$$

(This calculation is instructive in coordinates and begins to reveal the relation between the basis of $V_p{}^* = T_p{}^* M$ called dx^α and elements of the "infinitesimals" dx^α from calculus.)

$$df = \frac{\partial f}{\partial x^i}\, dx^i$$

Suppose $A = A_{\alpha_1 \cdots \alpha_r}\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_r}$ is an r-form field. Then, we have the formula

$$dA = (d\, A_{\alpha_1 \cdots \alpha_r})\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_r}.$$

We verify that this is an $(r+1)$ - form by checking how it transforms under coordinate change.

Suppose in new coordinates

$$A = A_{\beta_1' \cdots \beta_r'} \, dx'^{\beta_1} \wedge \cdots \wedge dx'^{\beta_r} ,$$

where $A_{\beta_1' \cdots \beta_r'}$ is to be determined from the transformation $x^\alpha \longrightarrow x'^\beta$ and the old coefficients $A_{\alpha_1 \cdots \alpha_r}$.

We know from the tensor transf. law that

$$A_{\beta_1' \cdots \beta_r'} = \frac{\partial x^{\alpha_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} A_{\alpha_1 \cdots \alpha_r} .$$

We need to calculate

$$d \left(\frac{\partial x^{\alpha_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} A_{\alpha_1 \cdots \alpha_r} \right)$$

$$= \frac{\partial x^{\alpha_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} \, d A_{\alpha_1 \cdots \alpha_r} + \frac{\partial^2 x^{\alpha_1}}{\partial x'^{\beta_1} \partial x^{e'}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} A_{\alpha_1 \cdots \alpha_r} \times de'$$

Thus

only one that survives

$$dA = \frac{\partial x^{\alpha_1}}{\partial x \beta_1'} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} \, d A_{\alpha_1 \cdots \alpha_r} \wedge d x^{\beta_1'} \cdots dx^{\beta_r'}$$

$$+ \underbrace{\frac{\partial^2 x^{\alpha_1}}{\partial x'^{\beta_1} \partial x^{e'}}}_{\text{symmetric}} \cdot \frac{\partial x^{\alpha_2}}{\partial x'^{\beta_2}} \cdots \frac{\partial x^{\alpha_r}}{\partial x'^{\beta_r}} \, de' \wedge dx'^{\beta_1} \wedge \cdots \wedge dx'^{\beta_r} + \cdots$$

$\Longrightarrow$ cancellation

Examples

$$d(df) = \frac{\partial^2 f}{\partial x^i \partial x^j} \, dx^i \wedge dx^j = 0.$$

A, B are differential forms. Suppose A is an r-form

$$d(A \wedge B) = dA \wedge B + (-)^r A \wedge dB$$

Thus, we have the Poincaré lemma.

$$d(dA) = 0. \quad \forall \ r\text{-form field } A.$$

This is a nice derivative operator. But it only works on antisymmetric tensors. This operator arises naturally in the setting of Stokes' theorem.

Remarks: ① differential forms are important for defining integration on manifolds.

② Forms α which satisfy $d\alpha = 0$ are called closed. Forms α which are d of something else $\alpha = d\beta$ are called exact. exact $\subset$ closed

$$\text{modulo} \longrightarrow \frac{\text{closed } p \text{ forms}}{\text{exact } p \text{ forms}} = p\text{th Betti number}$$

[de Rham cohomology]

Maps between manifolds induce maps between tensor bundles

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[drr, bend right] & N & \\
 & N \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$\phi^*f = f \circ \phi$ We "pullback" f from N to M.

$$\begin{tikzcd}
T_p M \arrow[r, "\phi_*"] & T_{\phi(p)} N \\
V \arrow[u, phantom, sloped, "\in"] & \phi_*(V) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

How? $(\phi^* v)(f) = v(\phi_* f)$

We "pushforward" v from M to N

$\phi_* : T_p M \longrightarrow T_{\phi(p)} N$

$$\begin{tikzcd}
T_p^* M \arrow[r, "\phi^*"] & T^*_{\phi(p)} N \\
V & df \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

How?

We "pullback" df from N to M.

$$\phi^*(df)(v) = df(\phi_* V)$$

$$\phi^* : T^*_{\phi(p)} N \longrightarrow T_p^* M.$$

Since we can pullback covectors, we can pullback lower-index-tensors from N to M. Suppose T_{abc} on N. We define on M

$$\underbrace{\phi^*(T_{abc})}_{\text{tensor}} \underbrace{(V^a\ V^b\ V^c)}_{\text{acting upon}} := T_{abc}(\phi_* V^a, \phi_* V^b, \phi_* V^c).$$

Similarly, suppose A^{abc} on M, we can define on N

$$\phi_*(A^{abc})(\omega_a, \omega_b, \omega_c) := A^{abc}(\phi^*\omega_a, \phi^*\omega_b, \phi^*\omega_c).$$

Similarly, we can define $(\phi^* T)$ via

$$(\phi^* T)^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}\left((\mu_1)_{a_1} \cdots (\mu_k)_{a_k}\ (t_1)^{b_1} \cdots (t_\ell)^{b_\ell}\right)$$

$$= T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}\left((\phi_* \mu_1)_{a_1} \cdots (\phi_* \mu_k)_{a_k} \cdots ([\phi^{-1}]^* t_\ell)^{b_\ell}\right)$$

~~Note It can~~

Exercise Show that $\phi_* = (\phi^{-1})^*$.

Lie Derivative

Let ϕ_t be a 1-parameter group of diffeos on a manifold $\mathcal{M}$:

$$\forall t \qquad \phi_t : \mathcal{M} \longrightarrow \mathcal{M} \qquad \text{(diffeo)}$$

$$\phi_{t+s} = \phi_t \phi_s \qquad \text{(group)}.$$

We have seen that ϕ_t may be viewed as a flow on $\mathcal{M}$ generated by a vector field V^a.

Suppose we define

$$\mathcal{L}_V T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \lim_{t \to 0} \frac{\left[\phi_{-t}^{*}\left(T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right) - T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\right]}{t} .$$

Note: No index.

This operation $\mathcal{L} : \mathcal{T}(k,l) \longrightarrow \mathcal{T}(k,l)$

Example

$$\mathcal{L}_V(f) = V(f).$$

$$\mathcal{L}_V(w^\mu) = ?$$

Exercise : Show that $\mathcal{L}_V(w^\mu) = [V,w]^\mu$.

Most tensors do not, in general, behave well under pushforward and pullback. However if ϕ is a diffeomorphism (requiring M, N to have same dimension) then $\exists$ $\phi^{-1}: N \longrightarrow M$ so we can use the inverse map to move any tensor.

$\phi: M \rightarrow N$
$M \leftarrow N : \phi^{-1}$

$$\phi_* : T_pM \longrightarrow T_{\phi(p)}N$$

$$\phi^* : T^*_{\phi(p)}N \longrightarrow T^*_pM$$

$$(\phi^{-1})_* : T_{\phi(p)}N \longrightarrow T_pM$$

$$(\phi^{-1})^* : T^*_pM \longrightarrow T^*_{\phi(p)}M.$$

Given $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ at $p \in M$ we can define $(\phi_* T)^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ at $\phi(p) \in N$ via

$$(\phi_* T)^{a_1\cdots a_k}{}_{b_1\cdots b_l} \underbrace{(\mu_1)_{a_1}\cdots(\mu_k)_{a_k}}_{\text{k covectors on } N} \underbrace{(t_1)^{b_1}\cdots(t_l)^{b_l}}_{l \text{ vectors on } N}$$

$$= T^{a_1\cdots a_k}{}_{b_1\cdots b_l} (\phi^*\mu_1)_{a_1}\cdots(\phi^*\mu_k)_{a_k}\left[(\phi^{-1})_* t_1\right]^{b_1}\cdots\left[(\phi^{-1})_* t_l\right]^{b_l}$$

III. Curvature

A. Covariant Derivative

B. Parallel transport

C. Curvature

3 March 2003

(M,g) Riemann manifold of dimension n.

x^α local coordinates on M.

$t \longmapsto x^\alpha(t)$ particle trajectory

Classical force-free particle motion on M.

(we might imagine M imbedded in a higher dim'l $\mathbb{R}^K$ and we consider constrained dynamics.)

$$L = \frac{m}{2} |v|^2 = \frac{m}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

$$\text{E-L}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L}{\partial x^\nu} = 0.$$

$$\downarrow$$

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

$$\Gamma^\mu_{\alpha\beta} = g^{\mu\nu} \frac{1}{2} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right).$$

special relativistic force-free particle motion

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

(M, η) Minkowski spacetime.

$$L = -mc^2 \sqrt{1 - \frac{|v|^2}{c^2}} = -mc \frac{ds}{dt} = -mc^2 \sqrt{\frac{-\eta_{\mu\nu}\, dw^\mu dw^\nu}{dt^2}}$$

$$\downarrow$$

$$\frac{d}{ds}\left(\frac{dw^\mu}{ds} \right) = 0 \implies \frac{dw^\mu}{ds} = U^\mu$$

4-velocity

Covariant derivative

V v.f. on M.

x^α, x'^β two sets of local coords. on M near p.

$$V(f)(p) = V^\alpha \frac{\partial}{\partial x^\alpha}\Big|_p f = V'^\beta \frac{\partial}{\partial x'^\beta}\Big|_p f.$$

invariance w.r.t. coordinates requires $\frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta}$, $\frac{\partial}{\partial x'^\beta} = \frac{\partial x^\alpha}{\partial x'^\beta} \frac{\partial}{\partial x^\alpha}$.

Thus

$$V^\alpha = V'^\beta \frac{\partial x^\alpha}{\partial x'^\beta}.$$

(vector transformation law)

partial derivative of a vector field d/n transform tensorially.

In a local coordinate, consider $\frac{\partial}{\partial x^\alpha} V^\gamma$.

Represent this in another local coordinate:

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta}$$

$$V^\gamma = \frac{\partial x^\gamma}{\partial x'^\beta} V'^\beta.$$

We thus have

$$\frac{\partial}{\partial x^\alpha} V^\gamma = \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta}\left(\frac{\partial x^\gamma}{\partial x'^\mu} V'^\mu\right)$$

$$= \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial V'^\mu}{\partial x'^\beta} \frac{\partial x^\gamma}{\partial x'^\mu} + \underbrace{\frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial^2 x^\gamma}{\partial x'^\beta \partial x'^\mu}}_{\text{non-tensorial.}} V'^\mu$$

We introduce ∇_μ on v.t. with the formula

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + C^\nu_{\mu\lambda} V^\lambda$$

with $C^\nu_{\mu\lambda}$ specified in a chart and subject to the transformation law

$$C^{\nu'}_{\mu'\lambda'} = - \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \right) + \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} C^\nu_{\mu\lambda}$$

to enforce

$$\nabla_\mu V^\nu = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} \nabla_{\mu'} V^{\nu'} \qquad \text{(1-1 - tensor transf. law.)}$$

We restrict our attention to initial choices of $C^\gamma_{\alpha\beta}$ implying

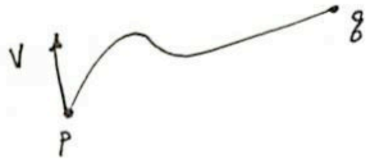
$$\nabla_a \nabla_b f = \nabla_b \nabla_a f \qquad \forall f \in \mathfrak{F}.$$

This is saying that the covariant derivative ∇_a defined using the connection $C^\gamma_{\alpha\beta}$ is <u>torsion free</u>, $\Rightarrow$ $\boxed{C^\gamma_{\alpha\beta} = C^\gamma_{\beta\alpha}}$

The tangent spaces $T_p(M)$ and $T_q(M)$ for $p, q \in M$, $p \neq q$ are naturally identified with:

- the connection $\Gamma^{\alpha}_{\beta\gamma}$
- a curve $c : [0,1] \longrightarrow M$, $c(0) = p$, $c(1) = q$.

This curvedependent identification of tangent spaces at distinct points on M explains the usage of "connection" as the descriptor for $\Gamma^{\alpha}_{\beta\gamma}$.



How? Suppose we are given the curve c, and a vector V^a at p. We parallel transport V_a along c to q by finding vectors V^a along the curve satisfying

$$\underset{\substack{\uparrow \\ \text{tangent to} \\ \text{curve.}}}{t^a} \nabla_a V^b = 0 \qquad \text{(along curve)}.$$

Similarly, we can parallel transport a tensor from p to q along the curve C by requiring

$$t^a \nabla_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_l} = 0 .$$

We express parallel transport of V^a along the curve c in coordinates:

$$t^a \nabla_a V^b = t^a \frac{\partial}{\partial x^a} V^b + t^a \Gamma^b_{ac} V^c$$

which is an ode system

$$\frac{dV^b}{dt} + \Gamma^b_{ac} t^a V^c = 0. \qquad \text{(parallel transport)}$$

<u>Remark</u>: Suppose we change perspective. The parallel transport equation describes how V^b should change along the curve c to ensure $t^a \nabla_a V^b = 0$, that is parallel transport. Suppose we have the additional miracle

$$t^a \nabla_a t^b = 0.$$

Then the tangent vector to c is parallel transported along c. This places a restriction on the curve. Along such a curve, in a local coordinate, we have

$$\frac{dt^b}{dt} + \Gamma^b_{ac} t^a t^c = 0$$

or

$$\ddot{x}^b + \Gamma^b_{ac} \dot{x}^a \dot{x}^c = 0.$$

Curves whose tangent vectors are parallel transported along the curve are called <u>geodesics</u>.

Theorem Let g_{ab} be a metric. Then $\exists!$ covariant derivative operator ∇_a s.t. $\nabla_a g_{bc} = 0$. (presumes torsion free)

proof We let C^d_{ab} be a connection which we try to find in terms of the metric g_{ab} s.t. $\nabla_a g_{bc} = 0$. We suppose that $\nabla_a = \partial_a + C^d_{ab}$, etc. Then

$$\nabla_a g_{bc} = \partial_a g_{bc} - C^d_{ab}\, g_{dc} - C^d_{ac}\, g_{bd}$$

$$= \partial_a g_{bc} - \underline{C_{abc}} - \underset{''''}{C_{acb}}$$

$$\nabla_b g_{ac} = \partial_b g_{ac} - \underline{C_{bac}} - \underset{\sim}{C_{bca}}$$

$$\nabla_c g_{ab} = \partial_c g_{ab} - \underset{''''}{C_{cab}} - \underset{\sim}{C_{cba}}.$$

$$\left[\underbrace{\nabla_a g_{bc} + \nabla_b g_{ac} - \nabla_c g_{ab}}_{=0}\right] = \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} - 2C_{abc}.$$

$$2C_{abc} = \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}$$

$$\boxed{\Gamma^c_{ab} := C^c_{ab} = \frac{1}{2} g^{cd}\left[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right]}$$

∎

Thus, the metric determines the connection coefficients.

Curvature

1. definition of Riemann Curvature tensor
2. Algebraic symmetry properties of Riemann Curvature tensor
3. Differential identity for curvature tensor
4. Contractions of curvature tensor

Let $f \in \mathfrak{F}$. ω_a $(0,1)$-field on M. $f\omega_a$ is also a $(0,1)$-field.

A calculation shows that

$$\nabla_a \nabla_b (f\omega_c) - \nabla_b \nabla_a (f\omega_c) = f\,(\nabla_a \nabla_b - \nabla_b \nabla_a)\,\omega_c .$$

Exercise: Verify this.

Thus, $(\nabla_a \nabla_b - \nabla_b \nabla_a)\Big|_p$ is linear and depends only upon $\omega_c|_p$.

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \; : \; (0,1)\text{-tensors} \longrightarrow (0,3)\text{-tensors.}$$

$$\omega_d \qquad\qquad R_{abc}{}^{d}\,\omega_d$$

Tensor Algebra $\Longrightarrow$ the action of $\nabla_a \nabla_b - \nabla_b \nabla_a$ on higher tensors may also be represented w. $R_{abc}{}^{d}$.

$R_{abc}{}^{d}$ is called the Riemann Curvature Tensor

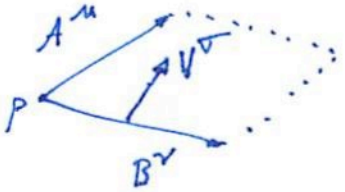
has $\sim n^4$ components on M^n.

$$\text{Curvature quantifies} \begin{cases} \text{change of vector under parallel transport on loops} \\ \text{noncommutativity of covariant differentiation} \\ \text{divergence of initially parallel geodesics.} \end{cases}$$

We developed this earlier and
may revisit it later.

We look at: $\dfrac{\text{change of vector under parallel transit round loop}}{\text{size of loop}}$.

Restrict attention to loops obtained naturally as "parallelograms" using two vectors:



data:
loop from A^μ, B^ν
vector to push V^σ

output:
$(\Delta V)^\rho$, change in V under transit.

Fact

$$R_{\mu\nu\sigma}{}^{\rho} A^\mu B^\nu V^\sigma = (\Delta V)^\rho.$$

So, the (1,3) tensor which characterizes noncommutativity of covariant derivatives also characterizes closed loop parallel transport.

How do we find $R_{\mu\nu\sigma}{}^{\rho}$?

The connection determines the curvature.
The metric determines the connection.
The metric determines the curvature.

Connection determines curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = (\partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e{}_{bd} - \Gamma^e{}_{bf} \Gamma^f{}_{ad}) V^d$$

Metric determines Connection

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right]$$

We sometimes say the metric <u>has</u> curvature.

Remark: $R_{abc}{}^d$ is a complicated beast involving $\sim n^4$ coefficients on M^n. We seek algebraic or differential symmetry properties of the curvature tensor and identities which reduce the complexity of $R_{abc}{}^d$.

Algebraic symmetry properties of curvature tensor

① $R_{abc}{}^d = - R_{bac}{}^d$.

② $R_{[abc]}{}^d = 0$.

③ $R_{abcd} = - R_{abdc}$ $\rightarrow$ (See III. 21 below)

④ $R_{abcd} = R_{cdab}$

A counting argument shows $\exists$ $\frac{1}{12} n^2 (n^2 - 1)$ independent components in curvature tensor on M^n.

$n=1$	0
$n=2$	1
$n=3$	6
$n=4$	20

Differential identity for $R_{abc}{}^{d}$ / Bianchi identity

$$\nabla_{[a} R_{bc]d}{}^{e} = 0.$$

These identities can be verified using Brute Force with the formula for curvature tensor in terms of metric and/or connection.

Contractions

$$R_{aac}{}^{d} = 0$$

$$R_{abcd} = - R_{abdc} \Longrightarrow R_{abcc} = 0.$$

$R_{abc}{}^{b} := R_{ac}$ <u>Ricci Tensor</u>

Note: $R_{abcd} = R_{cdab} \Longrightarrow R_{ac} = R_{ca}.$

$g^{ac}R_{ac} = R_{aa}{}^{a} = R$ <u>Scalar Curvature</u>

Contract Bianchi identity with $a = e$:

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0.$$ (1st Contracted Bianchi)

Bianchi identity

$$\nabla_{[a} R_{bc]d}{}^{e} = 0.$$

$+\ abc \quad -\ bac \quad +\ cab$

$-\ acb \quad +\ bca \quad -\ cba$

$\Longrightarrow$

$$\nabla_a R_{bcd}{}^{e} + \nabla_b R_{cad}{}^{e} + \nabla_c R_{\underset{\substack{\swarrow\searrow \\ b \;\; a \\ \;\;\;\; \| \\ \;\;\;\; e}}{ab} d}{}^{e} = 0.$$

a = e contraction:

$$\boxed{\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0.}$$

first contracted Bianchi identity.

b = d contraction

First raise the d index. (Note that metric compatibility allows us to multiply with g^{db} on left or right as the terms like $\nabla_a g^{db} = 0$.)

$$\underset{\uparrow}{\nabla_a R_c{}^{a}} + \underset{\nearrow}{\nabla_b R_c{}^{b}} - \nabla_c R = 0$$

same

$$\boxed{\nabla_a R_c{}^{a} - \frac{1}{2} \nabla_c R = 0}$$

Second contracted Bianchi identity

The second contracted Bianchi identity may be written as a divergence.

$$\nabla_a = g_{ad} \nabla^d . \qquad \nabla_c = g_{cd} \nabla^d .$$

Thus

$$\nabla^d \left(R_{cd} - \frac{1}{2} R g_{cd} \right) = 0 .$$

This motivates us to define

$$G_{cd} = R_{cd} - \frac{1}{2} R g_{cd}$$

since it has the nice property that

$$\nabla^d G_{cd} = 0 .$$

Remark: Einstein's Field equations are $R_{cd} - \frac{1}{2} R g_{cd} = T_{cd}$ where T_{cd} is the stess-energy-momentum tensor of some matter field. The geometry of curvature dictates that

$$\nabla^d T_{cd} = 0$$

which has the consequence that the matter obeys certain conservation properties.

2002 Notes

A. Covariant Derivative
B. Parallel transport
C. Curvature
D. Geodesics

III Curvature

A. Covariant Derivative

Manifold $M \ni p \in \mathcal{O}_\alpha \xrightarrow{\psi_\alpha} U_\alpha \Longrightarrow$ coordinate bases $\{ \frac{\partial}{\partial x^\mu} \}$, $\{ dx^\mu \}$:

$$\frac{\partial}{\partial x^\mu} f \Big|_p := \frac{\partial}{\partial x^\mu} \left(f \circ \psi_\alpha^{-1} \right) \Big|_{\psi(p)} , \qquad \left(f \in \mathfrak{F} = C^\infty(M; \mathbb{R}) \right)$$

$$dx^\nu \left(\frac{\partial}{\partial x^\mu} \right) = \delta^\nu{}_\mu .$$

For fixed $\mu \in \{1, \dots, n\}$, $\frac{\partial}{\partial x^\mu} f \in \mathfrak{F}$. With μ free,

$$\partial_\mu f := \frac{\partial f}{\partial x^\mu} dx^\mu$$

is naturally a dual vector.

$$\partial_\mu : (0,0)\text{-tensors} \longrightarrow (0,1)\text{-tensors}.$$

Change basis: $\{ \frac{\partial}{\partial x^{\mu'}} \}$, $\{ dx^{\mu'} \}$ is a new basis.

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial}{\partial x^{\mu'}} , \qquad \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}$$

$$\partial_{\mu'} f = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} f \quad \left(= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu f \; dx^{\mu'} \right) \quad (0,1)\text{-tensor}$$

We see that $\partial_\mu f$ transforms covariantly under a change of basis so it really does define a $(0,1)$-tensor.

Q: How does $\partial_\mu f$ act as a $(0,1)$-Tensor?

Given $V = V^\mu \frac{\partial}{\partial x^\mu}$, the action of $\partial_\mu f$ on V is

$$(\partial_\mu f)(V) = V^\mu \frac{\partial}{\partial x^\mu} f$$

so we obtain the directional derivative of f in the direction V. Therefore $\partial_a f$ is the "gradient" of f.

Q: How does ∂_a act on a vector field V^b?

We saw $\partial_a : \underset{\text{functions}}{(0,0)\text{-tensors}} \longrightarrow \underset{\text{covector fields}}{(0,1)\text{-tensors}}$.

We expect $\partial_a : \underset{\text{vector fields}}{(1,0)\text{-tensors}} \longrightarrow (1,1)\text{-tensors}$.

How? We want to define $\partial_a V^b$. Let's follow our nose... Given a chart, we obtain coordinate bases $\{\frac{\partial}{\partial x^\mu}\}, \{dx^\mu\}$. We then define the (1,1)-tensor component

$$\partial_\mu V^\nu := \frac{\partial V^\nu}{\partial x^\mu}. \left(\Longrightarrow \partial_\mu V^\nu = \sum \frac{\partial x^\nu}{\partial x^\mu} dx^\mu \frac{\partial}{\partial x^\nu}. \right)$$

For this to be valid, we must verify that $\partial_a V^b$, defined in this way, transforms as a (1,1)-tensor under change of coordinate bases.

As above $\partial_\mu \longrightarrow \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}$.

$$V^\nu \longrightarrow \frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu = V^{\nu'}.$$

$$\partial_\mu V^\nu \longrightarrow \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \right) \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) = \partial_{\mu'} V^{\nu'}.$$

We check:

$$\underbrace{\underset{\text{covariant}}{\frac{\partial x^\mu}{\partial x^{\mu'}}} \ \underset{\text{contravariant}}{\frac{\partial x^{\nu'}}{\partial x^\nu}} \ \underset{\text{seen above}}{\frac{\partial V^\nu}{\partial x^\mu}}}_{\text{tensorial}} + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} \right) V^\nu}_{\boxed{\text{nontensorial}}}$$

= 0 under linear c.o.v.

$x' = \Lambda x$,

otherwise changes at 2nd order.

Punchline: The partial derivatives given by chart d/n behave tensorially on vector fields.

Naturally, we will want to differentiate tensors. The partial derivatives given to us by a chart lead to a <u>coordinate dependent</u> way of defining derivatives of vector fields. The partial derivatives work tensorially on functions ((0,0)-tensors) but <u>nontensorially</u> on higher order tensors.

We want to define a derivative operator which allows us to differentiate tensors and obtain tensors. The partial derivative operator doesn't work so we endeavour to fix it.

The new and improved coordinate independent derivative operator we will define is called the <u>covariant derivative operator</u>. In coordinates, we will denote such an operator by ∇_μ. The tensorial property we want is that for vector field V^a

$$\circledast \qquad \boxed{\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu ,}$$

with similar tensorial transformation properties when ∇_μ acts on a tensor.

Recall our formula, $\partial_\mu V^\nu$ transformed under basis change into

$$\partial_{\mu'} V^{\nu'} = \left[\underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}}}_{\text{covariant}} \underbrace{\frac{\partial x^{\nu'}}{\partial x^\nu}}_{\text{contravariant}} \partial_\mu + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}}}_{\text{covariant}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} \right) \right] V^\nu .$$

$$\downarrow$$

$$\bigcirc\, {}^{\nu'}_{\mu\nu}$$

This motivates trying to "correct" $\partial_\mu V^\nu$ by adding on a (necessarialy nontenserial) $C^{\nu}_{\mu\lambda} V^\lambda$ term which cancels away bad term in $\partial_{\mu'} V^{\nu'}$ formula resulting in a tensorial derivative.

DATE III . 4

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + C^{\nu}_{\mu\lambda} V^\lambda .$$

Change basis, express left-side of ⊛

$$\nabla_{\mu'} V^{\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu\right)\left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) + C^{\nu'}_{\mu'\lambda'}\left(\frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda\right)$$

$$= \underline{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu}} + \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}\right) V^\nu + C^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda .$$

Now, expand out the right-side of ⊛.

$$\underline{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} C^\nu_{\mu\lambda} V^\lambda$$

To have equality, we require ∀ vectors $V = V^\lambda$:

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^{\boxed{\nu}\lambda}}\right) V^{\overset{\lambda}{\boxed{\nu}}} + \underbrace{C^{\nu'}_{\mu'\lambda'}}_{\text{isolate}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} C^\nu_{\mu\lambda} V^\lambda .$$

The correction terms may be required to transform under change of coordinates so that

$$(\#) \qquad C^{\nu'}_{\mu'\lambda'} = -\frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}\right) + \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} C^\nu_{\mu\lambda} ,$$

which is non-tensorial. However, with this requirement on the correction terms

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + C^\nu_{\mu\lambda} V^\lambda$$

transforms as a (1,1) – tensor.

We saw last time that the "most natural" derivative operators $\{\frac{\partial}{\partial x^\mu}\}$ we obtain from the chart do <u>not</u> map (1,0)-tensors to (1,1)-tensors:

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^{\nu'} + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}\right) V^{\nu'}}_{\text{non-tensorial.}} .$$

So we introduced a "corrected" derivative operator

$$\nabla_\mu = \partial_\mu + \Gamma^\nu_{\mu\lambda}$$

whose action on vector fields is defined by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

and we hoped that $\Gamma^\nu_{\mu\lambda}$ may be chosen to transform so that

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\nabla_\mu V^\nu .$$

This condition led us to require that $\Gamma^\nu_{\mu\lambda}$ transform as

(#) $$\Gamma^{\nu'}_{\mu'\lambda'} = -\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}\right) + \frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} .$$

The improved derivative operator is called a <u>covariant derivative operator</u>. The $\Gamma^\nu_{\mu\lambda}$ are called <u>connection coefficients</u> or <u>Christoffel symbols</u> and specifying a choice is referred to as defining a <u>connection on $\mathcal{M}$</u>.

The coefficients $C_{\mu\lambda}^{\nu}$ are called connection coefficients., and, in the discussion just presented which starts from ∂_a, are also called Christoffel symbols.

Definition A covariant derivative operator, ∇_a, on a manifold takes smooth $\mathcal{T}_M(k,\ell)$-fields to smooth $\mathcal{T}(k,\ell+1)$-fields on M satisfying the following 5 conditions:

① linearity: $\forall\ A, B \in \tilde{\mathcal{T}}_M(k,\ell)$ + $\forall\ \alpha, \beta \in \mathbb{R}$

$$\nabla_c(\alpha A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} + \beta B^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}) = \alpha \nabla_c A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} + \beta \nabla_c B^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}.$$

② Leibnitz Rule: $\forall\ A \in \mathcal{T}_M(k,\ell)$, $B \in \mathcal{T}_M(k';\ell')$

$$\nabla_e\left[A^{a_1\cdots a_k}{}_{b_1\cdots b_k} B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{\ell'}}\right] = \left[\nabla_e A^{\cdots}{}_{\cdots}\right]B + A^{\cdots}{}_{\cdots}\left[\nabla_e B^{\cdots}{}_{\cdots}\right].$$

③ Commutativity w. contraction: $\forall\ A \in \mathcal{T}_M(k,\ell)$

$$\nabla_d\left(A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_k}\right) = \nabla_d A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_k}.$$

④ Consistency w. tangent vectors as directional derivatives on scalar fields:

$$\forall f \in \tilde{\mathcal{T}}_1 = C^\infty(M;\mathbb{R}) + \forall\ t^a \in V_p$$

$$t(f) = t^a \nabla_a f.$$

⑤ Torsion Free: $\forall\ f \in \tilde{\mathcal{T}}_1 = C^\infty(M;\mathbb{R})$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f.$$

Remark Commutator of 2 vector fields expressed via covariant derivative.

$$[V,W](f) = V\{W(f)\} - W\{V(f)\}$$

$$= V^a \nabla_a \{ W^b \nabla_b f \} - W^a \nabla_a \{ V^b \nabla_b f \}$$

Leibnitz Leibnitz

$$\nabla_a \nabla_b - \nabla_b \nabla_a = 0$$

$$= \left[V^a \nabla_a W^b - W^a \nabla_a V^b \right] \nabla_b f .$$

$$\longrightarrow \quad [V,W]^b = \left[V^a \nabla_a W^b - W^a \nabla_a V^b \right].$$

Covariant derivative operators exist.

We showed above that if we define some numbers $\Gamma^a{}_{bc}$ which transform under changes of basis according to ⊕ then

$$\nabla_b = \partial_b + \Gamma^a{}_{bc}$$

defines a covariant derivative operator. (Actually, we only verified this to be the case on functions [trivial] and (1,0)-tensors.)

Exercise: Describe the action of a covariant derivative operator on a (1,1)-tensor by using components. Verify, using ⊕, that the (1,2)-tensor obtained from covariant differentiation transforms as a tensor.

Q: How many covariant derivative operators on M exist.?

Good guess is number of slots in Christoffel symbol $\Gamma^c{}_{ab}$ $\sim n^3$.

Claim: Torsion Free $\longrightarrow \Gamma^c{}_{ab} = \Gamma_{ba}^c$.

Fact: The action of the covariant derivative we constructed above on a 1-form = (0,1)-tensor = covector ω_b:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\, \omega_\lambda .$$

Since we have restricted $\Gamma^\lambda_{\mu\nu}$ to transform by (#), we have a tensorial equation so we can write

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c_{ab}\, \omega_c .$$

Now suppose $\tilde{\nabla}$ is another covariant operator on M.
Then $\forall f \in \mathfrak{F}$ $\tilde{\nabla}_b f$ is a (0,1)-tensor.
Set $\omega_b = \tilde{\nabla}_b f$.

$$\nabla_a \tilde{\nabla}_b f = \partial_a \tilde{\nabla}_b f - \Gamma_{ab}^c\, \tilde{\nabla}_c f .$$

By property ④ $\tilde{\nabla}_b f = \partial_b f = \nabla_b f$, so

$$\nabla_a \nabla_b f = \partial_a \partial_b f - \Gamma_{ab}^c \nabla_c f .$$

By ⑤ torsion free, 1st two terms symmetric under $a \to b$, $b \to a$ symmetry and therefore

$$\Gamma^c_{ab} = \Gamma^c_{ba} .$$

$\exists\ n\,\frac{n(n+1)}{2}$ distinct covariant derivative operators.
at each $p \in M$.

B. Parallel transport

Suppose we have a curve C with tangent vector t^a in our manifold $\mathcal{M}$. A vector V^a given at each point on the curve is said to be parallelly transported as one moves along the curve if

$$t^a \nabla_a V^b = 0 \text{, along the curve.}$$

More generally, the parallel transport of a tensor is

$$t^a \nabla_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_k} = 0 .$$

If we choose a coordinate system, we can express the parallel transportation of a vector eq. above as

$$t^a \partial_a V^b + t^a \Gamma^b_{ac} V^c = 0 ,$$

which unravels to read

parallel transportation equation

$$\boxed{\frac{dV^\nu}{dt} + \sum_{\mu,\lambda} t^\mu \Gamma^\nu_{\mu\lambda} V^\lambda = 0 .}$$

This ode in V^ν shows that we may consider parallel transport properties of vectors defined only along the curve rather than requiring global vector fields. Moreover, if we specify a vector along C, solving the above ODE-system with that initial data defines a vector at each point along C and the vector is parallelly transported.

This means that there is a natural way to identify tangent spaces at V_p, $p \in M$ and V_q, $q \in M$ if we also have

- a connection coefficient system $\Gamma^{\nu}_{\mu\lambda}$
- a curve C with $p \in C$, $q \in C$

  e.g. $C : [0, 1] \to M$

  $c(0) = p$, $C(1) = q$.

Such a curve-dependent ~~A~~ way of identifying tangent spaces at distinct points in a manifold is called a connection.

One can also define a covariant derivative by first giving the connection and going backwards . . .

Metric Compatibility

Suppose our manifold M has a metric g_{ab}.
We know $\exists \sim n^3$ different systems of connection coefficients leading to $\sim n^3$ different covariant derivative operators. The metric gives a natural condition to impose on the covariant derivative under which the derivative is unique.
Fix a point $p \in M$
Given V^a, w^b vectors $\in V_p$ we can consider $g_{ab} V^a w^b$ (inner-product). If we parallel transport V^a, w^b from $p \in M$ to $q \in M$ along some curve, we can impose the natural requirement that
$g_{ab} V^a w^b$ remain unchanged during the parallel transportation.

Using Leibnitz rule, we calculate the requirement $t^a \nabla_a (g_{bc} v^b w^c) = 0$.

$$t^a(\nabla_a g_{bc}) v^b w^c + t^a g_{bc} (\nabla_a v^b) w^c + t^a g_{bc} v^b (\nabla_a v^c) = 0$$

$$= t^a (\nabla_a g_{bc}) v^b w^c + g_{bc} \overset{0}{\overset{\shortparallel}{(t^a \nabla_a v^b)}} w^c + g_{bc} v^b \overset{0}{\overset{\shortparallel}{(t^a \nabla_a v^c)}} = 0$$

$$= t^a (\nabla_a g_{bc}) v^b w^c.$$

If we require this to hold ∀ curves and all parallelly transported vector fields we encounter the condition

$$\boxed{\nabla_a g_{bc} = 0} \qquad \text{metric compatibility}$$

on the covariant derivative ∇_a.

<u>Theorem</u> Let g_{ab} be a metric. Then $\exists!$ covariant derivative operator ∇_a satisfying $\nabla_a g_{bc} = 0$.

<u>Proof</u> Let $\tilde{\nabla}_a$ be any deriv. operator, eg. an ordinary derivative operator associated w. a coordinate system. We attempt to solve for the Christoffel coefficients C^c_{ab} so that $\tilde{\nabla}_{ab} + C^c_{ab}$ satisfy the metric compatibility condition..

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - \underset{\substack{\shortparallel \\ [illegible]}}{C^d_{ab} g_{dc}} - \underset{\substack{\shortparallel \\ C_{acb}}}{C^d_{ac} g_{bd}}$$

~~C_{ab}~~ c

$$\Rightarrow \quad 0 = \tilde{\nabla}_a g_{bc} - C_{cab} - C_{bac}.$$

$$+ \quad \tilde{\nabla}_a g_{bc} = C_{cab} + C_{bac}.$$

$$+ \quad \tilde{\nabla}_b g_{ac} = C_{cba} + C_{abc}$$

$$- \quad \tilde{\nabla}_c g_{ab} = C_{bca} + C_{acb}$$

$$\left[\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} \right] = C_{c\dot{a}\dot{b}} + \underset{//}{C_{bac}} + C_{c\dot{b}\dot{a}} + \underset{=}{C_{abc}} + \underset{//}{C_{bca}} + \underset{=}{C_{acb}}$$

~~Now apply g~~

$$= 2 C_{cab}.$$

Now raise index c.

$$g^{cd} \left[\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right] = 2 C^c{}_{ab}.$$

$\Rightarrow$ $C^c{}_{ab}$ determined by the metric.

Note: We can choose $\tilde{\nabla}_a$ to be ∂_a in a fixed chart $\Rightarrow C^c{}_{ab} = C^c{}_{ab}(g)$.

Henceforth, we will restrict our covariant derivative to be <u>the</u> metric compatible covariant derivative operator. In terms of partial derivative operators we therefore have

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left\{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right\}.$$

We can thus compute the Christoffel symbols (and therefore ∇_a) by taking partial derivatives of coord. basis components of metric.

1st lecture after reading week

M $\quad C^\infty$ real manifold of dimension n

$$\mathcal{F} = C^\infty(M; \mathbb{R}) \qquad \forall \alpha \quad f \circ \psi_\alpha^{-1} : \underset{\overset{\cap}{\mathbb{R}^n}}{U_\alpha} \longrightarrow \mathbb{R}^n \text{ smooth.}$$

$$V_p = \{ v : \mathcal{F} \to \mathbb{R} \mid v \text{ linear} + \circledast \text{ Leibnitz} \}.$$

chart $\to$ coord. basis $\left\{ \frac{\partial}{\partial x^\mu} \right\}$.

smooth vector field: smooth assignment of $v \in V_p$ $\forall p \in M$.

$$V_p^* = \{ v^* : V_p \to \mathbb{R} \mid v^* \text{ linear} \}$$

dual basis $\{ dx^\nu \}$ $\qquad dx^\nu \frac{\partial}{\partial x^\mu} = \delta^\nu_\mu$.

smooth covector field: $\quad v^* : \{\text{smooth v.f}\} \longrightarrow \mathcal{F}$.

$$\mathcal{T}_p(k,l) = \{ T : \overbrace{V_p^* \times \cdots \times V_p^*}^{k} \times \overbrace{V_p \times \cdots \times V_p}^{l} \longrightarrow \mathbb{R} \mid \text{multilinear} \}.$$

basis $\left\{ \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l} \right\}$.

w.r.t. basis T has components

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}.$$

smooth tensor field

Tensors transform in a particular way under change of basis.

A symmetric nondegenerate $(0,2)$-tensor on M is called a metric g_{ab}.

Partial derivative operator ∂_a maps $\mathcal{F} = \mathcal{T}(0,0) \longrightarrow \mathcal{T}(0,1)$, tensorially.

But ∂_a maps $\mathcal{T}(1,0) \longrightarrow$ nontensorial.

$\exists\,!$ metric compatible covariant derivative operator ∇_a:

tensorial · nontensorial · notensorial

$$\nabla_a = \partial_a + \Gamma^a{}_{bc} .$$

covariant derivative operator · ordinary old partial derivative operator · Christoffel coefficient

metric compatible : $\nabla_a g_{bc} = 0$

This came out of $t^a \nabla_a (g_{bc} V^b W^c) = 0$, which was an infinitesimal condition that the inner product of V, W is unchanged during parallel transport.

The metric compatibility restriction leads to the formula.

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left\{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right\}$$

C. Curvature

- define Riemann Curvature tensor
- shows relationship w. failure of commutativity of differentiations.
- relate to path dependent nature of parallel transport.
- Initially parallel geodesics fail to remain parallel.

Let $f \in \mathfrak{F}$, $\omega_a \in \mathfrak{F}_M(0,1)$, ∇_a covariant derivative operator.

[illegible]

= [illegible]

$$\nabla_a \nabla_b (f \omega_c) = \nabla_a \{ \nabla_b f \, \omega_c + f \, \nabla_b \omega_c \}$$

$$= \underbrace{\nabla_a \nabla_b f}\, \omega_c + \nabla_b f \, \nabla_a \omega_c + \nabla_a f \, \nabla_b \omega_c + f \, \nabla_a \nabla_b \omega_c$$

$$\nabla_b \nabla_a (f \omega_c) = \underbrace{\nabla_b \nabla_a f}\, \omega_c + \nabla_a f \, \nabla_b \omega_c + \nabla_b f \, \nabla_a \omega_c + f \, \nabla_b \nabla_a \omega_c .$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f \omega_c) = f \, (\nabla_a \nabla_b - \nabla_b \nabla_a)(\omega_c).$$

$\Rightarrow (\nabla_a \nabla_b - \nabla_b \nabla_a)(\omega_c)\big|_p$ depends only upon $\omega_c\big|_p$, and defines a linear map:

$\therefore$ $\nabla_a \nabla_b - \nabla_b \nabla_a$ maps $(0,1)$-tensors to $(0,3)$-tensors,

Hence, the action of $\nabla_a \nabla_b - \nabla_b \nabla_a$ may be represented by a $(1,3)$-tensor field. (Actually, we have only shown this to be the case on $(0,1)$-tensors):

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = \underbrace{R_{abc}{}^{d}}\, \omega_d .$$

Riemann Curvature Tensor.

action

$$R_{abc}{}^{d}$$

loop output

Naively, the Riemann tensor has $\sim n^4$ components.

PROJECT ACTION NOTES

S. Carroll p. 72

PROJECT PLANNING NOTES

Manifestations of curvature include:
- change of vector under parallel transport on closed loop.
- covariant derivatives of tensors d/n commute.
- initially parallel geodesics become unparallel,

All these notions are quantified and "explained" with the Riemann curvature tensor.

Parallel transport of a vector on a closed loop on the sphere returns a different vector. The "amount of change of vector" depends upon "size of the loop." We wish to characterize this effect infinitesimally.

Take $p \in M$ and consider two tiny vectors A^μ, $B^\nu \in V_p$. These define a rectangle $\subset$ 2d plane with vertex at p. parallel transport a fixed vector V^σ around this rectangle. Eventually, V^σ comes back transformed and we characterize this transformation:

$$\text{Ingredients} \left\{ \begin{matrix} \text{loop} \\ \updownarrow \\ A^\mu, B^\nu \end{matrix} ; \begin{matrix} \text{initial vector} \\ V^\sigma \end{matrix} \right\} \rightsquigarrow \delta V^\rho$$

"change in V^σ after loop".

3 - vectors outputs vector.

should probably write $R^{\rho}{}_{\sigma\mu\nu}$

$$\underbrace{R_{\sigma\mu\nu}{}^{\rho}} A^\mu B^\nu V^\sigma = \delta V^\rho$$

Riemann Curvature Tensor
(1,3).

Changing $A \to B$, $B \to A$ switches transit direction $\Longrightarrow$

$R^{d}{}_{abc}$

$$R_{abc}{}^{d} = -R_{bac}{}^{d}$$

$\rightsquigarrow n^3 \frac{(n-1)}{2}$ independent components.

Implicit: commutator of ∇_a, ∇_b coincides with change under infinitesimal loop transit.

<u>Fact</u>: The connection determines the Riemann curvature tensor.

A calculation reveals

$$[\nabla_\mu, \nabla_\nu] V^\rho = \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma}\right) V^\sigma$$

$$- 2\, \Gamma^\lambda{}_{[\mu\nu]} \nabla_\lambda V^\rho .$$

see (3.65) in S. Carroll's notes.

$$= \underset{R_{\mu\nu\sigma}{}^{\rho} V^\sigma}{} - \underset{\substack{\uparrow \\ \text{Torsion Tensor.}}}{T_{\mu\nu}{}^{\lambda}} \nabla_\lambda V^\rho .$$

In particular, the connection coefficients $\Gamma^a{}_{bc}$ determine :

$$ = \left(\partial_a \Gamma^d{}_{bc} - \partial_b \Gamma^d{}_{ac} + \Gamma^d{}_{ae} \Gamma^e{}_{bc} - \Gamma^d{}_{be} \Gamma^e{}_{ac}\right).$$

$R_{abc}{}^d$

Note manifest assymetry under $a \leftrightarrow b$ interchange.

$$R_{\text{loop, vector}}^{\quad\text{output}}$$

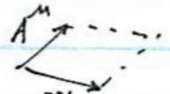
<u>Remark</u>

⊙ We have seen that $\nabla_a \nabla_b - \nabla_b \nabla_a : (0,1)\text{-tensors} \longrightarrow (0,3)\text{-tensors}$
s.t.
- The mapping property is local; depends only on $(0,1)$ tensor at $p \in M$
- The mapping is linear

$$\Rightarrow \left(\nabla_a \nabla_b - \nabla_b \nabla_a \right) W_c = R_{abc}{}^{d} W_d \quad \text{for some } (1,3)\text{-tensor } R.$$

⊙ We have seen that $R_{abc}{}^{d}$ may be understood as

$$\frac{\text{change of vector under closed loop transit}}{\text{size of closed loop.}}$$



$$R_{\mu\nu\sigma}{}^{\rho} \underbrace{A^{\mu} B^{\nu}}_{\text{loop data}} \underset{\substack{\uparrow \\ \text{initial vector}}}{V^{\sigma}} = \underset{\substack{\uparrow \\ \text{change in } V}}{(\delta V)^{\rho}}$$

A calculation in Wald shows the R's defined above in fact coincide.

⊙ We have seen that for $\nabla_a = \partial_a + \Gamma_{\cdots}$, metric compatible

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c =$$

$$\left(\partial_a \Gamma^{c}{}_{bd} + \partial_b \Gamma^{c}{}_{ad} + \Gamma^{c}{}_{ae} \Gamma^{e}{}_{bd} - \Gamma^{c}{}_{bf} \Gamma^{f}{}_{ad} \right) V^d$$

$$\Longrightarrow -R_{abd}{}^{c} = (\qquad\qquad\qquad\qquad)$$

Hence, the connection determines the curvature tensor.
In metric compatible setting, we know metric determines connection so we may say that the metric "has" curvature.

Claim $\nabla_{[a} \nabla_b w_{c]} = 0.$

We'll see. $\nabla_b w_c := \theta_{bc}$ (0,2) - tensor.

$$\nabla_a \theta_{bc} = \partial_a \theta_{bc} - \Gamma^{\mu}{}_{ab}\, \theta_{\mu c} - \Gamma^{\nu}{}_{ac}\, \theta_{b\nu}$$

But $\theta_{bc} = \nabla_b w_c = \partial_b w_c - \Gamma^{\alpha}_{bc}\, w_\alpha$

$$\nabla_a \theta_{bc} = \partial_a (\partial_b w_c - \Gamma^{\alpha}{}_{bc}\, w_\alpha) - \Gamma^{\mu}{}_{ab} (\partial_\mu w_c - \Gamma^{\alpha}{}_{\mu c}\, w_\alpha)$$

$$- \Gamma^{\nu}{}_{ac} (\partial_b w_\nu - \Gamma^{\alpha}{}_{b\nu}\, w_\alpha)$$

$$= \partial_a \partial_b w_c - \partial_a (\Gamma^{\alpha}{}_{bc}\, w_\alpha) - \Gamma^{\mu}{}_{ab} (\partial_\mu w_c)$$

$$+ \Gamma^{\mu}{}_{ab}\, \Gamma^{\alpha}{}_{\mu c}\, w_\alpha + \Gamma^{\nu}{}_{ac} (\partial_b w_\nu)$$

$$+ \Gamma^{\nu}{}_{ac}\, \Gamma^{\alpha}{}_{b\nu}\, w_\alpha$$

$$\nabla_a \nabla_b w_c. = \underline{\partial_a \partial_b w_c} - \underline{(\partial_a \Gamma^{\alpha}{}_{bc})\, w_\alpha} - \underset{\text{add}}{\underline{\Gamma^{\alpha}{}_{bc} (\partial_a w_\alpha)}} - \underline{\Gamma^{\mu}{}_{ab}} (\partial_\mu w_c)$$

$$+ \underline{\Gamma^{\mu}{}_{ab}}\, \Gamma^{\alpha}{}_{\mu c}\, w_\alpha + \underline{\Gamma^{\nu}{}_{ac}}\, \Gamma^{\alpha}{}_{b\nu}\, w_\alpha$$

$$+ \underset{\text{add}}{\underline{\Gamma^{\nu}{}_{ac} (\partial_b w_\nu)}}.$$

[abc] :

+ abc − bac + cab

− acb + bca − cba

$$\Longrightarrow \boxed{\nabla_{[a} \nabla_b w_{c]} = 0} \quad \checkmark$$

$$\cancel{\nabla_{[a} \nabla_b w_{c]} = -2 \left\{ \Gamma^{\alpha}{}_{ab} (\partial_c w_\alpha) + \Gamma^{\alpha}{}_{bc} (\partial_a w_\alpha) + \Gamma^{\alpha}{}_{ca} (\partial_b w_\alpha) \right\}}$$

But then,

$$0 = 2\nabla_{[a}\nabla_b w_{c]} = \nabla_{[a}\nabla_b w_{c]} - \nabla_{[b}\nabla_a w_{c]} = R_{[abc]}{}^{d}\, w_d .$$

$$\Longrightarrow \quad R_{[abc]}{}^{d} = 0 .$$

The next symmetry of $R_{abc}{}^{d}$ is revealed after lowering the upper index

$$R_{abcd} = g_{de}\, R_{abc}{}^{e} .$$

$$\boxed{\textcircled{3} \qquad R_{abcd} = -R_{abdc}\, .}$$

To validate this we will

- first identify how $R_{abc}{}^{d}$ acts on Tensors other than $(0,1)$-tenso
- second, apply $R_{abc}{}^{d}$ to $(0,2)$-tensor g_{ef} .

We have defined $R_{abc}{}^{d}$ by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, w_c = R_{abc}{}^{d}\, w_d .$$

How can we use this definition to determine the action of $R_{abc}{}^{d}$ on a $(1,0)$-tensor, that is on a vector field?

Let t^c be a vector field. Form the function $t^c w_c$.

① $R_{abc}{}^{d} = -R_{bac}{}^{d}$

② $R_{[abc]}{}^{d} = 0 \underset{①}{\Longrightarrow} 2(R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d}) = 0.$

③ $R_{abcd} = - R_{abdc}$.

$\Longrightarrow \quad R_{ab\,cd} = R_{cd\,ab}$

$$R_{abcd} = -R_{abdc} = R_{bdac} + R_{dabc}$$

$$= -R_{dbac} - R_{dacb}$$

$$= R_{badc} + R_{adbc} + R_{acdb} + \underline{R_{cdab}}$$ ①

~~suffices to show~~

~~$\boxed{R_{badc} + R_{adbc} + R_{acdb} = 0.}$~~

~~$+ R_{cdab} \approx R_{dacb}$~~

~~$R_{badc} + R_{adbc} - R_{adbc} - R_{cdab} = 0$, same as we've been hoping to show.~~

~~$-R_{bacd} + R_{adbc} + R_{acdb} = 0.$~~

$$R_{cdab} = -R_{dacb} - R_{acdb}$$

$$= +R_{dabc} + R_{acbd}$$

$$= -R_{abdc} - R_{bdac} - R_{cbad} - R_{bacd}$$

$$= \underline{R_{abcd}} - R_{bdac} - R_{cbad} - R_{bacd}$$ ②

$$(R_{abcd} - R_{cdab}) = \underline{R_{badc}} + \underline{R_{adbc}} + \underline{R_{acdb}}$$ ①

$$+ (\qquad\qquad) = \underline{R_{bdac}} + \underline{R_{cbad}} + \underline{R_{bacd}}$$ ②

$$0$$

$2(R_{abcd} - R_{cdab})$ = ~~R_{badc} +~~ R_{bacd}

$$2(\quad) = R_{adbc} + R_{acdb} + R_{bdac} + R_{cbad}$$

$$= -R_{adcb} + R_{acdb} + R_{bdac} + R_{cbad}.$$

$$= R_{dcab} + \cancel{\underline{R_{cadb}}} + \cancel{\underline{R_{acdb}}} + R_{bdac} + R_{cbad}$$

~~$= R_{dcab} - R_{bd}$~~

$$= -R_{dcba} - R_{bdca} - R_{cbda}$$

$$= 0.$$

Riemann Curvature Tensor

$R_{abc}{}^{d}$ is defined as the tensorial action of $\nabla_a \nabla_b - \nabla_b \nabla_a$ on w_c.

$$[\nabla_a \nabla_b - \nabla_b \nabla_a]\, w_c = R_{abc}{}^{d}\, w_d .$$

The action of $R_{abc}{}^{d}$ on $T \in \mathcal{T}(K,\ell)$ is given by the formula

$$[\nabla_a \nabla_b - \nabla_b \nabla_a]\, T^{c_1 \cdots c_K}{}_{d_1 \cdots d_\ell} = - \sum_{i=1}^{K} R_{abe}{}^{c_i}\, T^{c_1 \cdots \overset{i}{e} \cdots c_K}{}_{d_1 \cdots d_\ell}$$

$$+ \sum_{j=1}^{\ell} R_{abd_j}{}^{e}\, T^{c_1 \cdots c_K}{}_{d_1 \cdots e \cdots d_\ell} .$$

Algebraic symmetries of $R_{abc}{}^{d}$

① $R_{abc}{}^{d} = - R_{bac}{}^{d}$

② $R_{[abc]}{}^{d} = 0$

③ $R_{abcd} = - R_{abdc}$

$\Longrightarrow$ ④ $R_{abcd} = R_{cdab}$.

A counting argument in S. Carrol's notes shows that the number of independent components in $R_{abc}{}^{d}$ for an n-dimensional manifold is

$$\frac{1}{2}\left[\frac{1}{2}(n)(n-1)\right]\left[\frac{1}{2}n(n-1) + 1\right] - \frac{1}{4!}(n)(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1).$$

For $n=4$, $\frac{1}{12}\, 16(15) = 4 \cdot 5 = 20.$

A differential identity for $R_{abc}{}^{d}$

We calculate

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla_c w_d = R_{abc}{}^{e} \nabla_e w_d + R_{abd}{}^{f} \nabla_c w_f .$$

We also calculate

$$\nabla_a (\nabla_b \nabla_c - \nabla_c \nabla_b) w_d = \nabla_a R_{bcd}{}^{e} w_e = w_e \nabla_a R_{bcd}{}^{e} + R_{bcd}{}^{e} \nabla_a w_e .$$

$[[\nabla_a, \nabla_b], \nabla_c]_{\text{antisym}(a,b,c)}$

If we antisymmetrize left-sides in a, b, c we get same thing.

$$\left[\nabla_a \nabla_b \nabla_c - \nabla_b \nabla_a \nabla_c \right]_{\text{antisym}} = \left[\nabla_a \nabla_b \nabla_c - \nabla_a \nabla_c \nabla_b \right]_{\text{antisym}}.$$

$\Longrightarrow$ right-sides are equal upon antisymmetrizing over a, b, c.

$$\overset{\textcircled{2} \nearrow 0}{R_{[abc]}{}^{d}} \nabla_e w_d + \overset{\bigstar}{R_{[ab|d|}{}^{f} \nabla_{c]} w_f} = w_e \nabla_{[a} R_{bc]d}{}^{e} + \overset{\bigstar}{R_{[bc|d|}{}^{e} \nabla_{a]} w_e}$$

vertical bars means d/n antisymmetrize over d.

$$\Longrightarrow \quad w_e \nabla_{[a} R_{bc]d}{}^{e} = 0$$

Bianchi Identity

$$\boxed{\nabla_{[a} R_{bc]d}{}^{e} = 0.}$$

It is useful to decompose $R_{abc}{}^{d}$ into a "trace part" and "trace free" part.

Since $R_{abc}{}^{d} = -R_{bac}{}^{d}$, $R_{aac}{}^{d} = 0$.

similarly, $R_{abcd} = -R_{abdc} \Rightarrow R_{abcc} = 0$.

If we trace over 2 and 4th (equivalent to 3 + 4th slots)

~~R_{abad}~~

$$R_{abc}{}^{b} := R_{ac}.$$ Ricci Tensor

Since

$$R_{abcd} = R_{cdab},$$

the Ricci tensor is symmetric

$$R_{ac} = R_{ca}.$$

The scalar curvature, R, is the trace of the Ricci tensor

$$R = R_{a}{}^{a} = g^{ac} R_{ac}.$$

Next, we contract the Bianchi identity.

+ abc − bac + cab
− acb + bca − cba

$$\nabla_{[a} R_{bc]d}{}^{e} = 0.$$

$$\nabla_a R_{\underline{b}cd}{}^{e} - \nabla_a R_{\underline{cb}d}{}^{e} + \nabla_b R_{\underset{\cdot\cdot}{ca}d}{}^{e} - \nabla_b R_{\underset{\cdot\cdot}{ac}d}{}^{e} + \nabla_c R_{abd}{}^{e} - \nabla_c R_{bad}{}^{e} = 0$$

$$2\nabla_a R_{bcd}{}^{e} + 2\nabla_b R_{cad}{}^{e} + 2\nabla_c R_{\underset{\curvearrowright}{ab}d}{}^{e} = 0$$

$e = a$ contraction

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0.$$

Now, raise the index d using the metric and then contract on b, d indices:

$$(*) \quad \underbrace{\nabla_a R_c{}^{a}}_{\text{same}} + \underbrace{\nabla_b R_c{}^{b}}_{\text{same}} - \nabla_c R = 0$$

twice contracted Bianchi identity

$$\nabla_a R_c{}^{a} - \tfrac{1}{2}\nabla_c R = 0.$$

If we define

$$G_{ab} = R_{ab} - \tfrac{1}{2} R\, g_{ab},$$

Einstein Tensor

$(*)$ may be rewritten as

$$\nabla^a G_{ab} = 0.$$

Let's see this

$$g^{ea}\left(\nabla_a R_c{}^a - \frac{1}{2}\nabla_c R\right) g_{ae} = 0.$$

$$\nabla^e R_{ce} - \frac{1}{2}\nabla_c R = 0$$

$$\nabla^e R_{ce} - \frac{1}{2} g_{ce}\nabla^e R = 0$$

$$\nabla^e \underbrace{\left(R_{ce} - \frac{1}{2} g_{ce} R\right)}_{G_{ce}} + \frac{1}{2} R\,\nabla^e g_{ce} = 0$$

locally flattening process
"Riemann Normal Coordinates"
allows us to ignore this term.

The point of this calculation is that the Einstein Tensor is a "tensorial extraction" of data from the Riemann curvature tensor which satisfies a conservation law

$$\nabla^a G_{ab} = 0,$$ as a conseq. of Bianchi identity.

Einstein's field equations are $G_{ab} = c\,T_{ab}$ where T_{ab} is the stress-energy-momentum tensor of a matter field. Bianchi's identity implies that

$$\nabla^a T_{ab} = 0$$

which translates into a conservation of energy/momentum/mass.

Example Curvature Calculations

2-sphere of radius a

$$ds^2 = a^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Natural coordinates θ, ϕ. $\rightsquigarrow$ $\left\{ \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} \right\}$ and $\{ d\theta, d\phi \}$.

Metric is diagonal in natural dual basis.

$$g. = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix} = g_{ab} .$$

$$g^{ab} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2 \sin^2\theta} \end{pmatrix} .$$

The only nonzero 1st derivative of g_{ab} is $\partial_\theta g_{\phi\phi}$.

$$\partial_\theta g_{\phi\phi} = 2 a^2 \sin\theta \cos\theta .$$

Metric compatible Christoffel coefficients

$$\boxed{\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \}.}$$

$$\Gamma^\theta{}_{\theta\theta} = \frac{1}{2} g^{\theta\mu} \{ \partial_\theta g_{\theta\mu} + \partial_\theta g_{\theta\mu} - \partial_\mu g_{\theta\theta} \} = 0 .$$

$$\Gamma^\theta{}_{\theta\phi} = \frac{1}{2} g^{\theta\mu} \{ \partial_\theta g_{\phi\mu} + \overset{\nearrow 0}{\partial_\phi g_{\theta\mu}} - \overset{\nearrow 0}{\partial_\mu g_{\theta\phi}} \}$$

$$= \frac{1}{2} g^{\theta\phi} \{ \partial_\theta g_{\phi\phi} \} = \frac{1}{2} 0 .$$

$$\Gamma^\theta{}_{\phi\phi} = \frac{1}{2} g^{\theta\mu} \{ \overset{\nearrow 0}{\partial_\phi g_{\phi\mu}} + \overset{\nearrow 0}{\partial_\phi g_{\phi\mu}} - \partial_\mu g_{\phi\phi} \}$$

$$= \frac{1}{2} g^{\theta\theta} (- \partial_\theta g_{\phi\phi}) = -\frac{1}{2} \frac{1}{a^2} 2 a^2 \sin\theta \cos\theta$$

$$= - \sin\theta \cos\theta .$$

$$\Gamma^{\phi}{}_{\theta\theta} = \tfrac{1}{2} g^{\phi\mu}\{\partial_\theta g_{\theta\mu} + \partial_\theta g_{\theta\mu} - \partial_\mu g_{\theta\theta}\} = 0.$$

$$\Gamma^{\phi}{}_{\theta\phi} = \tfrac{1}{2} g^{\phi\mu}\{\partial_\theta g_{\phi\underset{\phi}{\mu}} + \partial_\phi \overset{\nearrow 0}{g_{\theta\mu}} - \partial_\mu \overset{\nearrow 0}{g_{\theta\phi}}\}$$

$$= \tfrac{1}{2} g^{\phi\phi}\,\partial_\theta g_{\phi\phi} = \cancel{\tfrac{1}{2}}\,\frac{1}{\cancel{a^2}\sin^2\theta}\,\cancel{2}\cancel{a^2}\sin\theta\cos\theta = \cot\theta.$$

$$\Gamma^{\phi}{}_{\phi\phi} = 0$$

Nonzero Christoffel symbols for 2 sphere:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.$$

Connection determines Riemann curvature tensor

$$\boxed{R_{abc}{}^{d} = \left(\partial_a \Gamma^{d}{}_{bc} - \partial_b \Gamma^{d}{}_{ac} + \Gamma^{d}{}_{ae}\Gamma^{e}{}_{bc} - \Gamma^{d}{}_{be}\Gamma^{e}{}_{ac}\right)}$$

Let's compute one of the coefficients:

$$R_{\theta\phi\phi}{}^{\theta} = \partial_\theta \Gamma^{\theta}{}_{\phi\phi} - \partial_\phi \overset{\nearrow 0}{\Gamma^{\theta}{}_{\theta\phi}} + \overset{\nearrow 0}{\Gamma^{\theta}{}_{\theta\lambda}}\Gamma^{\lambda}{}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}{}_{\theta\phi})$$

$$= \partial_\theta(-\sin\theta\cos\theta) - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} = \cancel{\cot\theta(\sin\theta\cos\theta)} =$$

$$= -\partial_\theta(\sin\theta\cos\theta) - (-\sin\theta\cos\theta)(\cot\theta)$$

$$= -\cos^2\theta + \sin^2\theta + \cancel{\sin\theta}\cos\theta\,\frac{\cos\theta}{\cancel{\sin\theta}}$$

$$= \sin^2\theta.$$

$$R_{\theta\phi\phi\theta} = R_{\theta\phi\phi}{}^{\lambda} g_{\lambda\theta} = R_{\theta\phi\phi}{}^{\theta} g_{\theta\theta} = a^2\sin^2\theta.$$

Note: # of independent coeff. in $R_{abc}{}^{d}$ on n-dim^l mfld is $\frac{1}{12}n^2(n^2-1)$. $n=2 \Rightarrow 1$. so done.

Ricci Tensor

$$R_{abc}{}^{b} = R_{ac} \qquad \text{symmetric.}$$

$$R_{\theta\theta} = R_{\theta\alpha\theta}{}^{\alpha} = R_{\theta\alpha\theta\beta}\, g^{\beta\alpha} = \overset{\nearrow 0}{R_{\theta\theta\theta\theta}}\, g^{\theta\theta} + R_{\theta\phi\theta\phi}\, g^{\phi\phi}$$

$$= \left(-a^2 \sin^2\theta\right)\left(\frac{1}{a^2 \sin^2\theta}\right) = -1 \quad ?$$

(lost sign somewhere ?)

$$R_{\phi\theta} = R_{\theta\phi} = R_{\theta\alpha\phi\beta}\, g^{\beta\alpha} = R_{\underbrace{\theta\theta}\phi\theta}\, g^{\theta\theta} + R_{\theta\phi\underbrace{\phi\phi}}\, g^{\phi\phi} = 0$$

$$R_{\phi\phi} = R_{\phi\alpha\phi}{}^{\alpha} = R_{\phi\alpha\phi\beta}\, g^{\beta\alpha} = R_{\phi\theta\phi\theta}\, g^{\theta\theta} + \overset{\nearrow 0}{R_{\phi\phi\phi\phi}}\, g^{\phi\phi}$$

$$= -a^2 \sin^2\theta \left(\frac{1}{a^2}\right) = -\sin^2\theta .$$

(lost sign ?)

Ricci Scalar

$$R_a{}^a = R_\theta{}^\theta + R_\phi{}^\phi = R_{\theta\lambda}\, g^{\lambda\theta} + R_{\phi\lambda}\, g^{\lambda\phi}$$

$$= R_{\theta\theta}\, g^{\theta\theta} + R_{\phi\phi}\, g^{\phi\phi}$$

$$= (-1)\left(\frac{1}{a^2}\right) + \left(-\sin^2\theta\right)\left(\frac{1}{a^2 \sin^2\theta}\right)$$

$$= -\frac{2}{a^2}$$

(same lost sign)

Find geodesics on S^2.

D. Geodesics

A *geodesic* is a curve whose tangent vector is parallel propagated along itself. That is, a curve whose tangent T^a satisfies

geodesic equation

$$T^a \nabla_a T^b = 0 .$$

Remark The geodesic equations is actually a specialization of the definition. Consider the weaker requirement

$$T^a \nabla_a T^b = \alpha T^b$$

where α is an arbitrary function of the curve. Then, the parallel property persists but the length may oscillate. This oscillation can be removed by reparametrizing to ensure the geodesic eq. holds. A parametrization in which the geodesic eq. holds is said to be an *affine parametrization.*

We unravel the geodesic equation by introducing a chart ψ. ψ maps the geodesic curve into $x^\mu(t) \in \mathbb{R}^n$. The components T^μ of the tangent vector T^a in this chart satisfy

$$\frac{dT^\mu}{dt} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} T^\sigma T^\nu = 0 .$$

But

$$T^\mu = \frac{dx^\mu}{dt}$$

so the geodesic eq. becomes

coupled system of n 2nd order ODEs in n functions $x^\mu(t)$.

$$\frac{d^2x^\mu}{dt^2} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} \frac{dx^\sigma}{dt} \frac{dx^\nu}{dt} = 0 .$$

Given an initial point $x^\mu(t_0)$ and initial velocity $\frac{dx^\mu}{dt}(t_0)$, this ODE has a unique solution. Translating this, we have:

Given $p \in M$ and any tangent vector $T^a \in V_p$, $\exists!$ geodesic passing through p with tangent T^a.

This structure lets us construct a useful (local) coordinate system.

Let $p \in M$. We define the exponential map from tangent space V_p to M by mapping

$$\begin{tikzcd} T^a \arrow[r] \arrow[d, phantom, sloped, "\in"] & r \arrow[d, phantom, sloped, "\in"] \\ V_p & M \end{tikzcd}$$

by setting r to be the point at unit affine parameter from p in the direction of T^a. One might encounter a singularity before unit affine parameter if Va is too large. Also geodesics may cross so the map may not be one-one. It can be shown that $\exists$ sufficiently small Nhd of origin in V_p s.t. exp. map is defined + one-one. Since V_p is an n-dim^l vector space, it may be identified w. $\mathbb{R}^n$. This leads us to Riemannian Normal coordinates at p.

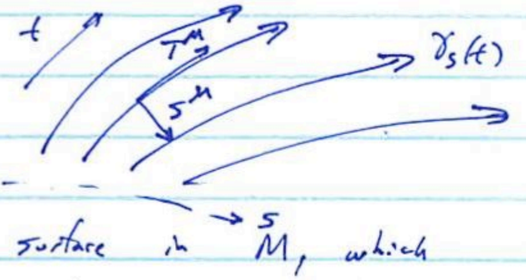
All geodesics get mapped under RNC into straight lines through origin of $\mathbb{R}^n$. The equation

$$\frac{d^2x^\mu}{dt^2} + \sum \Gamma^\mu{}_{\sigma\nu} \frac{dx^\sigma}{dt} \frac{dx^\nu}{dt} = 0$$

in these coordinates forces $\Gamma^\mu{}_{\sigma\nu}\big|_p = 0$.

Suppose we have a 1 parameter family of affine parametrized geodesics.

$$\gamma_{\underset{\text{geodesic index}}{s}}(\underset{\text{affine parameter along geodesic}}{t})$$



This 2-parameter family sweeps out a 2 d surface in M, which we may parametrize by s, t provided the geodesics d/n cross. We denote the surface by $x^\mu(s,t)$ with $x^\mu \in M$.

$\exists$ 2 natural vector fields on this 2d surface:

tangent vectors to geodesics $T^\mu = \dfrac{\partial x^\mu}{\partial t}$ "points along geodesics in family"

deviation vectors $S^\mu = \dfrac{\partial x^\mu}{\partial s}$. "points across geodesics in family"

These notions motivate us to introduce "relative velocity of the geodesics"

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$$

and the "relative acceleration of the geodesics"

$$a^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu.$$

Since S, T are basis vectors adapted to a coord. system, $[S,T]=0$. By a remark from way back (see III. 6), the commutator of 2 vector fields can be expressed using the covariant derivative operator

$$[v,w]^b = [v^a \nabla_a w^b - w^a \nabla_a v^b]$$

$$\Longrightarrow \qquad S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$$ ⓓ

Let's compute the accelaration.

$$a^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$

$$\overset{\textcircled{\#}}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= \underset{\textcircled{\#}}{T^\rho (\nabla_\rho S^\sigma)}(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu - R_{\sigma\rho\nu}{}^\mu T^\nu)$$

$$= \cancel{(S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu)} + S^\sigma \nabla_\sigma (\overbrace{T^\rho \nabla_\rho T^\mu}^{\text{geodesic} \to 0}) - S^\sigma \overset{\sigma \leftrightarrow \rho}{\cancel{(\nabla_\sigma T^\rho)(\nabla_\rho T^\mu)}}$$

$$- R_{\sigma\rho\nu}{}^\mu T^\rho S^\sigma T^\nu$$

$$= - R_{\sigma\rho\nu}{}^\mu T^\rho S^\sigma T^\nu.$$

punchline

$$\boxed{a^\mu = \frac{d^2}{dt^2} S^\mu = R_{\rho\sigma\nu}{}^\mu T^\rho S^\sigma T^\nu.}$$

geodesic deviation equation

IV. - General Relativity

Cartoon

special relativistic scalar field.

A. Structural Assumptions / Fundamental Objects.

- spacetime
- matter fields
- postulates
- volume element
- action principle

B. Lagrangian derivation of Einstein-Matter fields.

- Matter Field equations arise from matter field variations.
- Einstein Field equations arise from metric variations.
- Vanishing of boundary terms

C. Example Matter Fields.

- Vaccuum Einstein.
- Einstein-Scalar Field.

should also do Noether at top + infer various conservation properties within the examples.

Remark: The examples (e.g. Scalar Field, EM Field, etc.) should be verified to satisfy the matter field postulates. Gauge theory stuff at end should include more "choice of gauge" discussion.

IV General Relativity

Classical field theory on fixed background metric

A. Structural Assumptions

B. Matter fields

C. Einstein's Field equations

Lagrangian Formulation — Matter fields couple to metric.

D. Linearized gravity: Newtonian limit, gravitational radiation

pins down constants.

Remark The historical derivation of Einstein's Field eqs. emerges from the equivalence principle, aesthetic covariance properties and Gedanken experiments. We will not follow the historical derivation.

Cartoon $\mathcal{D} \subset M$.

No matter fields

$$\mathcal{L}_G(g_{ab}) = R.$$

depends upon metric

$$\mathcal{L}_m(\Psi, \nabla\Psi, \nabla\nabla\Psi, \ldots, g_{ab})$$ Matter field Lagrangian

Matter-field + gravity lagrangian

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_m.$$

Matter + Gravity Action

$$\frac{\partial \mathcal{L}_m}{\partial (g_{ab})} = T_{ab}$$

stress energy tensor

$$I = \int_{\mathcal{D}} \mathcal{L} \, dv.$$

$$\left.\begin{aligned} \frac{\delta I}{\delta(g_{ab})} &= 0 \iff \text{Einstein's eqs.} \\ \frac{\delta I}{\delta(\Psi)} &= 0 \iff \text{Matter field eqs.} \end{aligned}\right\} \text{Coupled}$$

(Special) relativistic scalar field

IV G.R.

$(\mathbb{R}^{1+3}, \eta)$ Minkowski spacetime.

global coordinate chart (t, x_1, x_2, x_3) write $\bar{x} = (t, x_1, x_2, x_3)$.

$$\eta_{ab} \sim \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

$\psi : \mathbb{R}^{1+3} \longrightarrow \mathbb{R}$ scalar field.

$$\mathcal{L}_{SF}(\psi) = -\frac{1}{2}\eta^{ab}\,\partial_a\psi\,\partial_b\psi + V(\psi).$$

Form the action on a spacetime domain $\mathcal{D} \subset \mathbb{R}^{1+3}$. Assume $\mathcal{D}$ bdd.

$$S_{SF}[\psi;\mathcal{D}] = \int_{\mathcal{D}} \mathcal{L}_{SF}(\psi)\, d\bar{x}.$$

Let φ be a compactly supported scalar field with support $\subset \mathcal{D}$.

Form the difference quotient

$$\frac{S_{SF}[\psi+\varepsilon\varphi] - S_{SF}[\psi]}{\varepsilon} = \frac{1}{\varepsilon}\int_{\mathcal{D}}\Big\{-\frac{1}{2}\eta^{ab}\partial_a(\psi+\varepsilon\varphi)\,\partial_b(\psi+\varepsilon\varphi) + V(\psi+\varepsilon\varphi)$$

$$+ \frac{1}{2}\eta^{ab}\partial_a(\psi)\,\partial_b(\psi) - V(\psi)\Big\}\, d\bar{x}$$

$$= \int_{\mathcal{D}} -\frac{1}{2}\eta^{ab}(\partial_a\varphi\,\partial_b\psi + \partial_a\psi\,\partial_b\varphi) + \frac{V(\psi+\varepsilon\varphi) - V(\psi)}{\varepsilon}\, d\bar{x}$$

$$+ O(\varepsilon).$$

Take $\lim_{\varepsilon\to 0}$ and demand that

$S_{SF}[\psi]$ be stationary w.r.t. perturbations of ψ.

$$0 = \int_{\mathcal{D}} \left[- \eta^{ab}\, \partial_a \psi\, \underset{\swarrow}{\partial_b} \varphi + V'(\psi)\, \varphi \right] dx$$

$$0 = \int_{\mathcal{D}} + \left[\partial_b \left(\eta^{ab}\, \partial_a \psi \right) + V'(\psi) \right] \varphi \; dx .$$

This is to be true $\forall\, \mathcal{D}$ and $\forall\, \varphi$ so we must have

$$\partial_b \left(\eta^{ab}\, \partial_a \psi \right) + V'(\psi) = 0 .$$

Since η^{ab} is independent of spacetime coordinates this reads

$$\boxed{\left(- \partial_t^2 + \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \right) \psi + V'(\psi) = 0}$$

- Spacetime
- Matter Fields
- Lagrangian Density/Action

→ Lagrangian Field Theory.

A. Structural Assumptions + Fundamental objects

spacetime

order of differentiability ?
- C^2, H^{4-}
- (perhaps) not physically relevant

spacetime is an $n+1$ dimensional manifold M with Lorentz metric g.
(M, g).

$n=3$ in physical case

g is a Lorentz metric on M means g is a symmetric nondegenerate $(0,2)$-tensor with signature $(-1, 1, \ldots, 1)$.

points in M are denoted x^α, $\alpha = 0, 1, \ldots, n$.

We will not distinguish between isometric manifolds in this theory.
$\Rightarrow$ $\exists$ intrinsic nonuniqueness for $(M, g) \sim (M', g')$
if $\exists$ diffeo $\theta : M \longrightarrow M'$ s.t. $\theta_* g = g'$.
This nonuniqueness demands special attention in the i.v.p. formulation of GR.

The Lorentz metric g naturally partitions $V_p \setminus \{0\}$ into 3 classes:

X timelike if $g(X, X) < 0$

X spacelike if $g(X, X) > 0$

X null if $g(X, X) = 0$.

A C^r pair (M', g') is a C^r-extension of (M, g) if $\exists$ isometric embedding $\mu : M \longrightarrow M'$. We may then view (M', g') as an extension of (M, g).

We hope to understand spacetime so well that we can show it is inextendible. (PDE, geometry issues emerge ...)

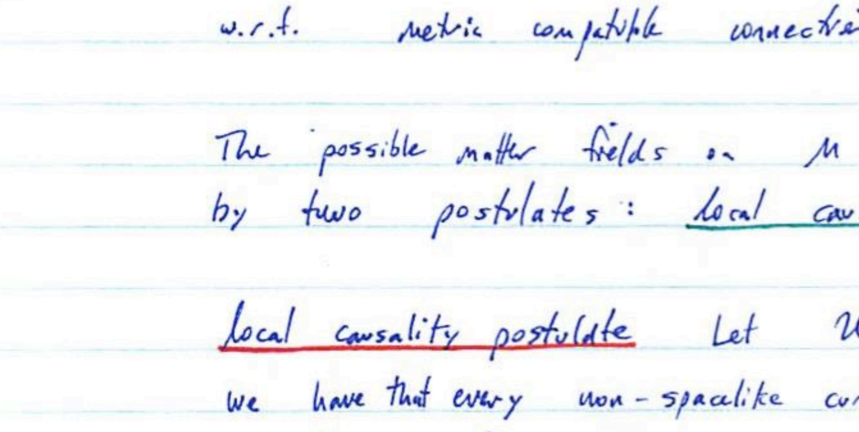
Matter Fields   more simply **Fields.**

denoted ψ which is shorthand for scalars, tensors, spinors, or other geometric objects defined on $\mathcal{M}$. We will restrict to those fields expressible with covariant tensors on $\mathcal{M}$.

Fields describe the matter content of spacetime. The fields obey equations expressed as relations between tensors on $\mathcal{M}$, in which all derivatives w.r.t. position are covariant w.r.t. metric compatible connection.

The possible matter fields on $\mathcal{M}$ will be further restricted by two postulates: local causality, local energy conservation.

local causality postulate Let $\mathcal{U} \subset \mathcal{M}$ be such that $\forall p \in \mathcal{U}$ we have that every non-spacelike curve passing through p intersects spacelike surface $x^0 = t = 0$ within $\mathcal{U}$. Let $\mathcal{F}$ be set of points in $x^0 = t = 0$ which can be reached by nonspacelike curves in $\mathcal{U}$ from p.



"domain of dependence"

We require that matter fields at p must be uniquely determined by the values of the fields and their derivatives up to some finite order on $\mathcal{F}$, and that the fields are not uniquely determined by those values on any proper subset of $\mathcal{F}$.

This postulate distinguishes g from other tensor fields on $\mathcal{M}$.

local energy conservation postulate

The eqs. governing the matter fields are such that $\exists$ symmetric tensor T^{ab} called the energy-momentum tensor which depends upon the fields, their covariant derivatives and the metric such that

energy positivity (i) T^{ab} vanishes on open $\mathcal{U}$ $\iff$ all matter fields vanish in $\mathcal{U}$.

local energy conservation (ii) T^{ab} satisfies $\nabla_b T^{ab} = T^{ab}{}_{;b} = 0$.

Remarks

- (i) precludes that $\exists$ distinct fields which cancel each other in some region. This may be the basis for an objection to (i).

- (ii) has extra impact if $\exists$ killing vector field K^a. With such a special v.f., we can form

$$P^b = T^{ab} K_a .$$

and (ii) + killing-property $\Longrightarrow$ $\int_{\mathcal{D} \subset M} \nabla_b P^b \, dv = 0$.

- The fields are often described using a Lagrangian and with that extra structure $\exists$ an algorithm for getting T^{ab}.

We will ultimately wish to include all known fields and develop extensions of the theory to incorporate (and predict the existence of) heretofore unobserved physical fields.

Lagrangian Density

$$\mathfrak{T}(\underline{k}, \underline{l})$$

Suppose $\psi \in \mathfrak{T}(k_1, l_1) \otimes \mathfrak{T}(k_2, l_2) \otimes \cdots \otimes \mathfrak{T}(k_n, l_n)$

e.g. $\psi = \left(\psi_{(1)}{}^{a_1^1 \cdots a_{k_1}^1}{}_{b_1^1 \cdots b_{l_1}^1}, \; \cdots\cdots , \; \psi_{(n)}{}^{a_1^{(n)} \cdots a_{k_n}^{(n)}}{}_{b_1^{(n)} \cdots b_{l_n}^{(n)}} \right).$

$$\mathfrak{T}(\underline{k}, \underline{l}+1)$$

$$\nabla\psi \in \mathfrak{T}(k_1, l_1+1) \otimes \cdots \otimes \mathfrak{T}(k_n, l_n+1).$$

We restrict attention to matter fields on M which are described with a scalar valued function

How ?

As discussed below.

$$\underset{\psi}{\mathcal{L} : \mathfrak{T}(\underline{k}, \underline{l})} \otimes \underset{\nabla\psi}{\mathfrak{T}(\underline{k}, \underline{l}+1)} \otimes \underset{\nabla\nabla\psi}{\mathfrak{T}(\underline{k}, \underline{l}+2)} \otimes \underset{g}{\mathfrak{T}(0,2)} \longrightarrow \mathbb{R}$$

which we call the Lagrangian density. (It will be required to tensorially transform in a while) development

requires by us.

The corresponding action is the integral

$$S = S[\psi, g ; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L} \; dV_g$$

where $\mathcal{U} \subset M$ is any relatively compact set of M and dV_g denotes the volume element on M generated by the metric g.

The action principle states that an acceptable solution of a physical system must be a stationary point of the action. w.r.t. variations of the solution.

explain

explain

Two Calculations

① $\{dx^\alpha\}$ 1-form basis on M, in a chart.

Form n-form

$\varepsilon = n!\ dx^1 \wedge \dots \wedge dx^n$.

change variables : $x'^\beta = x'^\beta(x^\alpha)$.

$$\varepsilon = n! \left(\frac{\partial x^1}{\partial x'^{\beta_1}}\, dx'^{\beta_1} \right) \wedge \cdots \wedge \left(\frac{\partial x^n}{\partial x'^{\beta_n}}\, dx'^{\beta_n} \right)$$

$$= n!\ \det\left(\frac{\partial x^\alpha}{\partial x'^\beta} \right) \quad dx'^1 \wedge \cdots \wedge dx'^n$$

$$= \det\left(\frac{\partial x^\alpha}{\partial x'^\beta} \right)\ \varepsilon'.$$

Thus, ε is nonunique. It may as well be

$\varepsilon = f\ dx^1 \wedge \cdots \wedge dx^n$ for some function f.

② Suppose M has metric g.

$$g_{ab}\, dx^a dx^b = g'_{cd}\, dx'^c dx'^d$$

where $$g'_{cd} = g_{ab} \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}.$$

$$\det(g'_{cd}) = \det(g_{ab}) \left[\det\left(\frac{\partial x^a}{\partial x'^c} \right) \right]^2.$$

Thus,

$$\left| \det(g'_{cd}) \right|^{\frac{1}{2}} = \left| \det\left(\frac{\partial x^a}{\partial x'^c} \right) \right| \ \left| \det(g_{ab}) \right|^{\frac{1}{2}}.$$

General Matter field Coupled to gravitational field

We use the shorthand notation ψ to denote matter fields of the form $\{ \psi_{(i)}{}^{a \cdots b}{}_{c \cdots d} \}_{i \in \mathcal{I}}$.

Suppose the matter fields are described by some Lagrangian

$$\mathcal{L}_M = \mathcal{L}_M(\psi, \nabla_e \psi)$$

(This is general except we have assumed $\mathcal{L}$ depends only upon 1st covariant derivatives. Since ψ_i for certain i could denote $\nabla_e \psi_j$ this is no real loss of generality.)

Consider the gravitational lagrangian defined as

Hilbert-Einstein Action

$$\mathcal{L}_G = \mathcal{L}_G(g) = R.$$

We couple the matter fields to gravity with coupling constant α by writing the coupled Lagrangian

$$\mathcal{L} = \mathcal{L}(\psi, g) = \mathcal{L}_M(\psi, \underset{\overset{\uparrow}{\text{g is in here.}}}{\nabla_e \psi}) + \alpha \mathcal{L}_G \underset{\overset{\uparrow}{\psi \text{ is } \underline{\text{not}} \text{ in here.}}}{(g)}.$$

The dynamical behavior of the fields ψ and spacetime metric g are encoded in equations derived from the Action principle.

The physical evolution of ψ and g is action stationary.

Thus, we wish to calculate, using the action

$$S = S[\psi, g; \mathcal{D}] = \int_{\mathcal{D}} \mathcal{L} \, dV_g$$

the variations

$$\frac{\delta S}{\delta(g)} = \cdots \longrightarrow \text{einstein's equation.}$$

$$\frac{\delta S}{\delta(\psi)} = \cdots \longrightarrow \text{matter field equations}$$

and stationarity requires $\frac{\delta S}{\delta(g)} = 0$, $\frac{\delta S}{\delta(\psi)} = 0$ giving us equations.

The matter field equations

Example (General Matter Field on a fixed background metric g)

Let ψ denote $\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d}$ where i indexes various tensor fields defined on the spacetime. We may replace ψ by $\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d}$ below to obtain a more complete description of the matter field properties.

Form the Lagrangian associated to the matter field(s) ψ:

$$\mathcal{L} = \mathcal{L}(\psi, \nabla_e \psi, x)$$

Note: This is "general" except it only depends upon 1st covariant derivatives of the tensor fields. This is not a real loss of power though ....

With $\mathcal{L}$ we form the associated action on a compact $\mathcal{D} \subset \mathcal{M}$.

$$S = S[\psi, g; \mathcal{D}] = \int_{\mathcal{D}} \mathcal{L} \, dV_g \, .$$

The action principle produces the matter field equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \nabla_e \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} = 0.$$

We carry out the calculation.

$$\lim_{\varepsilon \to 0} \frac{S[\psi+\varepsilon\varphi, g; \mathcal{D}] - S[\psi, g; \mathcal{D}]}{\varepsilon} = 0 .$$

We study the variation in the action due to a variation in the matter field.

Thus,

$$\lim_{\varepsilon \to 0} \int_{\mathcal{D}} \frac{\mathcal{L}(\psi+\varepsilon\varphi, \nabla_e(\psi+\varepsilon\varphi), x) - \mathcal{L}(\psi, \nabla_e\psi, x)}{\varepsilon} \, dV_g = 0$$

$$\int_{\mathcal{D}} \left\{ \frac{\partial \mathcal{L}}{\partial \psi} \varphi + \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \underset{\uparrow}{\nabla_e \varphi} \right\} dV_g = 0 .$$

expose the φ

$$\nabla_e \left(\frac{\partial \mathcal{L}}{\partial(\nabla_e\psi)} \varphi \right) = \left(\nabla_e \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \right) \varphi + \frac{\partial \mathcal{L}}{\partial(\nabla_e\psi)} \nabla_e \varphi$$

Leibniz/product rule.

So,

$$\int_{\mathcal{D}} \left[\frac{\partial \mathcal{L}}{\partial \psi} \varphi - \nabla_e \frac{\partial \mathcal{L}}{\partial(\nabla_e\psi)} \varphi \right] dV_g + \underbrace{\int_{\mathcal{D}} \nabla_e \left(\frac{\partial \mathcal{L}}{\partial(\nabla_e\psi)} \varphi \right) dV_g}_{\parallel} = 0 .$$

|| Stokes Theorem

$$\int_{\partial\mathcal{D}} \frac{\partial \mathcal{L}}{\partial(\nabla_e\psi)} \varphi \, d\sigma$$

$$\parallel$$

$$0$$

✓

A Matrix lemma

Let $M(t)$ be an $n \times n$ invertible matrix. Then

$$\frac{d}{dt}\left(\det M(t)\right)\Big|_{t=0} = \det M(0) \; tr\left(M^{-1}(0)\,\dot{M}(0)\right).$$

proof:

$$M^{-1}(0)\,M(t) = I + M^{-1}(0)\,\dot{M}(0)\,t + O(t^2)$$

$$\det\left(M^{-1}(0)\,M(t)\right) = \det\left(\qquad\qquad\right).$$

$$\det\left(M^{-1}(0)\right)\frac{d}{dt}M(t)\Big|_{t=0} = tr\left(M^{-1}(0)\,\dot{M}(0)\right)$$; as claimed.

Application:

$-g = -\det g_{ab}$. $\qquad g_{ab}(s) = g_{ab} + \dot{g}_{ab}\,s$.

$$\frac{d}{ds}(-g)^{\frac{1}{2}} = \frac{1}{2}(-g)^{-\frac{1}{2}}\left(-\frac{d}{ds}g\right) = \frac{1}{2}(-g)^{-1/2}(-g)\,g^{ab}\,\dot{g}_{ab}$$

$$= \frac{1}{2}(-g)^{1/2}\,g^{ab}\,\dot{g}_{ab}.$$

We apply the lemma...

$$g^{\beta\nu}\dot{R}_{\beta\nu} = g^{\beta\nu}\partial_\nu \dot{\Gamma}^\alpha{}_{\beta\alpha} - g^{\beta\nu}\partial_\alpha \dot{\Gamma}^\alpha{}_{\beta\nu}$$

rename $\nu \to \delta$ rename $\alpha \to \delta$

$$= g^{\beta\delta}\partial_\delta \dot{\Gamma}^\alpha{}_{\beta\alpha} - g^{\beta\nu}\partial_\delta \dot{\Gamma}^\delta{}_{\beta\nu}$$

metric compatibility

$$= \partial_\delta \underbrace{\left(g^{\beta\delta}\dot{\Gamma}^\alpha{}_{\beta\alpha} - g^{\beta\nu}\dot{\Gamma}^\delta{}_{\beta\nu} \right)}_{V^\delta}$$

V^δ, vanishes near $\partial\mathcal{D}$.

$\Longrightarrow$

$$g^{\beta\nu}\dot{R}_{\beta\nu} = \nabla_\delta V^\delta .$$

$$\int_{\mathcal{D}} g^{\beta\nu}\dot{R}_{\beta\nu}\, dV_g \underset{\substack{\uparrow \\ \text{stokes}}}{=} \int_{\partial\mathcal{U}} n_\delta V^\delta = 0 .$$

Action variation with respect to spacetime metric

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_m \; ; \; \mathcal{L}_G = R \, , \; \mathcal{L}_m = \mathcal{L}_m(\psi, \nabla_c \psi)$$

$$S[\psi, g_{ab}; \mathcal{D}] = \int_{\mathcal{D}} R + \mathcal{L}_m \, dV_g = \int_{\varphi(\mathcal{D})} [R + \mathcal{L}_m] \sqrt{-g} \, dx^1 \dots dx^n .$$

$$\left\langle \frac{\delta S}{\delta(g_{ab})} , \dot{g}_{ab} \right\rangle = ? \qquad g_{ab}(s) = g_{ab} + s \dot{g}_{ab} .$$

$$\int [\dot{R} + \dot{\mathcal{L}}_m] \sqrt{-g} + \int [R + \mathcal{L}_m] \dot{\sqrt{-g}}$$

$$\dot{\sqrt{-g}} = \tfrac{1}{2} \sqrt{-g} \, g^{ab} \dot{g}_{ab}$$

$$R = R_{\mu\nu} g^{\mu\nu} \, , \quad \dot{R} = \dot{R}_{\mu\nu} g^{\mu\nu} + R_{\mu\nu} \dot{g}^{\mu\nu}$$

$$= \dot{R}_{\mu\nu} g^{\mu\nu} + R_{\mu\nu} \left(-g^{\alpha\mu} g^{\beta\nu} \dot{g}_{\alpha\beta} \right)$$

$$\dot{\mathcal{L}}_m = \frac{\partial \mathcal{L}_m}{\partial g_{ab}} \dot{g}_{ab}$$

$$\int_{\varphi(\mathcal{D})} \dot{R}_{\mu\nu} g^{\mu\nu} \sqrt{-g} + \left[R^{ab} + \tfrac{1}{2} g^{ab} R + \underbrace{\frac{\partial \mathcal{L}_m}{\partial g_{ab}} + \tfrac{1}{2} \mathcal{L}_m g^{ab}}_{T^{ab}} \right] \dot{g}_{ab} \sqrt{-g} \, dx^1 \dots dx^n$$

Stoke's ⇒ zero

(boundary term)

$$\boxed{R^{ab} - \tfrac{1}{2} g^{ab} R = T^{ab} .}$$

We show this vanishes next ...

Vanishing of Boundary term

Lemma Let $g_{\mu\nu}(s) = g_{\mu\nu} + s\,\dot{g}_{\mu\nu}$. $\quad \cdot = \frac{d}{ds}$.

⊛ $$\dot{R}_{\mu\nu} = \nabla_\alpha \dot{\Gamma}^\alpha{}_{\mu\nu} - \nabla_\mu \dot{\Gamma}^\alpha{}_{\alpha\nu}$$

where $$\dot{\Gamma}^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}\left(\nabla_\beta \dot{g}_{\gamma\lambda} + \nabla_\gamma \dot{g}_{\beta\lambda} - \nabla_\gamma \dot{g}_{\beta\gamma}\right).$$

Riemannian Normal Coordinates

proof: In special coordinates which are always available at $p \in M$, we have $\nabla_a = \partial_a$ so $\Gamma = 0$ at p. We then have at p.

$$R_{\mu\beta\nu}{}^\alpha = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \underbrace{\Gamma^\alpha{}_{\delta\mu}\Gamma^\delta{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu}}_{\to 0.}$$

$$-R_{\beta\mu\nu}{}^\alpha =$$

contract $\mu = \alpha$

$$-R_{\beta\nu} = \partial_\alpha \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\alpha}.$$

Thus,

$$-\dot{R}_{\beta\nu} = \partial_\alpha \dot{\Gamma}^\alpha{}_{\beta\nu} - \partial_\nu \dot{\Gamma}^\alpha{}_{\beta\alpha} \quad \text{as claimed.}$$

Since this is tensorial, we also have ⊛.

Example (Vacuum Einstein)

$\mathcal{L}_m = 0 \longrightarrow$ no matter field equation

$$R^{ab} - \frac{1}{2} g^{ab} R = 0$$

$$R^{ab} g_{bc} - \frac{1}{2} g^{ab} g_{bc} R = 0$$

$$R^{a}{}_{c} - \frac{1}{2} \delta^{a}{}_{c} R = 0 .$$

$a = c$

$$R - \frac{1}{2} \cdot 4 \, R = 0 \; ; \quad \boxed{R = 0}$$

Example (Einstein – Scalar Field)

$$\mathcal{L}_{SF} = -\frac{1}{2} g^{ab} \nabla_a \psi \nabla_b \psi + V(\psi).$$

Matter Field

$$\frac{\partial \mathcal{L}}{\partial \psi} - \nabla_e \frac{\partial \mathcal{L}}{\partial (\nabla_e \psi)} = 0.$$

$$V'(\psi) - \nabla_e \left(-\frac{2}{2} g^{eb} \nabla_b \psi \right) = 0$$

$$\boxed{\nabla_e \; g^{eb} \nabla_b \psi + V'(\psi) = 0}$$

gravity field

$$R^{ab} - \frac{1}{2} R g^{ab} = T^{ab}$$

where $$T^{ab} = \frac{\partial \mathcal{L}_{SF}}{\partial (g_{ab})} + \frac{1}{2} \mathcal{L}_{SF} g^{ab}.$$

$$\begin{aligned} \mathcal{L}_{SF} &= -\frac{1}{2} g^{ab} \nabla_a \psi \nabla_b \psi + V(\psi) \\ &= -\frac{1}{2} \partial^b \psi \, g_{cb} \, \partial^c \psi + V(\psi) \end{aligned}$$

$$\frac{\partial \mathcal{L}_{SF}}{\partial (g_{\mu\nu})} = -\frac{1}{2} \partial^\mu \psi \, \partial^\nu \psi .$$

Thus,

$$\boxed{T^{\mu\nu} = -\frac{1}{2} \partial^\mu \psi \, \partial^\nu \psi + \frac{1}{2} g^{\mu\nu} \left(-\frac{1}{2} \partial^b \psi \, \partial_b \psi + V(\psi) \right)}$$

Example (Einstein – Maxwell)

Let A_α be a one-form field on $\mathcal{M}$ which we give a fancy name: the gauge potential. The gauge potential spawns the two form field

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,, \qquad (F = dA)$$

which is called the electromagnetic field.

Remark: $A_\alpha \rightsquigarrow A_\alpha + \nabla_\alpha \chi$ for scalar χ spawns the same em field. This is called gauge freedom.

The electromagnetic field Lagrangian is

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

We may view $\mathcal{L}_{EM} = \mathcal{L}_{EM}(A_\alpha)$.

Matter Field $\quad A_\alpha(s) = A_\alpha + \dot{A}_\alpha s \longrightarrow \dot{F}_{\mu\nu} = \nabla_\mu \dot{A}_\nu - \nabla_\nu \dot{A}_\mu .$

$$0 = \left\langle \frac{\delta S}{\delta A^\alpha}, \dot{A}_\alpha \right\rangle = -\frac{1}{2}\int_{\mathcal{O}} \dot{F}_{\mu\nu} F^{\mu\nu}\, dV_g$$

$$= -\frac{1}{2}\int_{\mathcal{O}} (\nabla_\mu \dot{A}_\nu - \nabla_\nu \dot{A}_\mu) F^{\mu\nu}\, dV_g$$

$\mu \leftrightarrow \nu$ then $F^{\nu\mu} = -F^{\mu\nu}$

$$= -\int_{\mathcal{O}} (\nabla_\mu \dot{A}_\nu) F^{\mu\nu}\, dV_g \qquad \text{product rule}$$

$$= \int_{\mathcal{O}} \dot{A}_\nu \nabla_\mu F^{\mu\nu}\, dV_g - \underbrace{\int_{\mathcal{O}} \nabla_\mu (\dot{A}_\nu F^{\mu\nu})\, dV_g}_{\text{stokes} \downarrow 0}$$

$$\Longrightarrow \boxed{\nabla_\mu F^{\mu\nu} = 0} \boxed{dF = 0}$$

Example (Complex scalar field / Maxwell – Klein – Gordon)

Fix $g_{ab} = \eta_{ab}$ for this example. So (M, g) is Minkowski spacetime.

$$\phi : M \longrightarrow \mathbb{C}.$$

$$\mathcal{L} = -\tfrac{1}{2}\, \eta^{ab}\, \partial_a \phi\, \partial_b \bar{\phi} \;-\; V(|\phi|).$$

This Lagrangian is invariant under the circle action $\phi(x) \longrightarrow e^{i\theta} \phi(x)$, $\forall\, e^{i\theta} \in \{ |z| = 1 \}$, $x \in M$. Note that θ is the same real number at all $x \in M$. This is an example of a global gauge transformation.

Remark: Special relativity precludes the activation of the global symmetry.

Localizing the global gauge transformation

We enhance the symmetry by considering $\phi(x) \longrightarrow e^{i\theta(x)} \phi(x)$. However, the x-dependence in θ breaks the symmetry of the Lagrangian:

$$\partial_\alpha \left(e^{i\theta(x)} \phi(x) \right) = e^{i\theta(x)}\, \partial_\alpha \phi(x) + \phi(x)\, e^{i\theta(x)}\, i\, \theta_\alpha(x).$$

$$= e^{i\theta(x)} \left[\partial_\alpha + i\,\theta_\alpha(x) \right] \phi(x).$$

$$\partial_\beta \left(\overline{e^{i\theta(x)} \phi(x)} \right) = e^{-i\theta(x)} \left[\partial_\beta - i\,\theta_\beta(x) \right] \bar{\phi}(x).$$

$$\partial_\alpha \left(e^{i\theta(x)} \phi(x) \right) \partial_\beta \left(\overline{e^{i\theta(x)} \phi(x)} \right) = \partial_\alpha \phi(x)\, \partial_\beta \bar{\phi}(x) + \underbrace{\text{other terms}}_{\text{symmetry breaking}}.$$

We resurrect the symmetry by introducing a compensating adjustment in the derivative operator. Define

$$D^A_\alpha(x) = \partial_\alpha + i A_\alpha(x)$$

where the 1-form field A_α is called the gauge potential or connection 1-form.

The local gauge transformation

$$⊛ \quad \phi(x) \longmapsto e^{i\theta(x)}\phi(x) \, , \quad A_\alpha(x) \longmapsto A_\alpha(x) - \partial_\alpha \theta(x)$$

is a symmetry of the (adjusted) Lagrangian

$$\mathcal{L} = -\frac{1}{2}\eta^{ab} D^A_\alpha \phi \, \overline{D^A_\beta \phi} - V(|\phi|).$$

proof :

$$D^A_\alpha \phi \longmapsto D^{A-\theta_\alpha}_\alpha \left(e^{i\theta(x)}\phi(x)\right) = \left[\partial_\alpha + i(A_\alpha - \theta_\alpha)\right]\left(e^{i\theta(x)}\phi(x)\right)$$

$$= \left\{\left[\partial_\alpha + i(A_\alpha \cancel{- \theta_\alpha})\right]\phi(x)\right\} e^{i\theta(x)}$$

$$+ \; \phi(x) e^{i\theta(x)} \cancel{i\theta_\alpha(x)}$$

$$= \left[D^A_\alpha \phi(x)\right] e^{i\theta(x)} .$$

$D^A_\alpha \phi(x) \overline{D^B_\alpha \phi(x)}$ transforms under ⊛ into itself.

2002 Notes subsumed
(essentially) by preceding notes

Therefore, the metric compatible volume element $\underline{\varepsilon}$ is expressed in terms of the *natural chart induced volume element* $\underline{e}$ as

$$\varepsilon_{\mu_1 \cdots \mu_n} = f\, e_{\mu_1 \cdots \mu_n} \, , \quad \text{where} \quad f = \sqrt{|g|}$$

and $$|g| = |\det g_{\mu\nu}| \, , \quad \text{so}$$

$$\varepsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\; dx^1 \wedge \cdots \wedge dx^n$$

A *tensor density* T is an object which can be expressed in the form

$$T^{a \cdots b}{}_{c \cdots d} = \sqrt{|g|}\; \tilde{T}^{a \cdots b}{}_{c \cdots d}$$

where $\tilde{T}^{a \cdots b}{}_{c \cdots d}$ does not depend upon the choice of e_{abcd}.
~~that is, where $\tilde{T}$ is a tensor~~.

Example The expression $\sqrt{|g|}\, R$ is a tensor density.

The nice thing about tensor densities is that they can be naturally integrated:

$$\int_U R \underbrace{dV_g}_{\substack{\text{natural}\\ \text{volume}\\ \text{element}}} = \int_U R\, \underline{\varepsilon} \underset{\substack{\uparrow\\ \text{chart}\\ \text{enters here}}}{=} \int_U R \sqrt{|g|}\, \underline{e} = \int_{\phi[U]} R \sqrt{|g|}\, dx^1 \cdots dx^n .$$

R

end of detour ... not really ...

A tensor density is a tensor-like object which transforms like a tensor apart from a Jacobian determinant.

Example (Scalar Field)

$$\psi : \overset{\mathcal{U}}{\mathcal{M}} \longrightarrow \mathbb{R}$$

$$\mathcal{L} = \left(-\tfrac{1}{2} \nabla_a \psi \nabla_b \psi \, g^{ab} - V(\psi)\right) \sqrt{|g|} \, .$$

Form the action

$$S[\psi, g ; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L} \, \underline{e} \, .$$

Let $(\psi_{(s)}, \mathcal{U})$ be a compact variation. We calculate $\frac{\delta S}{\delta \psi}$.

Write $\psi_{(s)} = \psi_{(0)} + \dot{\psi} s + O(s^2)$.

$$\frac{\delta S}{\delta \psi} = \lim_{s \to 0} \frac{S[\psi_{(s)}] - S[\psi_{(0)}]}{s}$$

$$= \lim \frac{1}{s} \left\{ \int_{\mathcal{U}} \left[-\tfrac{1}{2} \nabla_a (\psi_{(0)} + s\dot{\psi}) \nabla_b (\psi_{(0)} + s\dot{\psi}) g^{ab} - V(\psi_{(0)} + s\dot{\psi}) \right] \sqrt{|g|} \, \underline{e} \right.$$

$$\left. - \int_{\mathcal{U}} \left(-\tfrac{1}{2} \nabla_a \psi_{(0)} \nabla_b \psi_{(0)} g^{ab} - V(\psi_{(0)}) \right) \sqrt{|g|} \, \underline{e} \, . \right.$$

$$= \int_{\mathcal{U}} \left[-\tfrac{1}{2} \left(\underset{\swarrow \text{IBP}}{\nabla_a \psi_{(0)} \nabla_b \dot{\psi}} + \underset{\swarrow \text{IBP}}{\nabla_a \dot{\psi} \nabla_b \psi_{(0)}} \right) g^{ab} - V'(\psi_{(0)}) \dot{\psi} \right] \sqrt{|g|} \, \underline{e}$$

$$= \int_{\mathcal{U}} \frac{1}{\sqrt{g}} \left[+\tfrac{1}{2} \nabla_b \left(\sqrt{|g|} \nabla_a \psi_{(0)} \right) + \tfrac{1}{2} \nabla_a \left(\sqrt{|g|} \nabla_b \psi_{(0)} \right) \right] g^{ab} \dot{\psi} \, \underline{e} \, \sqrt{g}$$

$$- \int_{\mathcal{U}} V'(\psi_{(0)}) \dot{\psi} \sqrt{|g|} \, \underline{e}$$

$$= \int_{\mathcal{U}} \left(\frac{1}{\sqrt{g}} \left[\nabla_b \left(\sqrt{|g|}\, \nabla_a \phi \; g^{ab} \right) \right] - V'(\phi) \right) \dot{\phi} \sqrt{|g|} \; \underline{e} .$$

If $\frac{\delta S}{\partial \phi} = 0$ $\forall$ compact variations ϕ we infer that

$$\boxed{\begin{array}{c} \frac{1}{\sqrt{|g|}} \left[\nabla_b \left(\sqrt{|g|}\, \nabla_a \phi \; g^{ab} \right) \right] - V'(\phi) = 0 \\ \overset{\cdot\cdot}{\|} \\ \Box_g \phi \end{array}}$$

<u>Remark</u>: Because we are dealing with a scalar field the covariant derivatives ∇_b, ∇_a may be replaced by ∂_b, ∂_a.

Example (Vaccuum Einstein eqs.)

$g_{ab} : M \longrightarrow \mathcal{T}_M(0,2)$ symmetric, nondegenerate.

Form the Hilbert-Einstein action

$$S_G[g_{ab}; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L}_G \, dV_g = \int_{\mathcal{U}} R\sqrt{|g|} \, \underline{\epsilon}$$

where R is the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$.

Consider a compact variation $(g_{ab}(s), \mathcal{U})$ of the metric g. We derive a field equation for g_{ab} by looking for g_{ab} such that S_G is stationary w.r.t. compact variations of g_{ab}.

$$0 = \frac{\delta S_G}{\delta(g_{ab})} = \int_{M} \left(\dot{R} \sqrt{|g|} + R \dot{\sqrt{|g|}} \right) \underline{e} .$$

Claim $\quad \dot{|g|} = |g| \, g^{\alpha\beta} \, \dot{g}_{\alpha\beta}$

proof: Suppose $\Phi(t) = Id + t\dot{\Phi}$ is a time dependent 1st order deformation of the Identity matrix.

$$\frac{d}{dt}\left(\det \Phi(t)\right)\Big|_{t=0} = Tr\, \dot{\Phi} , \quad \text{when } \Phi(0) = Id.$$

Now, suppose A is a constant coefficient matrix. By formula above for an arbitrary (invertible) Φ,

$$\frac{d}{dt}\left(\det A\Phi(t)\right)\Big|_{t=0} = Tr\, A\dot{\Phi} \quad \text{when } A\Phi(0) = Id.$$

e.g. $A = \Phi(0)^{-1}$.

$$/\!/$$

$$\det A \left(\frac{d}{dt} \det \Phi(t) \right)$$

$$\Longrightarrow \quad \frac{d}{dt} \det \Phi(t)\Big|_{t=0} = \det \Phi(0) \;\; Tr\left(\Phi(0)^{-1} \dot{\Phi} \right).$$

$$\Longrightarrow$$

$$\dot{|g|} = |g| \, g^{\alpha\beta} \, \dot{g}_{\alpha\beta}$$

* * *

$$\overset{\circ}{\sqrt{|g|}} = - \frac{1}{2\sqrt{|g|}} \dot{|g|} = \frac{1}{2\sqrt{|g|}} - |g| \, g^{\alpha\beta} \, \dot{g}_{\alpha\beta}$$

$$= \frac{1}{2} \sqrt{|g|} \;\; g^{\alpha\beta} \; \dot{g}_{\kappa\beta}$$

Raising + lowering indices and metric perturbations

$$g^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu}$$

$$\cancel{\dot{g}^{\alpha\beta}} = \dot{g}^{\alpha\mu} g^{\beta\nu} g_{\mu\nu} + g^{\alpha\mu} \dot{g}^{\beta\nu} g_{\mu\nu} + g^{\alpha\mu} g^{\beta\nu} \dot{g}_{\mu\nu}$$

$$= \dot{g}^{\alpha}{}_{\nu} g^{\beta\nu} + g^{\alpha\mu} \dot{g}^{\beta}{}_{\mu} + g^{\alpha\mu} \dot{g}_{\mu}{}^{\beta}$$

$$= \cancel{\dot{g}^{\alpha\beta}} + \dot{g}^{\beta\alpha} + \dot{g}^{\alpha\beta}$$

$\Longrightarrow$ $\dot{g}^{\beta\alpha} = -\dot{g}^{\alpha\beta}$ but this is impossible since we are perturbing the metric

$g^{\alpha\beta}_{(s)} = g^{\alpha\beta}_{(0)} + s\dot{g}^{\alpha\beta}$ must remain symmetric.

$$\Longrightarrow \boxed{\dot{g}^{ab} = -g^{a\mu} g^{b\nu} \dot{g}_{\mu\nu}}$$

Lagrangian Field Equations

Basic ingredients

- spacetime (M, g) Lorentzian manifold
- fields $\Psi = (\psi^{(1)}, \dots, \psi^{(p)})$ (restrict to tensor case)
- Lagrangian density $\mathcal{L}\sqrt{g}$ scalar density depending upon ψ, covariant derivatives of ψ and the metric.

These ingredients are used to form the Action,

$$S = S[\psi, g; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L}\sqrt{g}\, \underline{e} = \int_{\mathcal{U}} \mathcal{L}\, dV_g,$$

where $\mathcal{U}$ is a relatively compact subset of M.

The action principle states that physical solutions are stationary points of the action. The calculus of variations recasts stationary points of the action as solutions of the Euler-Lagrange equations which are also known as the field equations, in this setting.

The equation for the metric may be different from the equations for the fields due to the distinguished role played by the metric in defining the spacetime.

Vanishing of Boundary Term

Lemma Let $g_{\mu\nu}(s)$ be a family of spacetime metrics with $g(0) = g$ and $\frac{d}{ds} g \Big|_{s=0} = \dot{g}$. Let $\frac{d}{ds} R_{\alpha\beta} \Big|_{s=0} = \dot{R}_{\alpha\beta}$.

Then

$$\dot{R}_{\mu\nu} = \nabla_\alpha \dot{\Gamma}^{\alpha}{}_{\mu\nu} + \nabla_\mu \dot{\Gamma}^{\alpha}{}_{\alpha\nu}$$

where

$$\dot{\Gamma}^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(\nabla_\beta \dot{g}_{\gamma\lambda} + \nabla_\gamma \dot{g}_{\beta\lambda} - \nabla_\lambda \dot{g}_{\beta\gamma} \right).$$

proof We wish to prove a tensor identity and so may choose coordinates at $p \in M$ s.t. $\nabla_a = \partial_a$ at p and hence $\Gamma = 0$ at p. In such coordinates, the formula for the Riemann curvature tensor,

$$R_{\mu\beta\nu}{}^{\alpha} = \partial_\mu \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\delta\mu} \underset{\nearrow 0}{\Gamma^{\delta}{}_{\beta\nu}} - \Gamma^{\alpha}{}_{\sigma\nu} \underset{\nearrow 0}{\Gamma^{\sigma}{}_{\beta\mu}}$$

$$- R_{\beta\mu\nu}{}^{\alpha} = \cdots$$

(contract μ, α) $\cdots >$

$$- R_{\beta\nu} = \partial_\alpha \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\alpha}.$$

$$\Longrightarrow \quad \dot{R}_{\beta\nu} = \partial_\nu \dot{\Gamma}^{\alpha}{}_{\beta\alpha} - \partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\nu},$$

which may be rewritten as claimed. ∎

We apply the lemma.

$$g^{\beta\nu}\dot{R}_{\beta\nu} = g^{\beta\nu}\partial_\nu \dot{\Gamma}^{\alpha}{}_{\beta\alpha} - g^{\beta\nu}\partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\nu}$$

$$= g^{\beta\delta}\partial_\delta \dot{\Gamma}^{\alpha}{}_{\beta\alpha} - g^{\beta\gamma}\partial_\delta \dot{\Gamma}^{\delta}{}_{\beta\gamma}$$

$$= \partial_\delta \Big(\underbrace{g^{\beta\delta}\dot{\Gamma}^{\alpha}{}_{\beta\alpha} - g^{\beta\nu}\dot{\Gamma}^{\delta}{}_{\beta\nu}}_{V^\delta} \Big).$$

vanishes outside support of compact variation.

$\Longrightarrow$

$$g^{\beta\nu}\dot{R}_{\beta\nu} = \nabla_\delta V^\delta.$$

$$\int_{\mathcal{U}} \nabla_\delta V^\delta \overset{\text{Stoke's Thm.}}{=} \int_{\partial\mathcal{U}} n_\delta V^\delta = 0 \quad \text{since } V^\delta \equiv 0 \text{ near } \partial\mathcal{U}.$$

Matrix Calculation

$$\frac{\partial (g^{ab})}{\partial (g_{\mu\nu})} = ?$$

$$\frac{\partial}{\partial(g_{\mu\nu})} : \quad g^{ab} g_{bc} = \delta^{a}{}_{c}.$$

$$\frac{\partial (g^{ab})}{\partial (g_{\mu\nu})} \underbrace{g_{bc} g^{cd}}_{\delta_{b}{}^{d}} + g^{ab} \frac{\partial (g_{bc})}{\partial (g_{\mu\nu})} g^{cd} = 0.$$

$$\frac{\partial g^{ad}}{\partial (g_{\mu\nu})} = - g^{ab} \frac{\partial (g_{bc})}{\partial (g_{\mu\nu})} g^{cd}$$

$$\delta^{b}{}_{\mu} \, \delta^{c}{}_{\nu}$$

$$= - g^{a\mu} g^{\nu d}.$$

$$\Longrightarrow \quad \frac{\partial g^{ad}}{\partial (g_{ad})} = - g^{aa} g^{dd}$$

2x2 case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \qquad A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{\partial}{\partial a}\left(\frac{d}{ad-bc} \right) = \frac{-d^{2}}{(ad-bc)} = - \left(\frac{d}{ad-bc} \right) \left(\frac{d}{ad-bc} \right).$$

$$\frac{\partial}{\partial c}\left(\frac{-c}{ad-bc} \right) = - \left(\frac{a}{ad-bc} \right) \left(\frac{d}{ad-bc} \right).$$

$$S_{SF}[\phi, g ; \mathcal{U}] = \int_{\mathcal{U}} \overbrace{\left[-\tfrac{1}{2} \nabla_a \phi \nabla_b \phi g^{ab} - V(\phi) \right]}^{L_{SF}} \overbrace{\sqrt{-g}\ \underline{e}}^{d V_g} .$$

$$\frac{\delta S_{SF}}{\delta (g_{\mu\nu})} = \int_{\mathcal{U}} \left\{ \frac{\partial L_{SF}}{\partial (g_{\mu\nu})} \left.\frac{d}{ds}\right|_{s=0} (g^{\mu\nu}) \sqrt{-g} \right.$$

$$\left. + L_{SF} \left(\left.\frac{d}{ds}\right|_{s=0} \sqrt{-g} \right) \right\} \underline{e}$$

$$= \int_{\mathcal{U}} \left\{ \frac{\partial L_{SF}}{\partial (g_{\mu\nu})} \left(- g^{\mu\alpha} g^{\nu\beta} \dot{g}_{\alpha\beta} \right) \sqrt{-g} \right.$$

$$\left. + L_{SF} \left(\tfrac{1}{2} g^{\mu\nu} \dot{g}_{\mu\nu} \sqrt{-g} \right) \right\} \underline{e}$$

$$= \int_{\mathcal{U}} \underbrace{\left(- \frac{\partial L_{SF}}{\partial (g_{\mu\nu})} + \frac{1}{2} g_{\mu\nu} L_{SF} \right)}_{T_{\mu\nu}} g^{\mu\alpha} g^{\nu\beta} \dot{g}_{\alpha\beta} \sqrt{-g}\, \underline{e}$$

~~This can be calculated as~~

~~$T_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + 2 V(\phi) \right) \right)$~~.

This is good.

Lagrangian Formulation

$$S = S[\psi, g_{ab}; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L}\, dV_g = \int L \sqrt{-g}\, \underline{e} . \qquad \underset{\mathcal{L}}{L} = \underset{\mathcal{L}}{L}(\psi, \nabla_e \psi, g_{ab})$$

compact variation of fields

$$\psi_{(s)} = \psi_{(0)} + s\dot{\psi} , \qquad \psi_{(s)} = \psi_{(0)} \text{ on } \mathcal{M} \setminus \mathcal{U}, \quad \mathcal{U} \subset\subset \mathcal{M}.$$

$$\left\langle \frac{\delta S}{\delta \psi}, \dot{\psi} \right\rangle = \int_{\mathcal{U}} \frac{\partial \mathcal{L}}{\partial \psi} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} (\nabla_e \psi)^{\cdot} \quad dV_g$$

$$= \int_{\mathcal{U}} \frac{\partial \mathcal{L}}{\partial \psi} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} (\nabla_e \dot{\psi}) \quad dV_g$$

$$\nabla_e \left(\frac{\partial \mathcal{L}}{\partial \nabla_e \psi} \dot{\psi} \right) = \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} (\nabla_e \dot{\psi}) + \nabla_e \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \dot{\psi}$$

$$= \int_{\mathcal{U}} \left[\frac{\partial \mathcal{L}}{\partial \psi} - \nabla_e \frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \right] \dot{\psi}\, dV_g + \overset{\nearrow 0}{\int_{\mathcal{U}} \nabla_e \left(\frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \dot{\psi} \right) dV_g}$$

$$\Longrightarrow \quad \frac{\delta S}{\delta \psi} = \frac{\partial \mathcal{L}}{\partial \psi} - \nabla_e \left(\frac{\partial \mathcal{L}}{\partial(\nabla_e \psi)} \right) .$$

$$= \int_{M} \left[-R^{ab} + \frac{1}{2} R g^{ab} + \underbrace{\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{a\mu} g^{b\nu}}_{\partial^a \phi \partial^b \phi} - \frac{1}{4} g^{ab} \left(\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + 2V(\phi) \right) \right] \times \dot{g}_{ab} \sqrt{-g} \, \underline{e} .$$

Now lower the ab indices to observe

$$\int_{M} \left\{ - R_{ab} + \frac{1}{2} g_{ab} + \frac{1}{2} \left(\partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \left[\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + 2V(\phi) \right] \right) \right\} \dot{g}_{ab} \sqrt{-g} \, \underline{e} .$$

Since $\dot{g}_{ab}$ is arbitrary, we learn that the field equation

$$R_{ab} - \frac{1}{2} g_{ab} = T_{ab}$$ must hold

where

$$T_{ab} = \frac{1}{2} \left(\partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \left[\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + 2V(\phi) \right] \right)$$

is the scalar field's stress energy tensor.

Example   Maxwell's equations

A $(0,1)$-tensor field A_μ is given a new fancy names: gauge potential

An electromagnetic field on M is an antisymmetric $(0,2)$-tensor field

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu .$$

Note that the same electromagnetic field is obtained if we replace the gauge potential A_μ by $A_\mu + \nabla_\mu \chi$ for some scalar $\chi : M \longrightarrow \mathbb{R}$. Why?

$$\nabla_\mu (A_\nu + \nabla_\nu \chi) - \nabla_\nu (A_\mu + \nabla_\mu \chi)$$

$$= F_{\mu\nu} + \nabla_\mu \nabla_\nu \chi - \nabla_\nu \nabla_\mu \chi = F_{\mu\nu} \checkmark .$$

Form the Lagrangian

$$L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

and define the associated action

$$S_{EM}[F_{\mu\nu}, g; \mathcal{U}] = \int_{\mathcal{U}} \mathcal{L}_{EM}\, dV_g .$$

Compact variation of the EM field.

$$F_{\mu\nu}(s) = F_{\mu\nu}(0) + s\, \underset{\substack{\uparrow \\ \text{antisymmetric}}}{\dot{F}_{\mu\nu}}$$

$$\left\langle \frac{\delta S_{EM}}{\delta (F_{\mu\nu})}, \dot{F}_{\mu\nu} \right\rangle = \left. \frac{dF_{\mu\nu}}{ds} \right|_{s=0} = ?$$

We carry out the variation at the level of the gauge potential instead.

Write $\dot{A} = \left.\frac{d}{ds}\right|_{s=0} A(s)$ with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ so that

$$\dot{F}_{\mu\nu} = \nabla_\mu \dot{A}_\nu - \nabla_\nu \dot{A}_\mu .$$

$$\left.\frac{d}{ds}\right|_0 S(s) = -\frac{1}{2} \int_{\mathcal{M}} \dot{F}_{\mu\nu} F^{\mu\nu} \, dV_g$$

$$= -\frac{1}{2} \int_{\mathcal{M}} (\nabla_\mu \dot{A}_\nu - \nabla_\nu \dot{A}_\mu) \, F^{\mu\nu} \, dV_g$$

$$= - \int_{\mathcal{M}} \nabla_\mu \dot{A}_\nu F^{\mu\nu} \, dV_g = - \int_{\mathcal{M}} \nabla_\mu \dot{A}_\nu \, F^{\mu\nu} \sqrt{-g} \, \underline{e}$$

see Schutz

$$= \int_{\mathcal{M}} \dot{A}_\nu \left(\frac{1}{\sqrt{g}} \nabla_\mu \left(F^{\mu\nu} \sqrt{-g} \right) \right) \sqrt{-g} \; \underline{e}$$

$$= \left\langle \frac{\delta S}{\delta (A_\nu)} , \dot{A}_\nu \right\rangle .$$

Example *Complex scalar field*

Let $\eta_{\alpha\beta}$ denote the metric for Minkowski space.

$$\phi : M \longrightarrow \mathbb{C}$$

$$L = -\tfrac{1}{2}\, \eta^{\alpha\beta}\, \partial_\alpha \phi\, \partial_\beta \bar{\phi} - V(|\phi|).$$

L is invariant under the transformation $\phi(x) \longrightarrow e^{i\theta} \phi(x)$ for fixed $\theta \in \mathbb{R}$, $x \in M$. Note that this invariance property requires that θ be the same real number across M. This is an example of a *global gauge transformation*. The is a symmetry group action on ϕ described by points in the circle, parametrized here by $e^{i\theta}$, which leaves the complex scalar field Lagrangian invariant.

The global aspect of this transformation violates the basic tenets of relativity.

We enhance the symmetry by considering a *local gauge transformation* $\phi(x) \longrightarrow e^{i\theta(x)} \phi(x)$. The x-dependence in θ breaks the symmetry of the Lagrangian:

$$\partial_\alpha (e^{i\theta(x)} \phi)(x) = e^{i\theta(x)}\, \partial_\alpha \phi(x) + \phi(x) e^{i\theta(x)}\, i\theta_\alpha(x).$$
$$= e^{i\theta(x)} (\partial_\alpha + i\theta_\alpha(x))\, \phi(x).$$

$$\overline{\partial_\beta (e^{i\theta(x)} \phi)(x)} = e^{-i\theta(x)} (\partial_\beta - i\,\theta_\beta(x))\, \bar{\phi}(x)$$

$$(\partial_\alpha e^{i\theta(x)} \phi(x))\overline{(\partial_\beta e^{i\theta(x)} \phi(x))} = (\partial_\alpha \phi(x))\overline{(\partial_\alpha \phi(x))} + i\theta_\alpha(x)\, \phi(x)\, \partial_\beta \bar{\phi}(x)$$
$$- i\,\theta_\beta(x)\, \bar{\phi}(x)\, \partial_\alpha \phi(x)$$
$$- \theta_\alpha(x)\, \theta_\beta(x)\, |\phi|^2(x).$$

We resurrect the symmetry by introducing a compensating ~~to~~ adjustment in the derivative operator. Define

$$D_\alpha^{A(x)} = \partial_\alpha - A_\alpha(x)$$

where the covector field A_α is called the gauge potential.

aka connection 1-form

The local gauge transformation

$$\phi(x) \longrightarrow e^{i\theta(x)}\phi(x) \quad , \quad A_\alpha(x) \longmapsto A_\alpha(x) - \partial_\alpha\theta(x)$$

is a symmetry of the (adjusted) Lagrangian

$$L = -\tfrac{1}{2}\,\eta^{\alpha\beta}\, D^A_\alpha\phi\, \overline{D^A_\beta\phi} - V(|\phi|).$$

proof Introduce θ.

A_α is free as a bird at this point.

$$D^A_\alpha\phi \longmapsto D_\alpha^{A_\alpha - \partial_\alpha\theta}\left(e^{i\theta(x)}\phi(x)\right)$$

$$= \left[\partial_\alpha + i(A_\alpha - \partial_\alpha\theta)\right] e^{i\theta(x)}\phi(x)$$

$$= \left\{[\partial_\alpha + iA_\alpha]\phi(x)\right\} e^{i\theta(x)} + \cancel{\phi(x)\, e^{i\phi(x)}\, i\partial_\alpha\theta(x)} - \cancel{\phi(x)\, e^{i\theta(x)}\, i\partial_\alpha\theta(x)}$$

$\longrightarrow L$ and $e^{i\theta(x)}e^{-i\theta(x)} = 1$ cancels away.

$$D_\alpha^{A_\alpha - \partial_\alpha\theta}\left(e^{i\theta(x)}\phi(x)\right) = \left(D^A_\alpha\phi\right)e^{i\theta(x)}.$$

How does A_α influence the Lagrangian? We want invariance under Gauge transformations

Define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (electromagnetic field).

and Form

$$L_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$

Consider the extended Lagrangian

$$L = -\frac{1}{2} \eta^{ab} D_\alpha^A \phi \overline{D_\beta^A \phi} - V(|\phi|) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$

Remark: The relativistic criticism of the global gauge transformation of the complex scalar field motivated the localization of the gauge transformation. The localitation broke the Lagrangian symmetry but this is resurrected by modifying the derivative operator to compensate. The compensating term introduces a new field (the gauge potential) which gives rise to the em field.

punchline [em field arises naturally by demanding invariance of complex scalar field Lagrangian under local gauge transformations.]

The resulting field equation is called the Maxwell-Klein-Gordon equation.

(2002 notes)

A detour

I skipped this in 2003.

→ Scalar field.

Integration on a Manifold.

M n dimensional manifold.

A totally antisymmetric $(0,n)$ tensor $T_{a_1 \cdots a_n} = T_{[a_1 \cdots a_n]}$ is called an *n-form*.

The vector space V_p of n-forms is 1 dimensional.

If $\exists$ cts. nowhere vanishing n-form field $\varepsilon = \varepsilon_{[a_1 \cdots a_n]}$ on M, then M is *orientable* and ε is said to provide an *orientation*.

Two orientations ε and ε' are equivalent if $\varepsilon = f \varepsilon'$ where f is a (strictly) positive function. Hence any orientable M has two inequivalent orientations.

We will define the integral of a continuous (or measurable) n-form field α over an n-dimensional manifold (w.r.t. orientation ε) as follows: Let $U \subset M$ be covered by a single coordinate chart ψ. We expand ε in the coordinate basis of ψ to get

$$\varepsilon = h \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

e.g.

$$\varepsilon_{a_1 \cdots a_n} = n! \, h \, dx^1_{[a_1} \cdots dx^n_{a_n]} \, , \qquad h \neq 0.$$

If $h > 0$, ψ is said to be *right handed*.
If $h < 0$, *left*.

Suppose $\alpha = a(x^1, \dots, x^n) \, dx^1 \wedge \cdots \wedge dx^n$

If ψ is right handed, we define the integral of α over U by

$$\int_U \alpha = \int_{\psi[U]} a \; \underbrace{dx^1 \cdots dx^n}_{\text{Riemann or Lebesgue integral on } \mathbb{R}^n}$$

If ψ is left handed, we insert a minus sign.

addendum to detour

We identified a natural coordinate based volume element we called e with the definition

$$e_{\mu_1\cdots\mu_n} = \begin{cases} (-1)^p & \mu_1, \ldots, \mu_n \text{ all distinct} \\ 0 & \text{otherwise,} \end{cases}$$

where p is signature of permutation $(1, \ldots, n) \longrightarrow (\mu_1, \ldots, \mu_n)$.

Happily, we have used this device several times to observe

$$e_{\mu_1' \cdots \mu_n'} \, |M| = \varepsilon_{\mu_1 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'} \cdots M^{\mu_n}{}_{\mu_n'} .$$

where $M^{\mu}{}_{\mu'}$ is a matrix and $|M|$ is its determinant.

Suppose $M^{\mu}{}_{\mu'}$ is the Jacobian matrix $M^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$.
Then

$$e_{\mu_1' \cdots \mu_n'} = \underbrace{\left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|}_{\text{non tensorial}} \underbrace{e_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}}_{\text{tensorial}}$$

Consider $g_{\mu\nu}$, the metric on M. Form $|g| = |g_{\mu\nu}| = |\det(g_{\mu\nu})|$.

Under a coordinate transformation $X^{\mu} \longrightarrow x^{\mu'}$,

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}$$

so

$$|g_{\mu'\nu'}| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^2 |g_{\mu\nu}| = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2} |g_{\mu\nu}| .$$

We see that $|g|$ does not transform as a scalar.

V. Exact Solutions

Einstein Field Equations $\quad R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}$.

Remarks

① Let (M, g) be an arbitrary Lorentzian manifold. With g, we calculate $R_{ab} - \frac{1}{2} R g_{ab}$ and define the result to be T_{ab}. This defines a "matter field" so that (M, g) solves the Einstein equations.

But T_{ab}, obtained in this way, may not satisfy natural requirements: local causality, energy positivity.

② Assuming T_{ab} is given, the Einstein equations involve 6 independent metric coefficients.

$$g_{ab} \quad \overset{\text{symmetry}}{\underset{g_{ab}=g_{ba}}{}} \quad g_{ab} = g_{ba} \quad \begin{matrix}\text{diffeos}\\ \text{adjust}\\ \text{4 coefficients.}\end{matrix} \longrightarrow 6.$$

$$16 \qquad\qquad\qquad 10$$

$$16 \Big/_{[g_{ab}=g_{ba}]} \Big/_{\text{diffeo}} = 6.$$

We say (M, g_{ab}) is an exact solution of Einstein's equations if the field eq. is satisfied & the tensor T_{ab} satisfies local causality & energy positivity conditions

Q: (A. Nachman) Does $\exists$ characterization of g which are compatible with local causality, energy positivity ?

Hawking & Ellis provide a taxonomy of some exact solutions

A. Constant Curvature metrics ; Minkowski, De-Sitter, Anti-deSitter
~~Friedman - Robertson - Walker~~

B. Spatially isotropic + homogeneous cosmological models
Friedman - Robertson - Walker

C. spherically symmetric metrics outside a massive body
Schwarzschild, Reissner - Nordstrom

D. Axially symmetric metrics outside rotating massive body.
Kerr

E. Exotic solutions which possess timelike closed geodesics.
Gödel Universe.

Minkowski spacetime is clearly a solution.

Remark ③ Solutions obtained under symmetry assumptions should be contrasted with solutions obtained by solving the initial value problem in GR.

Special solutions may reveal key phenomena in solutions of the field equations.

Spherically Symmetric Solutions of Vacuum Einstein Eqs.

Recall the metric on $\mathbb{R}^3$ in polar coordinates (r, θ, ϕ):

$$\mathbb{R}^3 = \{(x,y,z)\} = \{(r,\theta,\phi)\}$$

$$\underset{\uparrow}{dx^2} + \underset{\uparrow}{dy^2} + \underset{\uparrow}{dz^2} = \underset{\nearrow}{dr^2} + \underset{\substack{\uparrow \\ \text{magnification}}}{r^2} \underbrace{\{d\theta^2 + \sin^2\theta \, d\phi^2\}}_{\text{unit sphere element}}.$$

similar objects.

A <u>spherically symmetric solution</u> of the field equations is a metric of the form

$$ds^2 = -A(t,r)\,dt^2 + B(t,r)\,dr^2 + C(t,r)\{d\theta^2 + \sin^2\theta\, d\phi^2\}.$$

We are $\boxed{\text{implicitly assuming}}$ that t stays timelike and θ, ϕ, r spacelike by demanding $A > 0$, $B > 0$, $C > 0$ $\forall\, t, r$.

We can then write

$$A(t,r) = e^{\alpha(t,r)}, \quad B(t,r) = e^{\beta(t,r)}, \quad C(t,r) = e^{\gamma(t,r)}.$$

Introduce $\tilde{r} = \tilde{r}(t,r)$ by $(\tilde{r}(t,r))^2 = C(t,r)$. Recasting in terms of $\tilde{r}$ vs. r produces $C(t,r) = \tilde{r}^2$ and $\alpha(t,r) \rightsquigarrow \tilde{\alpha}(t,\tilde{r})$, $\beta(t,r) \rightsquigarrow \tilde{\beta}(t,\tilde{r})$.

So we may assume $C(t,r) = r^2$.

(reduced form) $$ds^2 = -e^{\alpha(t,r)}\,dt^2 + e^{\beta(t,r)}\,dr^2 + r^2\{d\theta^2 + \sin^2\theta\, d\phi^2\}.$$

The spherical symmetry assumption reduced the complexity of the metric from 6 unknown functions to 2 positive unknown functions $e^{\alpha(t,r)}$, $e^{\beta(t,r)}$.

We seek solutions of the vacuum einstein equation

$$R_{ab} - \frac{1}{2} R g_{ab} = 0$$

$$\Rightarrow R = 0$$

$$\Rightarrow \boxed{R_{ab} = 0} ,$$

of the reduced form:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & -e^{\alpha(t,r)} & 0 & 0 & 0 \\ 1 & 0 & e^{\beta(t,r)} & 0 & 0 \\ 2 & 0 & 0 & r^2 & 0 \\ 3 & 0 & 0 & 0 & r^2 \sin^2\phi \end{array}$$

$g_{\alpha\beta}$.

Curvature from connection

$$R_{abc}{}^{d} = \partial_a \Gamma^d{}_{bc} - \partial_b \Gamma^d{}_{ac} + \Gamma^d{}_{ae} \Gamma^e{}_{bc} - \Gamma^d{}_{be} \Gamma^e{}_{ac}$$

trace on b, d. $R_{ac} = 0$.

Connection from metric

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right)$$

$$= \text{function of } (\dot{\alpha}, \alpha', \dot{\beta}, \beta') \qquad \dot{} = \frac{d}{dt}, \quad ' = \frac{d}{dr}.$$

Thus, $R_{ab} = 0$ unravels as a 2nd order PDE system on (r, t) plane. ~ 16 2nd order pde with solution expressed with $\alpha(t,r)$, $\beta(t,r)$ given with integration constant(s). The system separates on certain coefficients of the Riemann tensor

Chasing this through is reported to imply

$$\frac{\dot{\beta}}{r} = 0 \qquad \Longrightarrow \dot{\beta} = 0 , \ \beta = \beta(r).$$

$$-e^{-\beta}\left[1 + \frac{r}{2}(\alpha' - \beta')\right] + 1 = 0 , \Longrightarrow \alpha' \text{ is } t \text{ indepnd.}$$

$$\alpha(t,r) = \alpha_0(r) + \underset{\substack{\Big\uparrow \\ \text{time} \\ \text{rescaling} \\ \text{removes this.}}}{\alpha_1(t)}$$

and other equations.

Thus, both α and β are independent of time for spherically symmetric vacuum einstein solutions.

Birkhoff's Theorem Every spherically symmetric vacuum einstein solution is time independent.

This shows there are no spherical waves or perturbations, even during an Explosion!

$\Longrightarrow$ $\dot{\alpha}$, $\dot{\beta}$ terms in other equations go away

$$R_{ab} = 0 \rightsquigarrow \begin{cases} \dfrac{\alpha''}{2} + \dfrac{(\alpha')^2}{4} - \dfrac{\alpha'\beta'}{4} - \dfrac{\beta'}{r} = 0 \\ (\alpha' + \beta')\dfrac{1}{r} = 0 \\ e^{-\beta}(1 - r\beta') - 1 = 0 \end{cases}$$

$$\boxed{\alpha' = -\beta'} \Longleftarrow (\alpha' + \beta')\frac{1}{r} = 0$$

$$\Downarrow$$

$$\alpha = -\beta \ \left(\text{absorb constant by scaling}\right) \quad A = B^{-1}$$

Third equation

$$e^{-\beta}(1 - r\beta') = 1$$

$$y = e^{-\beta} \quad ; \quad y' = e^{-\beta}(-\beta')$$

$$y(1 + ry') = 1$$

$$\frac{y}{r} + y' = \frac{1}{r}$$

Wait! $y = \frac{1}{B}$.

$$\left(\frac{1}{B}\right)' + \frac{\frac{1}{B}}{r} = \frac{1}{r}$$

which has solution

$$\frac{1}{B} = \left[1 - \frac{2m}{r}\right]. = A.$$

check: $\left(\frac{1}{B}\right)' = + \frac{2m}{r^2}$

$$\cancel{\frac{2m}{r^2}} + \frac{1 - \cancel{\frac{2m}{r}}}{r} = \frac{1}{r} \checkmark.$$

$$ds^2 = -\left[1 - \frac{2m}{r}\right] dt^2 + \left[1 - \frac{2m}{r}\right]^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

is the Schwarzschild Solution (1916).

$\underline{\mu=2}$ $\quad \dfrac{\partial L}{\partial \dot\theta} = 2r^2\dot\theta \,, \quad \dfrac{d}{ds}\dfrac{\partial L}{\partial \dot\theta} = 4r\dot r\dot\theta + 2r^2\ddot\theta$

$$\frac{\partial L}{\partial \theta} = 4r \sin\theta \cos\theta\, \dot\theta \dot\phi \,.$$

$$\ddot\theta + \underset{\substack{\shortparallel \\ 2\Gamma^2_{12}}}{2\,\frac{\dot r \dot\theta}{r}} - \underset{\Gamma^2_{23}}{\frac{2\sin\theta\cos\theta}{r}\,\dot\theta\dot\phi} = 0.$$

$\underline{\mu=3}$ $\quad \dfrac{\partial L}{\partial \dot\phi} = 2r^2\sin^2\theta\,\dot\phi \;; \quad \dfrac{d}{ds}\dfrac{\partial L}{\partial \dot\phi} = 4r\dot r \sin^2\theta\,\dot\phi$

$$+ 2r^2 \sin\theta\cos\theta\,\dot\theta\dot\phi$$

$$\frac{\partial L}{\partial \phi} = 0 \qquad\qquad + 2r^2\sin^2\theta\,\ddot\phi$$

$$\ddot\phi + \underset{\Gamma^3_{23}}{\frac{1}{2}\cot\theta\,\dot\theta\dot\phi} + \underset{2\Gamma^3_{13}}{\frac{2}{r}\,\dot r\dot\phi} = 0.$$

We observe that ?

$$\Gamma^\mu_{\mu 1} = \Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} = \frac{A_r}{2A} + \frac{B_r}{2B} + \frac{2}{r} + \frac{2}{r}$$

$$\Gamma^\mu_{\mu 2} = \Gamma^0_{02} + \Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} = 0 + 0 + 0 + \frac{1}{2}\cot\theta$$

One also can verify that

$$\sqrt{-g} = r^2 \sin\theta \sqrt{AB} \,.$$

Fact: $\quad \Gamma^\mu_{\mu\beta} = \partial_\beta \log\sqrt{-g} = \dfrac{\partial_\beta \sqrt{-g}}{\sqrt{-g}} \,.$

we have so far determined that

$$\Gamma^0_{00} = \frac{1}{2}\frac{A_t}{A} \qquad \Gamma^0_{10} = \frac{1}{2}\frac{A_r}{A} \qquad \Gamma^0_{11} = \frac{B_t}{2A}$$

$$\Gamma^1_{11} = \frac{B_r}{2B} \qquad \Gamma^1_{10} = \frac{B_t}{2B} \qquad \Gamma^1_{22} = \frac{-r}{B} \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^2_{12} = \frac{1}{r}$$

$$\Gamma^3_{13} = \frac{1}{r} \qquad \Gamma^3_{23} = \cot\theta$$

and the rest are zero.

We can then calculate using the formula for $R_{abc}{}^d$ to find that the only nonzero Ricci tensor coefficients are, using

$$A = e^{-\alpha}, \quad B = e^{-\beta}, \quad \dot{} = \frac{d}{dt}, \quad {}' = \frac{d}{dr}$$

$$R_{11} = -\frac{\alpha_{rr}}{2} - \frac{\alpha_r^2}{2} + \frac{\alpha_r\beta_r}{4} + \frac{\beta_r}{r} + \frac{B}{A}\left[\frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4} - \frac{\dot{\beta}\dot{\alpha}}{4}\right]$$

$$R_{00} = \frac{A}{B}\left[\frac{\alpha_{rr}}{2} + \frac{\alpha_r^2}{4} - \frac{\alpha_r\beta_r}{4} + \frac{\alpha_r}{r}\right] - \frac{\ddot{\beta}}{2} - \frac{\dot{\beta}^2}{4} + \frac{\dot{\beta}\dot{\alpha}}{4}$$

$$R_{01} = \frac{\dot{\beta}}{r} \Longrightarrow \dot{\beta} = 0 \text{ for } R_{ac} = 0.$$

$$R_{22} = -\frac{1}{B}\left[1 + \frac{r}{2}(\alpha' - \beta')\right] + 1 \longrightarrow$$

α' is time independent
↳ $\alpha = \alpha(r)$ via rescaling

$$R_{33} = \sin^2\theta \, R_{22}.$$

Geodesics of the Schwarzschild metric

These notes follow a nice presentation by Gilbert Weinstein I found on the web, combined with the usual sources.

Suppose $\phi_t : M \to M$ is a one-parameter group of isometries. Then $\phi_t^* g_{ab} = g_{ab}$. The vector field ξ^a which generates ϕ_t is called a Killing Vector Field.

Since $\phi_t^* g_{ab} = g_{ab}$ we have, essentially by definition, that $\mathcal{L}_\xi g_{ab} = 0$. The Lie derivative acts on (k,l) tensor to produce a (k,l) tensor. For the $(0,2)$-tensor g_{ab}, we have that

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi^b + \nabla_b \xi^a .$$

Proposition (M, g) Lorentz manifold. Let ξ be a Killing vector field on (M,g) and let γ be a geodesic of (M,g) with tangent vector $\dot\gamma$. Then $g(\xi, \dot\gamma)$ is constant along the geodesic flow.

~~Proof Denote $\dot\gamma, \xi$ in coordinates as $\dot\gamma^a, \xi^b$.~~

We ~~calculate~~

~~$\dot\gamma^c \nabla_c (g_{ab} \xi^a \dot\gamma^b) = \dot\gamma^c (\nabla_c g_{ab}) \xi^a \dot\gamma^b + \dot\gamma^c g_{ab} (\nabla_c \xi^a) \dot\gamma^b + \dot\gamma_c g_{ab} \xi^a \nabla_c \dot\gamma$~~

~~$= \dot\gamma_c (\nabla_c g_{ab}) \xi^a \dot\gamma^b + \dot\gamma_c g_{ab} (\nabla_c \xi^a) \dot\gamma_b + \dot\gamma_c$ [illegible]~~

Proof Write $\dot{\gamma}^a$, ξ^b in coordinates. We calculate

$$\underbrace{\dot{\gamma}^c \nabla_c}_{\text{along curve change}} \left(g_{ab} \xi^a \dot{\gamma}^b \right) = \underbrace{\dot{\gamma}^c (\nabla_c g_{ab}) \xi^a \dot{\gamma}^b}_{\substack{0 \\ \text{(metric compatible} \\ \text{covariant derivative.)}}} + \dot{\gamma}^c g_{ab} (\nabla_c \xi^a) \dot{\gamma}^b + \dot{\gamma}^c g_{ab} \xi^a \times \underbrace{\nabla_c \dot{\gamma}^b}_{\substack{\dot{\gamma}^c \nabla_c \dot{\gamma}^b \\ \| \\ 0 \\ \text{(geodesic)}}}.$$

$$\dot{\gamma}^c \dot{\gamma}^a (\nabla_c \xi^a)$$

but $\nabla_c \xi^a = - \nabla_a \xi^c$ since ξ is Killing.

$$\Downarrow$$

$$0 \, . \qquad ∎$$

The Schwarzschild metric is

$$g = ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

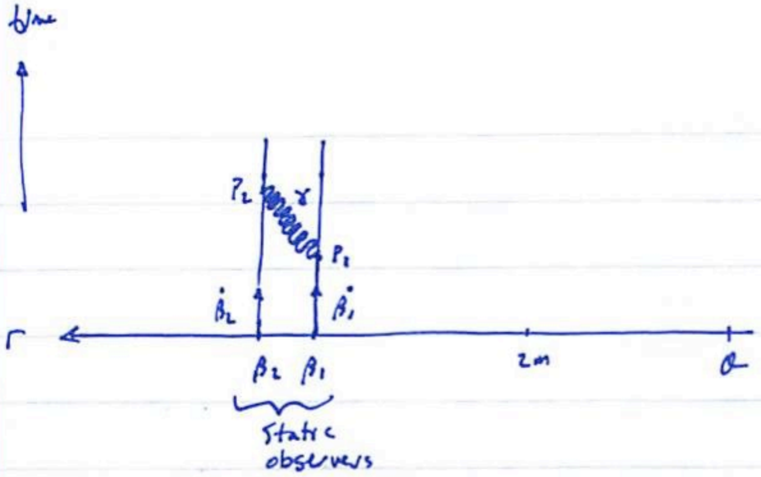
This metric was constructed to have an $SO(3)$ invariance and is manifestly invariant under $t \mapsto t + t_0$ so there is an $\mathbb{R} \times SO(3)$ invariant action. The invariance here means that $\mathbb{R} \times SO(3)$ is in fact a group of isometries for the Schwarzschild metric.

The one-parameter subgroup $\mathbb{R}$ of the isometry group of the Schwarzschild metric is generated by the vector field ∂_t, thus ∂_t is a killing vector field. We renormalize ∂_t by the natural condition

$$-1 = g_{ab}\left((\theta 1, 0, 0, 0), (\partial 1, 0, 0, 0)\right) = -(1-\frac{2m}{r})\theta^2 \Rightarrow \theta = \frac{1}{\sqrt{1-\frac{2m}{r}}}$$

so $\dfrac{\partial_t}{\sqrt{1-\frac{2m}{r}}}$ is proper-time reparametrization of the static killing field ∂_t.

The integral curves of this vector field are the world lines of static observers in the Schwarzschild spacetime.



light signal of frequency $-g(\dot\beta_1, \dot\gamma) = w_1$ is emitted by β_1 along γ at $p_2 \in M$. The signal is received by β_2 at $p_2 \in M$ with measured frequency $-g(\dot\beta_2, \dot\gamma). = w_2$

Since ∂_t is Killing, we know that $g(\partial_t, \dot\gamma) =$ constant. We look at

$$\frac{w_1}{w_2} - 1 = \frac{-g(\dot\beta_1, \dot\gamma)}{-g(\dot\beta_2, \dot\gamma)} - 1$$

$$= \frac{-\frac{1}{\sqrt{1-\frac{2m}{r_1}}} \cancel{g(\partial_t, \dot\gamma)}}{-\frac{1}{\sqrt{1-\frac{2m}{r_2}}} \cancel{g(\partial_t, \dot\gamma)}} - 1$$

$$\boxed{\frac{w_1}{w_2} - 1 = \frac{\sqrt{1-\frac{2m}{r_2}}}{\sqrt{1-\frac{2m}{r_1}}} - 1}$$

Thus, $w_1 < w_2$ if $r_1 < r_2$. This is gravitational redshift.

It has been measured within 0.01 % to be correct. This is an experimental validation of GR.

Integration of the geodesic equations

Let γ be any geodesic. Then $g(\dot\gamma, \dot\gamma)$ is constant along γ. (Why? $(\dot\gamma^c \nabla_c) g_{ab} \dot\gamma^a \dot\gamma^b = \dots$). Thus, if γ is not null, it may be parametrized by proper time τ so that $g(\dot\gamma, \dot\gamma) = -1$. For γ null, we assume τ is an affine parameter along γ.

Lemma (M, g) Lorentz manifold. Let φ be an isometry of (M, g). Let $\Sigma = \{ p \in M : \varphi(p) = p \}$, the set of fixed points of φ. Then, any geodesic γ initially on Σ and tangent to Σ remains in Σ. In particular, Σ is totally geodesic.

~~proof Suppose $\exists \tau$ s.t. $\gamma(\tau) \notin \Sigma$ but $\gamma(0) \in \Sigma$ and $\dot\gamma(0)$ is tangent to Σ. Then,~~ [illegible] ~~geodesic with $\varphi(\gamma(0)) = \gamma(0)$ and $\varphi(\dot\gamma(0)) = \dot\gamma(0)$. By ODE uniqueness, φ~~

proof Suppose not. Then $\exists$ geodesic γ and a proper time τ s.t. $\gamma(0) \in \Sigma$, $\dot\gamma(0)$ is tangent to Σ but $\gamma(\tau) \notin M$. We must have then that $\gamma(\tau) \neq \varphi(\gamma(\tau))$. However, $\varphi \circ \gamma$ is also a geodesic with $\varphi(\gamma(0)) = \gamma(0)$ and $\varphi(\dot\gamma(0)) = \dot\gamma(0)$ so, by ODE uniqueness, $\varphi \circ \gamma$ and γ coincide. Ⓒ!

For Schwarzschild, consider the map $(t,r,\theta,\phi) \longrightarrow (t,r,\pi-\theta,\phi)$. This is an isometry whose fixed points are the equatorial plane $\Sigma = \{ (t,r,\theta,\phi) : \theta = \frac{\pi}{2} \}$.
If γ is any geodesic $\exists$ isometry $\varphi \in SO(3)$ s.t. $\varphi(\gamma(0)) \in \Sigma$ and $\varphi(\dot\gamma(0)) \in T\Sigma$.
We may therefore restrict attention to geodesic motions lying in the equatorial plane.

Let $\gamma = (t, r, \frac{\pi}{2}, \phi)$ be a geodesic. Then (t, r, ϕ) is a critical point of the Lagrangian

$$S = \int \left\{ -\left(1-\frac{2m}{r}\right)\dot t^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot r^2 + r^2\dot\phi^2 \right\} d\tau .$$

The killing vectors $\xi_1 = \partial_t$ and $\xi_2 = \partial_\phi$ imply that

energy →

$$\left.\begin{aligned} E &= \left(1-\frac{2m}{r}\right)\dot t = g_{ab}(\xi, \gamma) \\ L &= r^2\dot\phi \end{aligned}\right\} \text{ conserved.}$$

angular momentum.

Thus, $\dot\phi$ = constant if the angular momentum L is zero. These geodesics are called <u>radial geodesics</u>

We'll analyze these soon.

We also have that $g(\dot\gamma, \dot\gamma)$ = constant:

$$-\left(1-\frac{2m}{r}\right)\dot t^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot r^2 + r^2\dot\phi^2 = -k$$

and $k = \begin{cases} 0 & \text{if } \gamma \text{ is null} \\ 1 & \text{if } \gamma \text{ is timelike.} \end{cases}$

Null nonradial geodesics

$$L \neq 0, \quad \kappa = 0. \implies \dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = E^2.$$

rescale affine parameter

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = 1.$$

Equations of motion

$E = \left(1 - \frac{2m}{r}\right)\dot{t}$ and $E = 1$ by rescaling so

* $$\boxed{\frac{dt}{d\tau} = \frac{1}{1 - \frac{2m}{r}}}$$

☆ $$\boxed{\frac{dr}{d\tau} = \sqrt{1 - \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right)}}$$

\# $$\boxed{\frac{d\phi}{d\tau} = \frac{L}{r^2}}$$

The effective potential is $\frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right)$. Take ∂_r, find critical point(s):

$$\frac{L^2}{r^2}\left(\frac{2m}{r^2}\right) - \frac{2L^2}{r^3}\left(1 - \frac{2m}{r}\right) = 0$$

$$\times \frac{r^2}{L^2} \to \quad \frac{2m}{r^2} - \frac{2}{r} + \frac{4m}{r^2} = 0$$

$$\times r^2 \to \quad 2m - 2r + 4m = 0$$

$\boxed{r = 3m}$ is critical

$$\dot{r}^2 \underset{\searrow\, -E^2}{-\left(1-\frac{2m}{r}\right)^2 \dot{t}^2} + \left(1-\frac{2m}{r}\right) r^2 \dot{\phi}^2 = K\left(1-\frac{2m}{r}\right)$$

$$\boxed{\dot{r}^2 + \left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+K\right) = E^2.}$$

We analyze this in <u>2 subcases</u>: null; $K=0$ and timelike; $K=1$.

~~Thus, there are 3 families of geodesics for us to understand:~~

- ~~radial geodesics (null + timelike)~~
- ~~nonradial null geodesics ↔ bending of light rays~~
- ~~nonradial timelike geodesics ↔ perihelon precession.~~

The geodesics separate into 4 categories

	radial	nonradial
null		(bending of light)
timelike		(perihelon precession)

Radial Null geodesics

$L = 0, \ K = 0 \implies \dot{r}^2 = E^2.$

rescale affine parameter: $\dot{r}^2 = 1 \implies \dot{r} = \pm 1.$ $\overset{\nearrow +}{\underset{\searrow -}{}}$

outgoing radial null geodesics: $\dot{r} = +1$

$$\implies \boxed{r = \tau} \quad \text{(after shift)}$$

Recall that $E = (1 - \frac{2m}{r})\dot{t}$ and $E = 1$ here.

Thus, $\dot{t} = \dfrac{1}{1 - \frac{2m}{\tau}}$ so

$$\boxed{(t - t_0) = \int \frac{d\tau}{1 - \frac{2m}{\tau}} = r + 2m \log (r - 2m)}$$

incoming radial null geodesics: $\dot{r} = -1$

$$(t - t_0) = -(r + 2m \log (r - 2m))$$

Note: All incoming null (radial) geodesics reach $r = 2m$ within finite affine parameter even though the Schwarzschild time t is infinite upon arrival. The set $\{(r, \theta, \phi, t) : r = 2m\}$ is the event horizon.

Radial timelike geodesics

$L = 0, \quad \kappa = 1. \implies \dot{r}^2 + \left(1 - \frac{2m}{r}\right) = E^2.$

recast as

$$\frac{1}{2}\dot{r}^2 + \underbrace{\left(\frac{1}{2} - \frac{m}{r}\right)}_{V(r)} = \frac{1}{2}E^2$$

$\ddot{r} = \nabla V = \frac{m}{r^2}$, so this coincides with Newtonian case.

If $E^2 < 1$, the orbit is a crash orbit.

If $E^2 \geq 1$, the orbit is an escape orbit.

At $r = 3m$, what is the value of the potential?

$$\frac{L^2}{27m^2}.$$

Thus, an orbit with $r < 3m$ can escape if and only if $L^2 < 27m^2$. Conversely, an orbit initially in $r > 3m$ will not get trapped if and only if $L^2 > 27m^2$.

Why? Energy $= 1$. $\frac{L^2}{27m^2} < 1 \Rightarrow$ orbit in $r < 3m$ gets over the bump. For $\frac{L^2}{27m^2} > 1$, not enough energy to get trapped.

Consider an orbit coming in from infinity, i.e. $r \to \infty$ as $t \to -\infty$. WLOG assume $L > 0$.

Combining $\# \cancel{\ast}^{-1}$ yields

$$\frac{d\phi}{dr} = \frac{L}{r^2\sqrt{1 - \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right)}}.$$

It follows immediately that

$$\frac{d\phi}{dr} \le \frac{L}{r^2}.$$

Hence $\phi \to \phi_0 =$ constant as $\tau \to -\infty$.

If $L^2 > 27m^2$, this orbit will not get trapped and $r \to \infty$ as $\tau \to \infty$. Thus $\phi \mapsto \phi_1 =$ const as $\tau \to \infty$.

We wish to compute the deflection angle

$$\delta = \phi_1 - \phi_0 - \pi$$

of the orbit from a linear orbit in flat space.

A perturbation analysis of an ode / elliptic integral shows that

$$\delta \approx \frac{4GM}{c^2 r_0}$$

where $r_0 = r_{min} =$ perihelon. This is within 1% of observed data.

In summary, Schwarzschild's null geodesic flow deflects from straight line motion so GR predicts the bending of light.

timelike nonradial geodesics

$$L \neq 0, \quad K = 1 \qquad \dot{r}^2 + \underbrace{\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + 1\right)}_{V(r).} = E^2.$$

$$V(r) = \underbrace{1 - \frac{2m}{r} + \frac{L^2}{r^2}}_{\text{Newtonian.}} - \underbrace{\frac{2mL^2}{r^3}}_{\substack{\text{relativistic} \\ \text{correction}}}$$

Let's look for critical points.

$$V'(r) = \frac{2m}{r^2} - \frac{2L^2}{r^3} + \frac{6mL^2}{r^4}.$$

$$0 = \cancel{2}mr^2 - \cancel{2}L^2 r + 3mL^2$$

$$0 = r^2 - \frac{L^2}{m} r + 3L^2.$$

$$r = \frac{\frac{L^2}{m} \pm \sqrt{\frac{L^4}{m^2} - 4(3L^2)}}{2}$$

$$r_{\text{crit}} = \frac{\frac{L^2}{m} \pm \frac{L^2}{m}\sqrt{1 - 12\frac{m^2}{L^2}}}{2}$$

$$\boxed{r_{\text{crit}} = \frac{L^2}{2m}\left[1 \pm \sqrt{1 - 12\frac{m^2}{L^2}}\right]}$$

If $1 - 12\frac{m^2}{L^2} < 0 \iff L^2 < 12m^2$ no critical points. The analysis of the orbit is then as in the radial case. $E^2 < 1$ vs. $E^2 \geq 1$

Crash Crash/escape

If $L^2 = 12m^2$ we have same situation except $r = 6m$ is an unstable critical point. So $r = 6m$ is an unstable circular orbit.

If $L^2 > 12m^2$ $\exists$ two critical points $r_1 < 6m < r_2$. The $r = r_2$ circular orbit is stable.

Suppose we start an orbit at r with r very near r_2 and initial energy E^2 slightly larger than $V(r_2)$. Then r will oscillate around r_2 between r_{min} and r_{max}.

When $L^2/m^2 >> 12$, then r_2 will be large so these orbits will forever have $r(\tau)$ large and the relativistic correction term will be small. Thus, these orbits with large L^2/m^2 will by close to Keplerian ellipses, with small correction due to the small relativistic correction.

Perihelon Precession

A local minimum in $r(\tau)$ will be called a (local) perihelon. Suppose we start the orbit at perihelon so

$$r = r_{min} \text{ at } \tau = 0, \ \phi = 0.$$

At some proper time $\tau = \tau_0$, and some angle $\phi = \phi_0$, the orbit will reach perihelon again. For a kepler ellipse, we have that $\phi_0 = 2\pi$. We wish to compute the perihelon precession

$$\psi = \phi_0 - 2\pi.$$

We can compute

$$\psi = 2 \int_{r_{min}}^{r_{max}} \frac{L\,dr}{r^2\sqrt{E^2 - V(r)}} - 2\pi.$$

For our $V(r)$, this results in an elliptic integral which cannot be explicitly evaluated. It suffices, however, to calculate the first order term in the parameter M. A (supressed) calculation shows

$$\psi \approx \frac{6\pi G M}{c^2 a(1-e^2)}$$

"... the universe whispered in huse ear..."

where a, e are such that $r_{min} = a(1-e)$

$$r_{max} = a(1+e).$$

2002 Notes
follow ...

V. Exact Solutions

Let g_{ab} be any Lorentz metric on $\mathbb{R}^{1+n}$ and calculate

$$R_{ab} - \frac{1}{2} R g_{ab}.$$

Define the result to be T_{ab} and we have found a solution of Einstein's equation A.

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}.$$

However, the energy-momentum tensor T_{ab} may not satisfy the natural requirements: local causality and energy positivity.

We say that (M, g_{ab}) is an <u>exact solution</u> of Einstein's equation if the field equation is satisfied & the energy momentum tensor T_{ab} satisfies local causality & energy positivity conditions.

~~One can find~~

There are various exact solutions known. A breakdown of solution types in Chapter 5 of Hawking - Ellis goes like this.

A. constant curvature metrics
① Minkowski spacetime
② De Sitter & Antidesitter spacetimes

B. spatially isotropic & homogeneous cosmological models (& certain anisotropic generalizations)
D. Denisenko's paper
- Robertson - Walker spacetimes

C. Spherically symmetric metrics outside of a massive body.
Spanling, Soda Kibitchenko, P. Lee
- Schwarzschild, Reissner - Nordström

D. axially symmetric metrics outside rotating massive body.
Regev
- Kerr

E. Exotic solutions which possess closed timelike curves.
A. Mumi.
- Gödel universe

Note: A. Nachmann asked:

Is there a characterization of metrics g which are compatible with the local causality + energy positivity postulates on T_{ab}?

I still don't know in Spring 2003.

Minkowski spacetime $(\mathbb{R}^4, \eta)$

Coordinatize $\mathbb{R}^4$ as (t, x^1, x^2, x^3) so $t = x^0$.

Then η may be expressed as

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Change to polar coordinates $(x^1, x^2, x^3) \longrightarrow (r, \theta, \phi)$ by writing

$x^0 = t$ unchanged, $x^3 = r\cos\theta$, $x^2 = (r\sin\theta)\cos\phi$, $x^1 = (r\sin\theta)(\sin\phi)$.

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,(d\phi)^2\right).$$ (nice exercise)

In these new coordinates, the metric is singular for $r = 0$ and for $\sin\theta = 0$ but these singularities are merely due to a failure of the coordinates.

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 \{ d\theta^2 + \sin^2\theta \, d\phi^2 \} .$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta .$$

$$dx = \sin\theta\cos\phi \, dr + r\cos\theta\cos\phi \, d\theta - r\sin\theta\sin\phi \, d\phi$$

$$dy = \sin\theta\sin\phi \, dr + r\cos\theta\sin\phi \, d\theta + r\sin\theta\cos\phi \, d\phi$$

$$dz = \cos\theta \, dr - r\sin\theta \, d\theta$$

$$\begin{aligned} dx^2 = {} & \sin^2\theta\cos^2\phi \, dr^2 + r^2\cos^2\theta\cos^2\phi \, d\theta^2 + r^2\sin^2\theta\sin^2\phi \, d\phi^2 . \\ & + 2r\cos\theta\sin\theta\cos^2\phi \, dr \, d\theta \\ & - 2r\sin^2\theta\cos\phi\sin\phi \, dr \, d\phi \\ & - r^2\sin\theta\cos\theta\cos\phi\sin\phi \, d\theta \, d\phi \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} dx^2 + dy^2 + dz^2 = {} & dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2 . \\ & + \left(2r\sin\theta\overset{2}{\cos\phi}\cos\theta\,\cancel{\cos\phi} + \overset{\nearrow 0}{2r\sin\theta\cos\theta\sin^2\phi} \right. \\ & \qquad \left. + 2r\cos\theta\cdot\sin\theta \right) dr \, d\theta \\ & + \left(-2r\sin^2\theta\cos\phi\sin\phi + \overset{\nearrow 0}{2r\sin^2\theta\sin\phi\cos\phi} \right) dr \, d\phi \\ & + \left(-2r^2\cos\theta\sin\theta\cos\phi\sin\phi + 2r^2 c_\theta s_\theta s_\phi c_\phi \right) d\theta \, d\phi \end{aligned}$$

$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2 .$$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, \underset{d\phi^2 .}{d\cancel{\phi}} \right)$$

✓

Schwarzschild Solution

We look for a solution of Einstein's field equations in the empty space surrounding a static star or planet. One has

$$T_{\mu\nu} = 0$$

in that region (so the causality + energy positivity conditions are trivial.)

Assume the star or planet does not rotate very fast ~~by~~ assuming the solution is spherically symmetric. Take

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$$

spherical symmetry of g_{ab} implies

$$g_{02} = g_{03} = g_{12} = g_{13} = g_{23} = 0,$$

and

$$g_{33} = \sin^2\theta \, g_{22} .$$

and time-reversal symmetry

$$g_{01} = 0 .$$

The metric tensor is then specified by writing down the infinitesimal line element

$$ds^2 = - A\, dt^2 + B\, dr^2 + C r^2 \left(d\theta^2 + \cancel{r} \sin^2\theta \, d\phi^2 \right)$$

where $A = A(r)$, $B = B(r)$, $C = C(r)$ are positive functions depending only upon r. At large distance from the planet or star we expect:

$$r \to \infty , \qquad A, B, C \longrightarrow 1$$

(Minkowski-like at large distance)

We can scale away $C(r)$.

Introduce $\tilde{r} = \sqrt{C(r)}\, r$. Then $Cr^2 = \tilde{r}^2$.

We then have

$$B\, dr^2 = B\left(\sqrt{C} + \frac{r}{2\sqrt{C}}\frac{dC}{dr}\right)^{-2} d\tilde{r} := \tilde{B}\, d\tilde{r}^2.$$

reduced form

$$\Longrightarrow \boxed{ds^2 = -A\,dt^2 + \tilde{B}\,d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

Henceforth, drop the ~. We have $A, B \to 1$ as $r \to \infty$. The signature of the metric is $(-,+,+,+)$ so

$$A > 0, \quad B > 0.$$

The metric is specified if we choose A, B. So, the affine connection, e.g. the Christoffel coefficients are determined and hence the metric compatible covariant derivative and the curvature tensor are all specified by A, B.

We are demanding that g satisfy Einstein's field equations in vaccuum

$$R_{ab} - \frac{1}{2} g_{ab} R = 0$$

and this should pin down our choice of A, B.

Note that if we apply g^{ab} to this field equation

$$g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R = 0$$

$$R - \frac{1}{2} R = 0$$

$$\frac{1}{2} R = 0$$

$\Rightarrow$ solution is ~~Ricci flat~~ scalar-curvature-flat.

Furthermore, using the field equation shows solution is Ricci flat:

$$R_{ab} = 0.$$

My task for Wednesday is to express R_{ab} in terms of A, B for a metric of the reduced form and use the equation $R_{ab} = 0$ to solve for A, B.

Schwarzschild Solution

$(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, is a global choice of coordinates.

We seek spherically symmetric solutions to the vacuum einstein eqs.

simplification due to vacuum assumption

$$R_{ab} - \frac{1}{2} R g_{ab} = 0.$$

since $T_{\mu\nu} = 0$, this equation can be reduced.

$$g^{ab} R_{ab} - \frac{1}{2} g^{ab} R g_{ab} = 0$$

$$R - \frac{1}{2} R = 0 \Rightarrow R = 0$$

$$\Rightarrow \boxed{R_{ab} = 0}$$ is Einstein's eq. in vaccuum setting.

simplification due to spherical symmetry assumption

$$g_{02} = g_{03} = g_{12} = g_{13} = g_{23} = 0$$

$$g_{33} = \sin^2\theta \, g_{22}.$$

time reversal symmetry:

$$g_{01} = 0.$$

Subject to these symmetry assumptions, our task reduces to find a metric in the form:

$$ds^2 = -A\,dt^2 + B\,dr^2 + C r^2 (d\theta^2 + r^2 \sin^2\theta \, d\phi^2)$$

$$A = A(r), \quad B = B(r), \quad C = C(r)$$

As $r \to \infty$, $A \to 1$, $B \to 1$, $C \to 1$.

$\tilde{r} = \sqrt{c(r)}\, r$ so $c r^2 = \tilde{r}^2$.

$$B(dr)^2 = B\left(\sqrt{c} + \frac{r}{2\sqrt{c}}\frac{dc}{dr}\right)^{-2}(d\tilde{r})^2 := \tilde{B}(d\tilde{r})^2.$$

why ?

$=$ ~~[illegible]~~

$$d\tilde{r} = \frac{r}{2\sqrt{c}}\frac{dc}{dr}\,dr + \sqrt{c}\,dr$$

$$= \left(\frac{r}{2\sqrt{c}} + \sqrt{c}\right) dr. \quad \checkmark.$$

spherical symmetry. reduced form of metric.

$$(ds)^2 = -A\,dt^2 + \tilde{B}\,(d\tilde{r})^2 + r^2\left((d\theta)^2 + (\sin^2\theta)(d\phi)^2\right)$$

drop tildes henceforth.

We have recast the problem of determining the spacetime metric g, which involves 10 unknown functions, into a problem involving 2 unknown functions by making symmetry assumptions about the solution we seek. Thus we seek 2 functions of the variable r instead of 10 depending upon 4 coordinates.

Our goal is to choose $A = A(r)$, $B = B(r)$ so that

$$\boxed{R_{ab} = 0.},$$

which implies the metric obtained ⓢ vacuum einstein eqs.

We seek a spacetime metric w. signature $(-,+,+,+)$
so $A > 0$, $B > 0$.

- Christoffel coeffecients in terms of metric:

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \}.$$

- Riemann Tensor in terms of Christoffel symbols:

$$- R_{abd}{}^c = (\partial_a \Gamma^c{}_{bd} + \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e_{bd} - \Gamma^c_{bf} \Gamma^f_{ad})$$

- We seek A, B s.t.

$$- R_{abd}{}^b = 0.$$

To carry out the task of finding A, B we proceed with an analysis of geodesics following the lecture notes of t'Hooft.

The geodesic equation

$$\ddot{x}_\mu + \Gamma^\mu{}_{\kappa\lambda} \dot{x}^\kappa \dot{x}^\lambda = 0$$

involves the Christoffel coefficients Γ! If we know all the geodesics in our spacetime, we will also know $\Gamma^\mu{}_{\kappa\lambda}$, and will then be able to solve for A, B.

The variational principle for a geodesic is

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau$$

where τ is an arbitrary parametrization of the curve. If we choose τ to coincide with s then $d\tau = ds$ and $\sqrt{\ }$-term is 1. Thus, the variational problem is equivalent to

$$0 = \delta \int g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\, ds$$

For our reduced form metric, the integrand is

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = - A \dot{t}^2 + B \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2$$

where $\cdot$ stands for $\frac{d}{ds}$. Define this quantity to be $F(s)$.

$$\delta \int F(s)\, ds = 0$$

The associated EL equation is

$$\cdot = \frac{d}{ds}$$

$$\frac{d}{ds} \frac{\partial F}{\partial \dot{x}^\mu} = \frac{\partial F}{\partial x^\mu}$$

$\mu = 0$:

$$' = \frac{d}{dr}$$

$$\frac{d}{ds}\left(-2A\dot{t}\right) = 0$$

$$-2A\ddot{t} - 2\left(A'(r)\cdot\dot{r}\right)\dot{t} = 0$$

$$\ddot{t} + \frac{1}{A}\left(A'(r)\cdot\dot{r}\right)\dot{t} = 0$$

Comparing w. geodesic equation for $\mu = 0$.

$$\ddot{t} + \Gamma^{0}_{\kappa\lambda}\,\dot{x}^{\kappa}\,\dot{x}^{\lambda} = 0.$$

only get contribution for $\kappa = 0,\ \lambda = 1$
or $\kappa = 1,\ \lambda = 0$

$$\Rightarrow \boxed{\Gamma^{0}_{01} = \Gamma^{0}_{10} = \frac{A'}{2A}}$$

$\mu = 1$:

$$\frac{d}{ds}\left(2B\dot{r}\right) = 2r\,\dot{\theta}^{2} + 2r\sin^{2}\theta\,\dot{\phi}^{2} + A'(r)\dot{r}$$

$$2B'(r)\,\dot{r}\,\dot{r}^{2} + 2B\ddot{r} =$$

$$\ddot{r} + \frac{\cancel{2}B'}{B}\dot{r}^{2} - \frac{r}{B}\dot{\theta}^{2} - \frac{r\sin^{2}\theta}{B}\dot{\phi}^{2} = 0$$

$$\Rightarrow \Gamma^{1}_{00} = \frac{A'}{2B}$$

$$F(s) = -A(r)\dot{t}^2 + B(r)\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2.$$

$\underline{\mu = 1}$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{r}}\right) = \frac{\partial F}{\partial r}$$

$$\frac{d}{ds}\left(B(r)\,2\dot{r}\right) = \cancel{2r} - A'(r)\dot{t}^2 + B'(r)\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2.$$

$$\cancel{2}B'(r)\dot{r}^2 + 2B(r)\ddot{r} = -A'(r)\dot{t}^2 + \cancel{B'(r)\dot{r}^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2.$$

$$\ddot{r} + \frac{B'}{2B}\dot{r}^2 + \frac{A'}{2B}\dot{t}^2 - \frac{r}{B}\dot{\theta}^2 - \frac{r\sin^2\theta}{B}\dot{\phi}^2 = 0.$$

$$\Longrightarrow \quad \Gamma^1_{00} = \frac{A'}{2B} \qquad \Gamma^1_{11} = B'/2B$$

$$\Gamma^1_{22} = -r/B \qquad \Gamma^1_{33} = -\frac{r}{B}\sin^2\theta$$

and $\Gamma^1_{\mu\neq\nu} = 0.$

Similarly,

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

and others $= 0.$

A consequence of this analysis is that

$$\Gamma^{\mu}_{\mu 1} = \frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}$$

$$\Gamma^{\mu}_{\mu 2} = \cot\theta .$$

We also have $\sqrt{-g} = r^2 \sin\theta \sqrt{AB}$ and it is a fact that

$$\Gamma^{\mu}_{\mu\beta} = \frac{\partial_\beta \sqrt{-g}}{\sqrt{-g}} = \partial_\beta \log \sqrt{-g} .$$

Since $R^{\nu}{}_{\kappa\lambda\alpha} = \partial_\lambda \Gamma^{\nu\lambda}_{\kappa\alpha} - \partial_\alpha \Gamma^{\nu\lambda}_{\kappa\lambda} + \Gamma^{\nu\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\kappa\alpha} - \Gamma^{\nu\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\kappa\lambda}$

we can now express $R_{\kappa\alpha} = R^{\lambda}{}_{\kappa\lambda\alpha} = 0.$

$$R_{\mu\nu} = -\partial_\mu\partial_\nu(\log\sqrt{-g}) + \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\beta}_{\alpha\mu}\Gamma^{\alpha}_{\beta\nu} + \Gamma^{\alpha}_{\mu\nu}\partial_\alpha(\log\sqrt{-g}) = 0$$

$\mu = 0,\ \nu = 0.$

$$R_{00} = 0 + \partial_1 \Gamma^{1}_{00} - \Gamma^{\beta}_{\alpha 0}\Gamma^{0}_{\beta 0} + \Gamma^{\alpha}_{00}\,\partial_\alpha(\log\sqrt{-g})$$

$$= \partial_1 \Gamma^{1}_{00} - 2\Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{00}\,\partial_1(\log\sqrt{-g}).$$

$$= \left(\frac{A'}{2B}\right)' - \cancel{2}\,\frac{A'}{\cancel{2}B}\frac{A'}{2A} + \frac{A'}{2B}\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right)$$

⊛ $$= \cancel{\frac{1}{2B}}\left(A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right) = 0.$$

From the reduced form we have $\sqrt{-g} = r^2 \sin\theta \sqrt{AB}$.

It is a fact that

$$\Gamma^{\mu}{}_{\mu\beta} = \frac{\partial_\beta (\sqrt{-g})}{\sqrt{-g}} = \partial_\beta \log (\sqrt{-g}).$$

Our calculations have shown that

$$\Gamma^{\mu}{}_{\mu 1} = \frac{A'}{2A} + \frac{B'}{B} + \frac{2}{r}$$

$$\Gamma^{\mu}{}_{\mu 2} = \cot\theta.$$

$$-R_{ad} = -R_{abc}{}^{b} = \left(\partial_a \underline{\Gamma^{b}{}_{bd}} + \partial_b \Gamma^{b}{}_{ad} + \Gamma^{b}{}_{ae}\Gamma^{e}{}_{bd} - \underline{\Gamma^{b}{}_{bf}}\,\Gamma^{f}{}_{ad}\right)$$

$$= \partial_a \partial_d (\log \sqrt{-g}) + \partial_b \Gamma^{b}{}_{ad} + \Gamma^{b}{}_{ae}\Gamma^{e}{}_{bd} - \partial_f (\log \sqrt{-g})\, \Gamma^{f}_{ad}.$$

explicitly

$$-R_{00} = \partial_0 \partial_0 (\log \sqrt{-g}) + \partial_b \Gamma^{b}{}_{00} + \Gamma^{b}{}_{0e}\Gamma^{e}{}_{b0} - \Gamma^{f}_{00}\, \partial_f (\log \sqrt{-g}).$$

$$= 0 + \partial_1 \Gamma^{1}{}_{00} + \Gamma^{1}{}_{00}\Gamma^{0}{}_{10} + \Gamma^{0}{}_{01}\Gamma^{1}{}_{00}$$

Similar calculation unravells $R_{11} = 0$ to read

(#) $$\cancel{\frac{1}{2A}}\left(-A'' + \frac{A'B'}{2B} + \frac{(A')^2}{2A} + \frac{2AB'}{rB} \right) = 0$$

$$\Longrightarrow \quad \frac{2A'B}{rB} + \frac{2AB'}{rB} = 0$$

$$\frac{2}{rB}(AB)' = 0 .$$

$$\Longrightarrow \quad (AB)' = 0 \quad \Longrightarrow \quad \boxed{AB = \text{const.}}$$

Both $A + B \longrightarrow 1$ as $r \longrightarrow \infty$

so Const $= 1$.

$$\boxed{AB = 1}$$

Finally, $R_{22} = 0$ unravels to read.

$$-\frac{\partial}{\partial\theta}\cot\theta - \left(\frac{r}{B}\right)' + \frac{2}{B} - \cot^2\theta - \frac{r}{B}\left(\frac{2}{r} + \overset{\nearrow 0}{\frac{(AB)'}{2AB}}\right) = 0$$

$$+\cancel{\frac{2}{B}} \qquad\qquad -\cancel{\frac{2}{B}}$$

$$\Longrightarrow \quad \left(\frac{r}{B}\right)' = 1$$

$$\Longrightarrow \quad \frac{r}{B} = r - \underbrace{2M}$$ integration constant.

$$\boxed{\frac{1}{B} = 1 - \frac{2m}{R} = A.}$$

Hence, we have found A & B and see that

$$+ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

is a metric which Ⓢ vacuum einstein eqs.

What it means is the topic of the next lecture.

Exterior region $r > 2m$. & interior region $r < 2m$ must be considered separately since spacetime is hypothesized to be Lorentz. We focus on exterior region and will later consider extension.

The Schwarzschild geometry is the vaccuum spacetime outside a spherical star. It has line element

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

To understand these geometry we investigate its timelike and null geodesics. These are the paths of massive and massless particles.

parity reflection symmetry:

$\theta \longmapsto \pi - \theta$ leaves ds^2 unchanged.

If initial position & tangent vector of a geodesic lie in the "equatorial plane" $\theta = \pi/2$ then the entire geodesic must stay in this "plane"

$$F(s) = -A(r)\,\dot{t}^2 + B(r)\,\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2.$$

geodesic equation is described variationally:

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^\mu}\right) = \frac{\partial F}{\partial x^\mu}.$$

$\mu = 2 : \theta$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\theta}}\right) - \frac{\partial F}{\partial \theta}$$

$$\frac{d}{ds}\left(2r^2\dot{\theta}\right) = r^2\, 2\sin\theta\cos\theta\,\dot{\phi}^2$$

$\theta = \frac{\pi}{2}$ at $t = 0$

$\dot{\theta} = 0$ at $t = 0$ $\leftarrow$ No loss in assuming this. So all geodesics are motions in (some) equatorial plane.

$2r^2\dot{\theta} = \text{const.}$

$\dot{\theta} = 0$ at $t = 0 \Rightarrow \text{const.} = 0. \Rightarrow \dot{\theta} = 0$ for all time

Consider a geodesic curve with tangent vector u^μ and with parametrization given by τ

$$u^\mu = \frac{dx^\mu}{d\tau} = \dot{x}^\mu .$$

$$u^\mu \nabla_\mu u^\beta = 0 .$$

- tangent vector is parallel translated along curve.

For timelike geodesics we may choose τ to be proper time. and for null geodesics τ is an affine parameter.

Since $\theta = \frac{\pi}{2}$, we have

$$-K = g_{ab} u^a u^b = -\left(1 - \frac{2m}{r}\right) \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2$$

where

$$K = \begin{cases} 1 & \text{timelike geodesics} \\ 0 & \text{null geodesics} . \end{cases}$$

The metric is invariant under the transformation $t \mapsto t + \alpha$ which implies that the vector field

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a = (1, 0, 0, 0)$$

is Killing. The quantity

$$E = -g_{ab} \xi^a u^b = \left(1 - \frac{2m}{r}\right) \dot{t}$$

is a constant under the geodesic motion.

Similarly, the metric is invariant under $\phi \mapsto \phi + \beta$.

$\Rightarrow \psi^a = \left(\frac{\partial}{\partial \phi}\right)^a = (0, 0, 0, 1)$ is killing.

$$\Rightarrow \qquad L = g_{ab}\,\psi^a\,u^b = r^2 \dot{\phi}$$

is a constant under the geodesic motion.

We now know that

$$\dot{t} = \frac{E}{(1-\frac{2m}{r})}$$

$$\dot{\theta} = 0$$

$$\dot{\phi} = \frac{L}{r^2}$$

and matters hinge on understanding the radial motion of the particle. We substitute these expressions into the equation

$$-K = g_{ab}\,u^a u^b$$

to obtain

$$-K = -\left(1-\frac{2m}{r}\right)\left(\frac{E}{(1-\frac{2m}{r})}\right)^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\left(\frac{L}{r^2}\right)^2$$

$$= -\frac{E^2}{(1-\frac{2m}{r})} + \frac{\dot{r}^2}{(1-\frac{2m}{r})} + \frac{L^2}{r^2}$$

$$\left(-K - \frac{L^2}{r^2}\right)\left(1-\frac{2m}{r}\right) = -E^2 + \dot{r}^2$$

~~$\frac{1}{2}\dot{r}^2 \quad \frac{1}{2}($~~

$$\Rightarrow \quad \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(K + \frac{L^2}{r^2}\right) = \frac{1}{2}E^2 .$$

(We interpret this as a mechanical system.

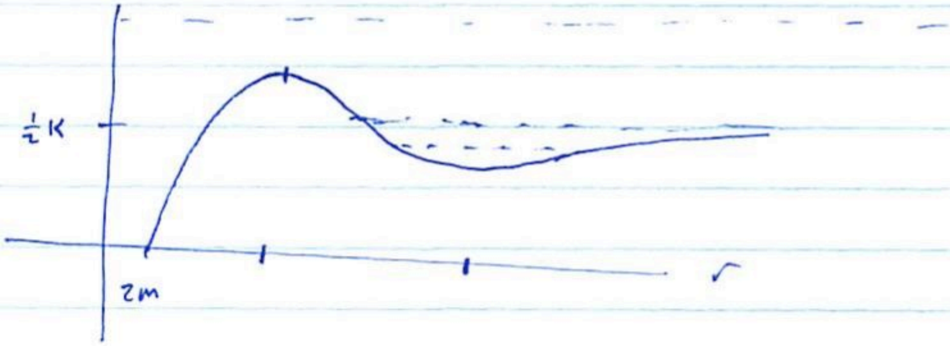
$$\frac{1}{2}\dot{x}^2 + V(x) = \text{Total energy.})$$

The "effective potential" is

$$V = \frac{1}{2}K - \underbrace{K\frac{M}{r}}_{\text{newtonian term}} + \underbrace{\frac{L^2}{2r^2}}_{\text{centrifugal barrier}} - \underbrace{\frac{ML^2}{r^3}}_{\text{new term due to GR.}}.$$

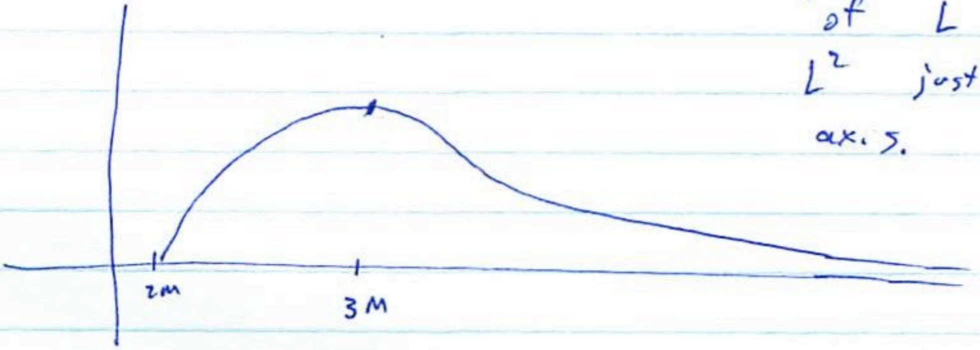
Typical forms

$K = 1$



$K = 0$

Shape is independent of L in this case, L^2 just scales vertical axis.



∃ unstable circular orbits of photons at $r = 3m$.

Write $\dot{\phi} = \frac{L}{r^2}$?

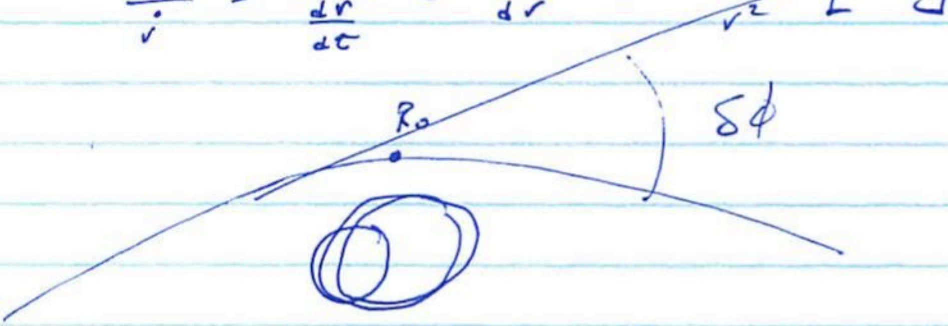
$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2} = \frac{1}{2}E^2.$$

Ⓢ for $\dot{r}$

$$\dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}$$

$$\dot{r} = \left[E^2 - \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}\right]^{+\frac{1}{2}}$$

$$\frac{\dot{\phi}}{\dot{r}} = \frac{\frac{d\phi}{d\tau}}{\frac{dr}{d\tau}} = \frac{d\phi}{dr} = \frac{L}{r^2}\left[\quad\right]^{-\frac{1}{2}}.$$



Some calculus reveals the bending of light in the Schwarzschild geometry.

$$\delta\phi \sim \frac{4GM\,\textcircled{E}}{\textcircled{L}\,c^2}$$

photon
○ particle properties

$$\cancel{\delta\phi \sim \frac{4GM}{3^{3/2}M c^2}} \ ?$$

$$\sim \frac{4G}{3^{3/2}c^2}$$

Schwarzschild spacetime extended from $r > 2m$ into $r < 2m$.

metric $\quad ds^2 = -(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + d\Omega^2$

where $d\Omega^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$.

Lorentz requirement of spacetime fails for $r=0$, $r=2m$. excise $r=0$, $r=2m$ from spacetime. We obtain 2 components:

interior region: $\quad 0 < r < 2m$

exterior region: $\quad 2m < r$.

We seek to extend the exterior region ~~beyond~~ into $r \le 2m$. Recall that (M', g') is a C^r-extension of (M, g) if $\exists$ C^r isometric embedding $\mu : M \to M'$.

The bad behavior of metric coefficients at $r = 2m$ suggests this may not be impossible. However, the metric coefficients don't always reveal the story so clearly.

Example $\quad$ 2-d metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2,$$

defined over $-\infty < x < 0$, $0 < t < \infty$. $\exists$ singular behavior in the coefficient $-\frac{1}{t^4}$ as $t \to 0$.

Change coordinates $\quad t' = \frac{1}{t}$, $\quad x' = x$.

$dt' = -\frac{1}{t^2}dt$

$$ds^2 = (dt')^2 + (dx')^2.$$

$t \to 0$ corresponds to $t' \to \infty$.

is defined for $0 < t' < \infty$, $-\infty < x' < \infty$.

As $t \to 0$, $t' \to \infty$ So the "singularity" at $t = 0$ corresponds to $t' \to \infty$ and merely corresponds to a representation of an infinite region of spacetime using the finite range of a coordinate. This metric is <u>geodesically complete</u> as $t \to 0$ ($t' \to \infty$). However, as $t \to \infty$, ($t' \to 0$) this metric is not geodesically complete. But, we can extend the spacetime $t' > 0$ into $t' \leq 0$ by simply adding the $t' \leq 0$ region of spacetime.

A calculation shows that, although the metric is singular at $r = 2m$ in the Schwarzschild coordinates (t, r, θ, ϕ), no scalar polynomial of the curvature tensor and the metric explodes as $r \to 2m$. This suggests $r = 2m$ is not a physical singularity but merely a problem with our coordinates.

Define

$$r^* = \int \frac{dr}{\left(1 - \frac{2m}{r}\right)} = r + 2m \log (r - 2m).$$

$$\uparrow$$

confirm

$$\underline{1} + 2m \frac{1}{r - 2m} = \frac{r - 2m + 2m}{r - 2m} = \frac{r}{r - 2m} \checkmark$$

Now introduce the advanced null coordinate

$$v = t + r^*$$

and the retarded null coordinate

$$w = t - r^*.$$

Change coordinates:

$$\underset{\text{Schwarzschild coords}}{(t, r, \theta, \phi)} \longrightarrow \underset{\text{Eddington - Finkelstein coords.}}{(v, r, \theta, \phi)}$$

Find ds^2

See

http://casa.colorado.edu/~ajsh/schwp.html

See also
www.fourmilab.ch/gravitation/orbits.

$$r^* = r + 2m \log(r-2m)$$

$$t = v - r^*$$

$$dt = dv - dr - 2m \frac{1}{(r-2m)} dr$$

$$= dv - \left[\frac{r-2m}{r-2m} + \frac{2m}{r-2m} \right] dr$$

$$= dv - \frac{1}{1-\frac{2m}{r}} dr.$$

$$ds^2 = -\left(1-\frac{2m}{r}\right) dt^2 + \cancel{\left(1-\frac{2m}{r}\right)^{-1} dr^2} + \text{same}.$$

$$dt^2 = dv^2 + 2\left(\frac{1}{1-\frac{2m}{r}}\right) dv\,dr + \frac{1}{(1-\frac{2m}{r})^2} dr^2$$

$$-\left(1-\frac{2m}{r}\right) dt^2 = -\left(\left(1-\frac{2m}{r}\right) dv^2 + 2\,dv\,dr - \cancel{\frac{1}{(1-\frac{2m}{r})} dr^2}.\right.$$

$$\boxed{ds^2 = -\left(1-\frac{2m}{r}\right) dv^2 + 2\,dv\,dr + d\Omega^2.}$$

$$\begin{bmatrix} -\left(1-\frac{2m}{r}\right) & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

nonsingular
w. negative
determinant, makes
sense for $0 < r < \infty$.

A similar extension of Schwarzschild geometry is obtained from

$$(t, r, \theta, \phi) \longrightarrow (w, r, \theta, \phi).$$

We perform a simultaneous combination of the v & w Eddington-Finkelstein extension by defining the Kruskal extension. Moreover, the Kruskal extension is the unique analytic and locally nextendible extension of the Schwarzschild solution.

Kruskal Extension

$$(t, r, \theta, \phi) \longleftrightarrow (v, w, \theta, \phi)$$

Schwarzschild coords

$$-\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + d\Omega^2$$

$$v = t + r^*$$

$$w = t - r^*.$$

$$-\left(1-\frac{2m}{r}\right)dv\,dw + \text{"}\ d\Omega^2$$

Introduce an "arbitrary" reparametrization of the v, w coordinates on the (v, w) plane.

$$v' = v'(v) \quad , \quad w' = w'(w)$$

$$(v, w) \longrightarrow (v', w').$$

$$-\left(1-\frac{2m}{r}\right)\frac{dv}{dv'}\frac{dw}{dw'}\,dv'dw' + d\Omega^2.$$

$$(v', w') \longrightarrow (x', t')$$

$$x' = \frac{1}{2}(v' - w') \qquad dx' = \frac{1}{2}(dv' - dw')$$

$$t' = \frac{1}{2}(v' + w') \qquad dt' = \frac{1}{2}(dv' + dw').$$

$$(dx')^2 - (dt')^2 = \frac{1}{4}\left((dv')^2 - 2dv'dw' + (dw')^2\right.$$

$$+ \left(\frac{1}{4} \cdots\right.$$

$$\mathbin{/\!/}$$

$$dv'dw'.$$

a

(x', t', θ, ϕ)

$$ds^2 = F^2(t', x')\left(-(dt')^2 + (dx')^2\right) + r^2(t', x')\left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

Kruskal chose $v' = \exp(v/4m)$, $w' = -\exp(-w/4m)$

with r determined implicitly by

$$r: \quad (t')^2 - (x')^2 = -(r-2m)\exp(r/2m).$$

and where

$$F^2 = \exp(-r/2m) \cdot 16\, m^2/r.$$