MATH 279: NONLINEAR WAVE EQUATIONS

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Abstract. These are notes in progress a graduate course in mathematics. Any suggestions or comments are welcome and can be sent to colliand@math.berkeley.edu.

Contents

1. **Introduction**

1.1. **Course Overview.** There are three main topics to be addressed in this course on nonlinear (usually dispersive) wave equations:

(1) Some **nonlinear wave equations will be derived** from physical and general principles. These derivations are intended to provide motivation

for studying these equations and to reveal their "universal" applicability in wave propagation problems.

(2) A special class of nonlinear wave equations, known as **completely integrable systems** will be studied. These equations often possess special travelling wave solutions called *solitons* and can be solved via the *Inverse Scattering Transform.*

(3) **Analytical techniques** for solving certain nonintegrable nonlinear wave equations will be described. Techniques for investigating qualitative properties of solutions will also be introduced.

In each of the three topics, dispersive and nonlinear phenomena will be emphasized, sometimes rigorously and sometimes heuristically. Also, the Hamiltonian structure of these equations will play a basic role during the three main topics.

1.2. **Literature.** A (growing) list of references can be found at the end of this (also growing) document. The derivations of nonlinear wave equations will be taken from the lecture notes of Alan Newell [25] and the recent book of Catherine and Pierre-Louis Sulem [29]. Integrable models will be discussed following the textbook of George Lamb [20] with extensions from [25] and elsewhere. The analytical techniques topic will be presented from various recent research papers and [29]. Hopefully, these notes will also be useful.

1.3. **Wave Propagation.** This section will mostly follow Logan [22] Chapter 6 (first section). We will briefly describe basic properties of wave propagation.

- What is a wave?

A *wave* is an identifiable signal or disturbance in a medium, propagated in time, carrying energy with it. Examples include sound waves, electromagnetic waves, surface waves on water, earthquake waves, etc. A simple model of a wave is

$$u(x,t) = f(x - ct) \tag{1.1}$$

where f is some given function and c is a constant. The graph of the function f is shifted to the right at speed c and this completely describes the function u. Notice that the "shape" of this wave moves to the right without any distortion. This is a special case. Some wave propagations may distort the shape of the disturbance due to nonlinear or other effects.

Notice that (1.1) satisfies the PDE

$$u_t + cu_x = 0. \tag{1.2}$$

This PDE is a one dimensional *transport equation.* Similarly, $f(x + ct)$ solves $u_t + cu_x = 0$.

There are many motivations, including the theory of Fourier series and the output of oscillators, to consider waves of the form

$$u(x,t) = Ae^{i(kx-\omega t)}. \tag{1.3}$$

The constant A is the *amplitude.* The parameter k is the *wave-number* and ω is the *frequency.* As a function of x, u displays a characteristic spatial oscillation at scale $\sim \frac{1}{k}$. So, very large wave-numbers correspond to very small scale oscillation in space or small scale structure in the solution.

Suppose now that $\omega = \omega(k)$. That is, the temporal oscillation is linked to the spatial oscillation. For example, suppose we experimentally determined

ω as a function of k for some system. One possibility is $\omega(k) = ck$. In this case, the time frequency of oscillation is the same as the space frequency oscillation for all values of k. So, $u(x,t) = e^{ik(x-ct)}$ and this wave is a special case of (1.1). Moreover, for certain types of functions f, we can write

$$f \sim \sum a_k e^{ikx}. \tag{1.4}$$

If each of the exponentials propagate in time according to $e^{ikx} \longmapsto e^{ik(x-ct)}$ with no interactions among these pieces then $f \longmapsto u(x,t)$ where

$$u(x,t) = \sum a_k e^{ik(x-ct)} \sim f(x-ct). \tag{1.5}$$

So, up to details regarding the representation of f using a sum of exponentials, we see the special case involving the wave of the form (1.3) with the assumption that $\omega(k) = ck$ reproduces the example (1.1).

- Dispersion
  Another possibility is for $\omega(k) = -k^3$ and there are many other possibilities to consider as well. Let's now assume for the moment that $\omega(k) = -k^3$ in (1.3). Notice that

$$(\partial_t + \partial_x^3)e^{i(kx+k^3t)} = 0 \tag{1.6}$$

Again, if we consider an initial profile f as given in (1.4) and assume that each of the exponential waves propagates independently of the others according to $e^{ikx} \longmapsto e^{i(kx+k^3t)}$, we find $f \longmapsto u(x,t)$ where

$$u(x,t) = \sum a_k e^{i(kx-k^3t)}. \tag{1.7}$$

We no longer have $u(x,t) \sim f(x-ct)$ for any value of c. The shape of the wave is distorted because different wave-numbers k propagate at different speeds. Notice that we can write (1.3) in this case as $e^{ik(x+k^2t)}$ which says that e^{ikx} propagates to the left at speed k^2. This is an example of *dispersion*.

The general definition of a *dispersive wave* of the form (1.3) is that $\omega(k)$ is real and satisfies $\omega''(k) \neq 0$. Dispersive waves occur in water waves and many other physical systems. We have seen that the effect of dispersion is that wave-numbers propagate at different speeds and this leads to a broadening of pulses; wave packets tend to spread out.

Exercise 1. **A.** *Find a PDE satisfied by functions of the form* $e^{i(kx-\omega(k)t)}$ *in case the dispersion relation is* $\omega(k) = k^m,\ m \in \mathbb{N}$.
B. *Find a PDE satisfied by funtctions of the form* $e^{i(kx+ny-\omega(k,n)t)}$ *where the dispersion relation is*

$$\omega(k,n) = k^3 \pm \frac{n^2}{k}. \tag{1.8}$$

This is the dispersion relation for the Kadomtsev-Petviashvili equation [12].

- Nonlinearity
  Consider the initial-value problem

$$\begin{cases} u_t + cu_x = 0, x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{1.9}$$

We have seen that the solution is given by $u(x,t) = \phi(x - ct)$. Let's gain some intuition from this problem which will generalize to some more complicated versions of (1.9). Consider the curve in the (x,t)-plane given by $x(t) = ct + x_0$. This curve is a straight line of slope c passing through the point $(x_0, 0)$. Now, calculate

$$\frac{d}{dt}u(x(t),t) = u_t + u_x x'(t) = u_t + cu_x = 0. \tag{1.10}$$

So, the function u does not change value along the curve $x(t)$ and this is true for the whole family of curves indexed by x_0. Evidently, these curves $x(t) = ct + x_0$ are quite special and are called *characteristic curves.*

Now, suppose we consider a more complicated version of (1.9):

$$\begin{cases} u_t + c(x,t)u_x = 0, x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases} \tag{1.11}$$

where c now depends upon x, t. We no longer have the explicit representation $u(x,t) = \phi(x - c(x,t)t)$ because the chain rule will bring out derivatives of c which don't cancel. However, we can cleverly use the characteristic curves to solve the initial value problem.

Consider the family of curves defined by the differential equation

$$\frac{dx}{dt} = c(x,t). \tag{1.12}$$

Notice this family is indexed by the integration constant. Along a member of the family, we have

$$\frac{du}{dt} = u_t + u_x\frac{dx}{dt} = u_t + c(x,t)u_x = 0. \tag{1.13}$$

This tells us that u does not change its value along the characteristic curves. So, if we knew the value at any point along a characteristic curve, we'd know the value of u all along the curve. But we know u along $t = 0$ and can therefore determine a value along each of the curves in the family and hence find the value of u at any point (x,t).

Finally, let's see that this method based on special curves can also work (in principle) for a nonlinear generalization of (1.9). Consider the initial value problem

$$\begin{cases} u_t + c(u)u_x = 0, x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases} \tag{1.14}$$

where c can now depend (nonlinearly) on the solution u. Define the *characteristic curves* as the solutions of the differential equation

$$\frac{dx}{dt} = c(u). \tag{1.15}$$

These curves can be indexed by a constant of integration, once again. Calculating $\frac{du}{dt}$ again reveals that u does not change along the characteristic curves so all we need is to move back along the characteristic curve to a point on $t = 0$ to determine the value of u along the curve.

However, characteristic curves may intersect. What can we say in that circumstance? Does the function u take on both values? We will see that this problem prevents the *method of characteristics* from describing the solution for all time. However, the method validates our intuition that signals should propagate under (1.14) at a speed determined by $c(u)$ or $c(\phi)$ in a certain sense. Thus, we can see that nonlinearity can have the effect of steepening a front and causing a pulse to compress.

- Balance
  It is remarkable that nonlinearity and dispersion can balance exactly allowing for pulse shapes to propagate as in (1.1) while dispersion tries to split it apart and nonlinearity keeps it together. This competition is a principal object of study in this course.

2. First Order PDE

2.1. **Method of Characteristics.** We have already discussed this method for studying transport equations in 1 dimension $u_t + c(u, x, t)u_x = 0$. The method may be developed for general first-order PDE with some gain in intuition. We follow the presentation of Section 3.2 from [9].

- General Derivation
  Consider the general first-order PDE

$$F(Du, u, x) = 0 \text{ in } U \subset \mathbb{R}^n \tag{2.1}$$

subject to the boundary condition

$$u = g \text{ on } \Gamma \tag{2.2}$$

where $\Gamma \subset \partial U$ and $g : \Gamma \to \mathbb{R}$ are given. Suppose F, g are nice smooth functions.

Here is our method for attacking the problem (2.1), (2.2). Suppose u solves (2.1), (2.2). We want to know the value of u at a given point $x \in U$. We hope to find some special curve lying in U connecting x with a point $x^0 \in \Gamma$ and along the curve we hope to calculate the values of u. Evidently, this curve must be quite **special**. How do we find it?

Describe the curve we seek parametrically $s \longmapsto (x^1(s), x^2(s), \dots, x^n(s))$ for the parameter s lying in some interval of $\mathbb{R}$. Assuming u is C^2, we also define

$$z(s) = u(\mathbf{x}(s)). \tag{2.3}$$

Set

$$\mathbf{p}(s) = Du(\mathbf{x}(s)) \tag{2.4}$$

so $p^i(s) = u_{x_i}(\mathbf{x}(s)),\ i = 1, \dots, n$. We wish to choose $\mathbf{x}(s)$ in such a way that we can compute $z(\cdot)$ and $\mathbf{p}(\cdot)$. Let's differential p^i w.r.t. s:

$$\dot{p^i}(s) = \sum_{i=1}^{n} u_{x_i x_j}(\mathbf{x}(s))\dot{x^j}(s).$$

The appearance of second deravitives does not look very good since we are studying first order PDE. But we can also differentiate the PDE (2.1) w.r.t. x_i to get

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x)u_{x_i x_j} + \frac{\partial F}{\partial z}(Du, u, x)u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0 \tag{2.5}$$

The "dangerous" 2nd derivative terms can be made to disappear provided we **choose**

$$\dot{x^j}(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)),\ j = 1, \ldots, n. \tag{2.6}$$

(This is the main step of the method.) Notice this ODE involves $\mathbf{x}$ as well as $\mathbf{p}$ and z. We evaluate (2.5) along $\mathbf{x}(s)$, assuming $\mathbf{x}(s)$ satisfies (2.6) to obtain

$$\sum_{j=1}^n \tfrac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) u_{x_i x_j}(\mathbf{x}(s)) + \tfrac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) + \tfrac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0.$$

$$\dot{p^i}(s) = -\tfrac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \tfrac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s).$$

Finally, we differentiate (2.3) to get

$$\begin{aligned} \dot{z}(s) &= \sum_{j=1}^n u_{x_j}(\mathbf{x}(s)) \dot{x}^j(s) \\ &= \sum_{j=1}^n p^j(s) \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{aligned}$$

by the choice of $\dot{\mathbf{x}}$ and the definition of $\mathbf{p}$.

Summarizing, we have found a system of $2n+1$ equations

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \mathbf{p}(s) \\ \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{2.7}$$

- F linear

Suppose for $x \in U$ that

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x) u(x) = 0.$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x) z$ and so

$$D_p F = \mathbf{b}(x). \tag{2.8}$$

Therefore, we set

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)). \tag{2.9}$$

In this case, we have no dependence upon $\mathbf{p}$ and z. The equation for z in this case is

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s). \tag{2.10}$$

Since $\mathbf{p}(s) = Du(\mathbf{x}(s))$, the PDE lets us write

$$\dot{z}(s) = -c(\mathbf{x}(s)) z(\mathbf{x}(s)). \tag{2.11}$$

Summarizing, we have

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \\ \dot{z}(s) = -c(\mathbf{x}(s)) z(s). \end{cases} \tag{2.12}$$

Example 2.1. *Use the method of characteristics to solve*

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u \text{ in } U \\ u = g \text{ on } \Gamma \end{cases} \tag{2.13}$$

where $U = \{x_1 > 0, x_2 > 0\}$, $\Gamma = \{x_1 > 0, x_2 = 0\} \subset \partial U$.

This is in the linear form with $\mathbf{b} = (-x_2, x_1)$ *and* $c = -1$. *Thus, the equations just derived for the linear case read*

$$\begin{cases} \dot{x}^1 = -x^2, \ \dot{x}^2 = x^1 \\ \dot{z} = z. \end{cases} \tag{2.14}$$

Therefore, we have

$$\begin{cases} x^1(s) = x^0 \cos s, \ x^2(s) = x^0 \sin s \\ z(s) = z^0 e^s = g(x^0) e^s, \end{cases} \tag{2.15}$$

where $x^0 \geq 0$, $0 \leq s \leq \frac{\pi}{2}$. *Fix any* $(x_1, x_2) \in U$. *Find* $s > 0, x^0 > 0$ *such that* $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$. *That is,* $x^0 = (x_1^2 + x_2^2)^{\frac{1}{2}}$ *and* $s = \arctan(\frac{x_1}{x_2})$. *We find that*

$$\begin{aligned} u(x_1, x_2) &= u(x^1(s), x^2(s)) = z(s) = g(x^0) e^s \\ &= g((x_1^2 + x_2^2)^{\frac{1}{2}}) e^{\arctan(\frac{x_1}{x_2})}. \end{aligned}$$

Exercise 2. A. *Write down the characteristic equations for the PDE*

$$u_t + b \cdot Du = f$$

in $\mathbb{R}^n \times (0, \infty)$ *where* $b \in \mathbb{R}^n$, $f = f(x, t)$.
B. *Use the characteristic ODE to solve this PDE subject to the boundary condition* $u = g$ *on* $\mathbb{R}^n \times \{t = 0\}$.

- F Quasilinear

  The first order PDE (2.1) is quasilinear when it is of the form

$$F(Du, u, x) = \mathbf{b}(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0. \tag{2.16}$$

So, here we have $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(z, x)$ and $D_p F = \mathbf{b}(x, z)$. The characteristic ODEs (for $\mathbf{x}$ and z) are therefore

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \\ \dot{z}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \cdot \mathbf{p}(s) = -c(\mathbf{x}(s), z(s)) \end{cases} \tag{2.17}$$

where we used the PDE (2.16).

Example 2.2. *Consider the boundary value problem*

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \text{ in } U \\ u = g \text{ on } \Gamma \end{cases} \tag{2.18}$$

where U *is the half-space* $\{x_2 > 0\}$ *and* $\Gamma = \{x_2 = 0\} = \partial U$. *Here* $\mathbf{b} = (1, 1)$ *and* $c = -z^2$ *and the ODEs are*

$$\begin{cases} \dot{x}^1 = 1, \ \dot{x}^2 = 1, \\ \dot{z} = z^2. \end{cases} \tag{2.19}$$

Solving the ODEs gives

$$\begin{cases} x^1(s) = x^0 + s, \ x^2(s) = s \\ z(s) = \frac{z^0}{1 - s z^0} = \frac{g(x^0)}{1 - s g(x^0)} \end{cases} \tag{2.20}$$

where $x^0 \in \mathbb{R},\ s \geq 0$ provided the denominator is not zero.

- F Nonlinear

  In the linear and quasilinear cases we have not needed to solve the characteristic ODE for $\mathbf{p}$. Here is a nonlinear example where the $\mathbf{p}$ equations need to be solved to carry out the method.

Example 2.3. *Consider the fully nonlinear problem*

$$\begin{cases} u_{x_1}u_{x_2} = u \text{ in } U \\ u = x_2^2 \text{ on } \Gamma \end{cases} \tag{2.21}$$

where $U = \{x_1 > 0\}$ and $\Gamma = \{x_1 = 0\} = \partial U$. here $F(p, z, x) = p_1p_2 - z$ and the characteristic ODE become

$$\begin{cases} \dot{p^1} = p^1,\ \dot{p^2} = p^2, \\ \dot{z} = 2p^1p^2, \\ \dot{x^1} = p^2, \dot{x^2} = p^1. \end{cases} \tag{2.22}$$

We can integrate these equations to find

$$\begin{cases} x^1(s) = p_2^0(e^s - 1),\ x^2(s) = x^0 + p_1^0(e^s - 1) \\ z(s) = z^0 + p_1^0p_2^0(e^{2s} - 1) \\ p^1(s) = p_1^0e^s,\ p^2(s) = p_2^0e^s \end{cases} \tag{2.23}$$

We need to determine the paramters $P_1^0,\ p_2^0$. Since $u = x_2^2$ on Γ, $p_2^0 = u_{x_2}(0, x^0) = 2x^0$. The PDE $u_{x_1}u_{x_2} = u$ implies $p_1^0p_2^0 = z^0 = (x^0)^2$ and so $p_1^0 = \frac{1}{2}x^0$. Now the formulas listed above explicitly give the values of $\mathbf{x}, z$ and $\mathbf{p}$.

Fix a point $(x_1, x_2) \in U$. We want to know the value of u at (x_1, x_2). Find s, x^0 such that $(x_1, x_2) = (x^1(s), x^2(s)) = (2x^0(e^s - 1), \frac{1}{2}x^0(e^s + 1))$. So $x^0 = \frac{4x_2 - x_1}{4}$ and $e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}$ and

$$\begin{aligned} u(x_1, x_2) &= u(x^1(s), x^2(s)) = z(s) = (x^0)^2e^{2s} \\ &= \frac{(x_1 + 4x_2)^2}{16}. \end{aligned}$$

Exercise 3. A. *Solve using characteristics the boundary value problem*

$$\begin{cases} uu_{x_1} + u_{x_2} = 1 \\ u(x_1, x_1) = \frac{1}{2}x_1. \end{cases}$$

B. *Solve using characteristics the initial value problem*

$$\begin{cases} u_t + uu_x = 1 \\ u(x, 0) = g(x) \end{cases}$$

where

$$g(x) = \begin{cases} 1, & -\infty < x < -1, \\ -x, & -1 \leq x \leq 0, \\ 0, & 0 < x < +\infty. \end{cases}$$

2.2. **Conservation Laws.**

- Characteristics for conservation laws

  We continue to follow [9]. Consider a quasilinear first order PDE of the form

$$G(Du, u_t, u, x, t) = u_t + \text{div } \mathbf{F}(u) = u_t + \mathbf{F}'(u) \cdot Du = 0$$

on $U = \mathbb{R}^n \times (0, \infty)$ subject to the initial condition

$$u = g \text{ on } \Gamma = \mathbb{R}^n \times \{t = 0\}.$$

Here $\mathbf{F} : R \longmapsto \mathbb{R}^n,\ \mathbf{F} = (F^1, \dots, F^n)$ and div denotes divergence with respect to the spatial variables $(x_1, \dots, x_n)$ and $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$.

We have written things a little differently in this example by distinguishing the variable $t = x_{n+1}$ in the notation. Define $q = (p, p_{n+1}),\ y = (x, t)$ so we now have notation similar to that used in the general derivation earlier. We have

$$G(q, z, y) = p_{n+1} + \mathbf{F}' \cdot p,$$

so

$$D_q G = (\mathbf{F}'(z), 1),\ D_y G = 0,\ D_z G = \mathbf{F}''(z) \cdot p.$$

The characteristic ODEs for $\mathbf{x}$ become

$$\begin{cases} \dot{x^i}(s) = (F^i)'(z(s)), i = 1, \dots n, \\ \dot{x}^{n+1}(s) = 1 \end{cases} \tag{2.24}$$

so $x^{n+1}(s) = s$, in agreement with our writing $t = x_{n+1}$ earlier. In particular, s and t may be identified here.

The ODE for z reads $\dot{z}(s) = 0$ so

$$z(s) = z^0 = g(x^0) \tag{2.25}$$

and (2.24) implies

$$\mathbf{x}(s) = \mathbf{F}'(g(x^0))s + x^0. \tag{2.26}$$

Remark 2.1. We observe that along (2.26) the value of $z(\cdot)$ given by (2.25) does not change. Similar reasoning could be applied to a different initial point $z^0 \in \Gamma$ with $g(x^0) \neq g(z^0)$ while the characteristic curves emanating from x^0 and z^0 may intersect. An apparent contradiction exists which is resolved by the fact that the i.v.p. does not in general have a smooth solution existing for all times $t > 0$. Note that the method of characteristics requires that we be able to differentiate u and lack of smoothness prevents us from doing so.

- Why are they called conservation laws?

  Equations of the form

$$\mathbf{u}_t + \text{div } \mathbf{F}(\mathbf{u}) = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \tag{2.27}$$

are called *conservation laws*. Here, $\mathbf{u} : \mathbb{R}^n \times (0, \infty) \longmapsto \mathbb{R}^m$ is a vector function and $\mathbf{F} : \mathbb{R}^m \longmapsto \mathbb{M}^{m \times n}$, the space of $m \times n$ matrices.

Given any smooth bounded domain U, note that the integral $\int_U \mathbf{u}(x, t) dx$ represents the total amount of the quantities (indexed by the vector $\mathbf{u}$) within the region U at time t. The rate of change within U is governed by a *flux* function $\mathbf{F} : \mathbb{R}^m \longmapsto \mathbb{M}^{m\times n}$, which controls the rate of loss or increase of $\mathbf{u}$ through the boundary ∂U. So, for each time t, we expect

$$\frac{d}{dt}\int_U \mathbf{u}(x,t)dx = -\int_{\partial U} \mathbf{F}(\mathbf{u})\nu dS,$$

where ν is the outward unit normal along U and dS is the natural surface area element on ∂U. Differentiating and applying the divergence theorem gives

$$\int_U \mathbf{u}_t dx = -\int_{\partial U} \mathbf{F}(\mathbf{u})\nu dS = -\int_U \text{div }(\mathbf{F}(\mathbf{u}))dx.$$

Since the region U was arbitrary, we derive (2.27) as a natural condition expressing the conservation of stuff indexed by $\mathbf{u}$.

Suppose we encounter a PDE in *conservation form*

$$\rho_t + \text{div }\Phi = 0. \tag{2.28}$$

Integrating over space, we observe

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho dx + \int_{\mathbb{R}^n} \text{div }\Phi dx = 0.$$

Provided Φ vanishes as $|x| \to \infty$, we learn that

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho dx = 0.$$

The quantity ρ is therefore called a *conserved density* and Φ is the *flux associated to* ρ.

We now collapse the discussion to $n = 1$ but leave the target dimension m free for a while. Our goal is to develop a notion of *solution* for conservation laws with the following properties:

1. The solutions we define should coincide with the solutions constructed using the method of characteristics when that method succeeds.
2. The solutions we define should make sense after the method of characteristics fails.
3. The solutions we define should be unique.

- Integral Solutions

Consider a class of smooth *test functions* $\mathbf{v} : \mathbb{R}\times[0,\infty) \longmapsto \mathbb{R}$ with compact support, $\mathbf{v} = (v^1, \dots, v^m)$. Temporarily assume $\mathbf{u}$ is a smooth solution of our problem

$$\begin{cases} \mathbf{u}_t + \partial_x \mathbf{F}(\mathbf{u}) \text{ in } \mathbb{R}^n \times (0,\infty), \\ \mathbf{u} = \mathbf{g} \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2.29}$$

Take the inner-product of the PDE with $\mathbf{v}$ and integrate by parts to get

$$\int_0^\infty \int_{-\infty}^{+\infty} \mathbf{u}\cdot\mathbf{v}_t + \mathbf{F}(\mathbf{u})\cdot\mathbf{v}_x dxdt + \int_{-\infty}^{+\infty} \mathbf{g}\cdot\mathbf{v}dx|_{t=0} = 0. \tag{2.30}$$

Notice that this identity makes sense if $\mathbf{u}$ is merely bounded (in particular, $\mathbf{u}$ does not have to be smooth to interpret (2.30)).

Definition 2.1. We say that $\mathbf{u} \in L^\infty(\mathbb{R}^n \times (0,\infty); \mathbb{R}^m)$ is an *integral solution* of the initial value problem (2.29) provided the identity (2.30) holds for all test functions $\mathbf{v}$.

- Rankine-Hugoniot Jump Condition

Suppose $\mathbf{u}$ is a solution of (2.29) which is smooth on either side of a curve C, along which u has a simple jump discontinuity. More precisely, assume $V \subset \mathbb{R} \times (0, \infty)$ is some region cut by a smooth curve C into a left-hand part V_l and a right-hand part V_r.

Assuming $\mathbf{u}$ is smooth in V_l, choose a test function $\mathbf{v}$ with compact support in V_l and we deduce from (2.30) that

$$\mathbf{u}_t + \partial_x \mathbf{F}(\mathbf{u}) = 0 \text{ in } V_l.$$

A similar statement holds in V_r.

Now choose $\mathbf{v}$ with compact support in V but which does not necessarily vanish along the curve C. We will see that the definition of integral solution forces the curve to have a particular shape. The identity (2.30) implies

$$0 = \int\int_{V_l} \mathbf{u} \cdot \mathbf{v}_t + \mathbf{F}(\mathbf{u}) \cdot \mathbf{v}_x dxdt + \int\int_{V_r} \mathbf{u} \cdot \mathbf{v}_t + \mathbf{F}(\mathbf{u}) \cdot \mathbf{v}_x dxdt. \tag{2.31}$$

As $\mathbf{v}$ has compact support in V, we deduce that

$$\begin{aligned} \int\int_{V_l} \mathbf{u} \cdot \mathbf{v}_t + \mathbf{F}(\mathbf{u}) \cdot \mathbf{v}_x dxdt, &= -\int\int_{V_l} [\mathbf{u}_t + \partial_x \mathbf{F}(\mathbf{u})] \cdot \mathbf{v} dxdt \\ &+ \int_C (\mathbf{u}_l \nu^2 + \mathbf{F}(\mathbf{u}_l)\nu^1) \cdot \mathbf{v} dl \\ &= \int_C (\mathbf{u}_l \nu^2 + \mathbf{F}(\mathbf{u}_l)\nu^1) \cdot \mathbf{v} dl \end{aligned}$$

where $\nu = (\nu^1, \nu^2)$ is the unit normal to C pointing from V_l into V_r and $\mathbf{u}_l$ is the limiting value of $\mathbf{u}$ as we approach the curve C from the left within V and $\mathbf{u}_r$ is similarly defined from the right.

Repeating the previous calculation in V_r and using (2.31) reveals that

$$\int_C [(\mathbf{F}(\mathbf{u}_l) - \mathbf{F}(\mathbf{u}_r))\nu^1 + (\mathbf{u}_l - \mathbf{u}_r)\nu^2] \cdot v dl = 0$$

holds for all test functions $\mathbf{v}$ as above. Therefore, we learn

$$(\mathbf{F}(\mathbf{u}_l) - \mathbf{F}(\mathbf{u}_r))\nu^1 + (\mathbf{u}_l - \mathbf{u}_r)\nu^2 = 0 \text{ along } C. \tag{2.32}$$

Suppose the curve C is parametrized as $\{(x, t) : x = s(t)\}$ for some smooth function $s : [0, \infty) \longmapsto \mathbb{R}$. Then $\nu = (\nu^1, \nu^2) = (1 + \dot{s}^2)^{\frac{1}{2}}(1, -\dot{s})$ so (2.32) reads

$$\mathbf{F}(\mathbf{u}_l) - \mathbf{F}(\mathbf{u}_r) = \dot{s}(\mathbf{u}_l - \mathbf{u}_r) \tag{2.33}$$

in V along the curve C. This vector inequality is called the *Rankine-Hugoniot Jump Condition.*

Example 2.4. *(Shock Waves) Consider the i.v.p.*

$$\begin{cases} u_t + \partial_x F(u) = 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u = g \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2.34}$$

with initial data

$$g = \begin{cases} 1 \text{ if } x \leq 0, \\ 1 - x \text{ if } 0 \leq x \leq 1, \\ 0 \text{ if } 1 \leq x. \end{cases}$$

Note: This problem was considered in Exercise 3 B.

The method of characteristics enables us to construct the solution for $t \in [0,1]$. *We find*

$$u(x,t) = \begin{cases} 1 \text{ if } x \leq t, 0 \leq t \leq 1, \\ \frac{1-x}{1-t} \text{ if } t \leq x \leq 1, 0 \leq t \leq 1, \\ 0 \text{ if } 1 \leq x, 0 \leq t \leq 1, \end{cases}$$

and we know the method of characteristics breaks down after $t \geq 1$ *since the characteristics cross. How should we define the solution of the i.v.p. after* $t = 1$?

Set $s(t) = \frac{1+t}{2}$ *and write*

$$u(x,t) = \begin{cases} 1 \text{ if } x < s(t), \\ 0 \text{ if } s(t) < x, \end{cases}$$

if $t \geq 1$. *Along the curve parametrized by* $s(\cdot)$, $u_l = 1$, $u_r = 0$ *and* $F(u_l) = \frac{1}{2}(u_l)^2 = \frac{1}{2}$, $F(u_r) = 0$. *Thus, the Rankine-Hugoniot Jump condition is satisfied and we have extended the solution obtained by the method of characteristics to an integral solution valid after the characteristics cross.*

Example 2.5. *(Rarefaction waves and nonphysical shocks) Consider again* (2.34) *but with the initial data*

$$g = \begin{cases} 0 \text{ if } x \leq 0, \\ 1 \text{ if } 0 \leq x. \end{cases}$$

There is no problem extending the characteristics but the method tells us nothing about how to prescribe the solution in the wedge $\{0 < x < t\}$. *Suppose we define*

$$u_1(x,t) = \begin{cases} 0 \text{ if } x \leq \frac{t}{2}, \\ 1 \text{ if } \frac{t}{2} \leq x. \end{cases}$$

Then it is easy to check that the R-H jump condition is satisfied, so that u_1 *is an integral solution of the i.v.p.. However, we can also define*

$$u_2(x,t) = \begin{cases} 0 \text{ if } x \leq 0, \\ \frac{x}{t} \text{ if } 0 \leq x < t, \\ 1 \text{ if } t \leq x, \end{cases}$$

which is called a rarefaction wave. This is a continuous integral solution of the i.v.p.

Remark 2.2. The preceding example reveals that *integral solutions are not in general unique.* We would like to shrink the class of integral solutions by imposing an extra condition upon our solution which permits us to prove uniqueness.

- Entropy Condition

  For the general scalar conservation law of the form $u_t + \partial_x F(u) = 0$, we know the solution takes the constant value $z^0 = g(x^0)$ along the characteristics

$$\mathbf{y}(s) = (F'(g(x^0))x + x^0, s), \ (s \geq 0).$$

We also know that we typically will encounter the crossing of characteristics and resultant discontinuities in the solution, if we move *forward* in time. We hope we can avoid the crossing of characteristics if we move *backward* in time. So, u_1 in the previous example has the crossing of characteristics as we move backward in time while u_2 does not.

Suppose that at some point on a curve C of discontinuities, the solution u has distinct left and right limits, u_l and u_r, and that a characteristic from the left and one from the right intersect at the point on C. The "no backwards-in-time intersection of characteristics" rule discussed above requires

$$F'(u_l) > \sigma > F'(u_r).$$

These inequalities are called the *entropy condition*. The entropy condition can be reexpressed in various ways provided F has certain properties (see [9]). In case $F : \mathbb{R} \longmapsto \mathbb{R}$ is smooth and uniformly convex then we can take the condition:

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z \tag{2.35}$$

for all $t > 0,\ z, x \in \mathbb{R}, z > 0$ as the *entropy condition*.

Definition 2.2. We say that $u \in L^\infty(\mathbb{R} \times (0,\infty); \mathbb{R})$ is an *entropy solution* of the initial value problem (2.34) provided it is an integral solution which satisfies the entropy condition (2.35).

Theorem 2.1. *Assume F is convex and smooth. Then, there exists–up to a set of measure zero–at most one entropy solution of* (2.34).

See [9] for the proof.

- Riemann Problem

The initial value problem (2.34) with piecewise constant initial function

$$g(x) = \begin{cases} u_l, \\ u_r. \end{cases} \tag{2.36}$$

is called *Riemann's problem* for the scalar conservation law. Here $u_l,\ u_r$ are called the left and right *initial states*, $u_l \neq u_r$. Assume that F is uniformly convex and C^2. Define $G = (F')^{-1}$.

Theorem 2.2. *(i) If $u_l > u_r$, the unique entropy solution of the Riemann problem* (2.34), (2.2) *is*

$$u(x,t) = \begin{cases} u_l \text{ if } \frac{x}{t} < \sigma, \\ u_r \text{ if } \frac{x}{t} > \sigma, \end{cases} \tag{2.37}$$

for $x \in \mathbb{R}, t > 0$ where

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}. \tag{2.38}$$

(ii) If $u_l < u_r$, the unique entropy solution of the Riemann problem (2.34), (2.2) *is*

$$u(x,t) = \begin{cases} u_l \text{ if } \frac{x}{t} < F'(u_l), \\ G(\frac{x}{t}) \text{ if } F'(u_l) < \frac{x}{t} < F'(u_r), \\ u_r \text{ if } \frac{x}{t} > F'(u_r). \end{cases} \tag{2.39}$$

Remark 2.3. In the first case of the theorem, the states u_l and u_r are separated by a *shock wave*. In the second case, the states u_l and u_r are separated by a *rarefaction wave*.

Proof. Consider first the case where $u_l > u_r$. The function u defined in the statement of the theorem certainly satisfies the Rankine-Hugonitot condition and is therefore an integral solution of the PDE. It remains to verify the entropy condition. Note that

$$F'(u_r) < \sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \frac{1}{u_l - u_r}\int_{u_l}^{u_r} F'(r)dr < F'(u_l)$$

using convexity. But this is precisely the (first version of the) entropy condition.

Now look at the case where $u_l < u_r$. We must verify that the function $u(x,t)$ defined in (ii) of the theorem solves the PDE in the region $\{F'(u_l) < \frac{x}{t} < F'(u_r)\}$. To verify this, we ask the general question: When does a function of the form

$$u(x,t) = v(\frac{x}{t})$$

satisfy the PDE? We compute

$$\begin{aligned} u_t + F(u)_x &= u_t + F'(u)u_x \\ &= -v'(\frac{x}{t})\frac{x}{t^2} + F'(v)v'(\frac{x}{t})\frac{1}{t} \\ &= v'(\frac{x}{t})\frac{1}{t}[F'(u) - \frac{x}{t}]. \end{aligned}$$

Thus, assuming v' never vanishes, we find $F'(v(\frac{x}{t})) = \frac{x}{t}$. Hence,

$$u(x,t) = v(\frac{x}{t}) = G(\frac{x}{t})$$

solves the PDE. One can check that $v(\frac{x}{t}) = u_l$ provided $\frac{x}{t} = F'(u_l)$ and a similar statement on the right to see that u as defined in (ii) is continuous.

The function u is a solution in each of its regions of definition and it is easy to check that it is in fact an integral solution. It remains to verify the entropy condition. But

$$u(x+z,t) - u(x,t) = G(\frac{x+z}{t}) - G(\frac{x}{t}) \leq \frac{\mathrm{Lip}(G)z}{t}$$

which is precisely the entropy condition (2.35). □

- Long time asymptotics in L^∞ and L^1
  We present two theorems from [9] which give us some detailed information about the solution of (2.34) as $t \to \infty$. From a physical point of view, the long-time asymptotics of solutions of evolution problems is often the main issue. What happens eventually?

Theorem 2.3. *(Asymptotics in L^∞) There exists a constant C such that*

$$|u(x,t)| \leq \frac{C}{t^{\frac{1}{2}}} \tag{2.40}$$

for all $x \in \mathbb{R}, t > 0$.

The estimate (2.40) shows that the L^∞-norm of u goes to zero as $t \to \infty$ (at a particular rate). On the other hand, the integral of u over $\mathbb{R}$ is conserved. Evidently, the solution has to "spread out" as it decays to zero to conserve the L^1 norm. In fact, assuming the initial data g has compact support, the solution u tends to a specific function as $t \to \infty$ in the L^1 norm.

Definition 2.3. For real parameters p, q, d, σ with $p, q \geq 0$, $d > 0$, the *N-wave* is the function

$$N(x,t) = \begin{cases} \frac{1}{d}(\frac{x}{t} - \sigma), & \text{if } -(pdt)^{\frac{1}{2}} < x - \sigma t < (qdt)^{\frac{1}{2}} \\ 0 & \text{otherwise.} \end{cases} \tag{2.41}$$

The constant σ is the *velocity* of the N-wave.

Define $\sigma = F'(0)$. Set $d = F''(0) > 0$ and write

$$p = -2 \min_{y \in \mathbb{R}} \int_{-\infty}^{y} g dx, \;\; q = 2 \max_{y \in \mathbb{R}} \int_{y}^{\infty} g dx. \tag{2.42}$$

Note that $p, q \geq 0$ and $G'(\sigma) = \frac{1}{d}$.

Theorem 2.4. *(Asymptotics in L^1) Assume that $p, q > 0$. Then there exists a constant C such that*

$$\int_{-\infty}^{+\infty} |u(x,t) - N(x,t)| dx \leq \frac{C}{t^{\frac{1}{2}}} \tag{2.43}$$

for all $t > 0$.

The preceding two theorems are models for "long time behavior" characterizations of other nonlinear evolution problems. Such results are not known in many physically important models and serve as a primary motivation for current research.

2.3. **Burgers' Equation.** The Navier-Stokes equations of fluid mechanics are

$$\textbf{typetheseout}. \tag{2.44}$$

These equations are extremely important in physics but are not yet well understood. A natural toy model which reveals some expected properties of the Navier-Stokes equations is the viscous Burgers' equation $u_t + uu_x = \nu u_{xx}$. The nonlinear term retains the formal structure as in the Navier-Stokes equations (but without the vectorial properties) and the diffusive properties of the equations are also retained. We will see shortly that the viscous Burgers' equation arises naturally in a simple model of traffic flow. Burgers' equation does not model turbulence in any significant manner but is a "fundamental evolution equation" which occurs in many unrelated applications where *viscous* and *nonlinear effects* are equally balanced. The material in this subsection is a combination of [9] and [16].

- Traffic Flow
  The PDE $\partial_t \rho + \partial_x \Phi = 0$ is the basic law of of "mass" conservation for a one-dimensional flow in the absence of physical sources. Here, as we've discussed, ρ is a "density" and Φ is the associated "flux".

  Let ρ represent the traffic density on a one-lane road with no on or off ramps–that is, the number of cars per unit distance on the road. Based on our intuition about driving, what should the flux function Φ look like? In particular, how should Φ depend upon the density ρ? Certainly, there is a maximum density $\rho_{\max}$ (e.g. bumper-to-bumper) when $\Phi = 0$. If there are no vehicles, then the associated flux must be zero. At some intermediate density $0 < \rho_0 < \rho_{\max}$, the flux will take on a maximum value. So, as a function of ρ, it is not unreasonable to imagine Φ to be of the form

$$\Phi(\rho, x, t) = R\rho(\rho_{\max} - \rho). \tag{2.45}$$

This is a quadratic function with maximum at $\frac{1}{2}\rho_{\max}$.

Notice that (2.45) could be refined to have $R = R(x,t)$ and $\rho_{\max} = \rho_{\max}(x,t)$ to allow for changes along the roadway due to weather, structural conditions, etc. These could in principle be determined by observing traffic. Another possible refinement would be to include dependence upon ρ_x to reflect the fact that drivers slow down when moving into a region of increasing traffic density and speed up when entering a region of decreasing traffic densities. We could write, for example

$$\Phi(\rho, x, t) = R\rho(x,t)(\rho_{\max} - \rho(x,t)) - k\rho_x(x,t). \tag{2.46}$$

The conservation law in this case becomes

$$\rho_t + [R\rho(\rho_{\max} - \rho)]_x = k\rho_{xx}$$

which after some nondimensionalizing may be reexpressed

$$u_t + uu_x = \epsilon u_x x \tag{2.47}$$

with $\epsilon \sim \frac{1}{R\rho_{\max}L_0}$ with L_0 some length scale along the road.

- Cole-Hopf Transformation

Suppose u is a nice smooth solution of the viscous Burgers' equation (2.47). Consider a transformation of dependent variable $u \longmapsto v$ defined by

$$u = -2\epsilon\frac{v_x}{v}. \tag{2.48}$$

We calculate

$$\begin{aligned} u_t &= -2\epsilon\frac{v_{xt}}{v} + 2\epsilon\frac{v_x v_t}{v^2} \\ u_x &= -2\epsilon\frac{v_{xx}}{v} + 2\epsilon\frac{v_x^2}{v^2} \\ u_x x &= -2\epsilon\frac{v_{xxx}}{v} + 6\epsilon\frac{v_x v_{xx}}{v^2} - 4\epsilon v_x^3 v^3. \end{aligned}$$

Subsituting these expressions into (2.47) reveals

$$\frac{v_x}{v}(\epsilon v_x x - v_t) - (\epsilon v_x x - v_t)_x = 0. \tag{2.49}$$

Thus, *any* solution $v(x,t)$ of (2.49), when used in (2.48) defines a function u which solves (2.47). In particular, if v solves the heat equation $v_t = \epsilon v_x x$, then v generates a solution to (2.47)! One might think that there could be other more complicated solutions in which the two terms in (2.49) cancel each other. However, as pointed out by Cameron Black in class, these terms scale differently with ϵ so there can not be such cancellation as $\epsilon \to 0$. Observe that we have transformed the *nonlinear* viscous Burgers' equation into the *linear* heat equation which is a spectacular simplification. A higher dimensional analog of this transformation is discussed in [9].

The preceding discussion may be used to solve the initial value problem

$$\begin{cases} u_t + uu_x - \epsilon u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u(x,0) = f(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2.50}$$

The change of variables (2.48) requires that the new variable $v(x,t)$ initially satisfy

$$f(x) = -2\epsilon \frac{v_x(x,0)}{v(x,0)}.$$

But this is just a linear first-order ODE for $v(x,0)$ which can be solved

$$v(x,0) = Ce^{-\frac{1}{2\epsilon}\int_0^x f(s)ds} := Cg(x) \tag{2.51}$$

where C is a constant of integration. Thus, for a given $f(x)$, we can compute $g(x)$ by calculating an integral.

So, we need to solve the initial value problem

$$\begin{cases} v_t - \epsilon v_{xx} = 0 \text{ in } \mathbb{R} \times (0,\infty), \\ v(x,0) = g(x) \text{ on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{2.52}$$

By the theory of the heat equation, we know how to solve the i.v.p. for the heat equation so we write

$$v(x,t) = \frac{C}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} g(y) e^{-\frac{(x-y)^2}{4\epsilon t}} dy. \tag{2.53}$$

We can then calculate

$$v_x(x,t) = \frac{-C}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} g(y) \frac{x-y}{2\epsilon t} e^{-\frac{(x-y)^2}{4\epsilon t}} dy.$$

Therefore, using (2.48), reveals that the solution of (2.50) is

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} g(y) \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\epsilon t}} dy}{\int_{-\infty}^{+\infty} g(y) e^{-\frac{(x-y)^2}{4\epsilon t}} dy}$$

in which the undetermined constant C cancels away. Recalling the definition of g given in (2.51) we can express u in terms of it initial data f. Define $h(x) = \int_0^x f(s)ds$ to be the antiderivative of the initial data f. Then,

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} g(y) \frac{x-y}{t} e^{-\frac{1}{2\epsilon}[\frac{(x-y)^2}{2t} + h(y)]} dy}{\int_{-\infty}^{+\infty} e^{-\frac{1}{2\epsilon}[\frac{(x-y)^2}{2t} + h(y)]} dy}, \tag{2.54}$$

solves (2.50).

- Asymptotics
  In the limit $\epsilon \to 0$, the viscous Burgers' equation $u_t + uu_x = \epsilon u_{xx}$ collapses to the scalar conservation law $u_t + uu_x = 0$. Notice that the order of the viscous PDE changes upon setting $\epsilon = 0$ so we may in fact have very bad behavior in the solutions $\{u^\epsilon\}$ as $\epsilon \to 0$. An analysis of the solution formula (2.54) shows this not to be the case. We actually have that the $\epsilon \to 0$ limit of (2.54) is the unique entropy solution of the scalar conservation law initial value problem obtained by setting $\epsilon = 0$ in (2.50).

The following lemma, an example of Laplace's "steepest descent" or "stationary phase" method, allows us to study the $\epsilon \to 0$ limit of (2.54).

Lemma 2.1. *Suppose $k, l : \mathbb{R} \longmapsto \mathbb{R}$ are continuous functions, that l grows at most linearly and that k grows at least quadratically. Assume also there exists a unique point $y_0 \in \mathbb{R}$ such that*

$$k(y_0) = \min_{y\in\mathbb{R}} k(y).$$

Then

$$\lim_{\epsilon\to 0}\frac{\int_{-\infty}^{+\infty} l(y)e^{-\frac{k(y)}{\epsilon}}dy}{\int_{-\infty}^{+\infty} e^{-\frac{k(y)}{\epsilon}}dy}=l(y_0).$$

Proof. Multiply the ratio above in the numerator and denominator by $e^{\frac{k_0}{\epsilon}}$ with $k_0=k(y_0)$. Then, we expect that $e^{\frac{k_0-k(y)}{\epsilon}}$ is vanishingly tiny if ϵ is small and $y\neq y_0$ so the only contribution to the integration should be from right near $y=y_0$. To validate our expectation, we note that the function

$$\mu_\epsilon(x)=\frac{e^{\frac{k_0-k(y)}{\epsilon}}}{\int_{-\infty}^{+\infty} e^{\frac{k_0-k(y)}{\epsilon}}dy}$$

satisfies

$$\begin{cases}\mu_\epsilon\geq 0, & \int_{-\infty}^{+\infty}\mu_\epsilon(y)dy=1,\\ \mu_\epsilon\to 0 & \text{exponentially fast for } y\neq y_0 \text{ as } \epsilon\to 0.\end{cases}$$

Consequently,

$$\lim_{\epsilon\to 0}\frac{\int_{-\infty}^{+\infty} l(y)e^{-\frac{k(y)}{\epsilon}}dy}{\int_{-\infty}^{+\infty} e^{-\frac{k(y)}{\epsilon}}dy}=\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} l(y)\mu_\epsilon(y)dy=l(y_0).$$

□

To apply the lemma to (2.54), we need to know that $y\longmapsto[\frac{(x-y)^2}{2t}+h(y)]$ attains its minimum at a unique point $y(x,t)$ (for all but at most countably many points x). Then the lemma gives

$$\lim_{\epsilon\to 0}u^\epsilon(x,t)=\frac{x-y(x,t)}{t}. \tag{2.55}$$

Remark 2.4. Define $K(x,y,t)=[\frac{(x-y)^2}{2t}+h(y)]$. We can write $K(x,y,t)=tL(\frac{x-y}{t})+h(y)$ where $L=F^*$ for $F(z)=\frac{1}{2}z^2$ and F^* represents the Legendre transformation of F. For Lipschitz h (corresponding to bounded f), it can be shown that a unique $y(x,t)$ exists (for at most a countable set of excepions x). Further pursuit of related ideas led to the Lax-Oleinik formula for solutions of the scalar conservation law and to the entropy condition [9].

Exercise 4. *(Method of Stationary Phase)*
Let $\psi\in C_0^\infty(\mathbb{R})$, so you can assume that ψ and all of its derivatives are continuous and $\psi=0$ for x outside some interval $[a,b]$. For $\lambda>1$, consider the oscilliatory integral

$$I_\lambda=\int_{\mathbb{R}} e^{i\lambda x^2}\psi(x)dx. \tag{2.56}$$

Prove that

$$|I_\lambda|\leq C\lambda^{-\frac{1}{2}} \text{ as } \lambda\to\infty. \tag{2.57}$$

This is Van der Corput's Lemma and the proof is an example of the "method of stationary phase".

3. Nonlinear Wave Equations

3.1. **Universal Derivation of NLS as an Amplitude equation.** The nonlinear Schrödinger equation (NLS) is ubiquitous as a model for nonlinear waves in a variety of physical settings. It is relevant in optics, plasma physics, fluid dynamics of water waves to superfluids, and recently in Bose-Einstein condensation. This wide array of physical systems motivates the question: Why is the nonlinear Schrödinger equation so universal? This section begins to provide an answer by showing that NLS arises in modelling the behavior of certain "natural" waves emerging from the study of a (rather) *general* nonlinear wave equation. The following discussion is modelled on a similar presentation in [29].

Consider a scalar nonlinear wave equation written symbolically

$$L(\partial_t, \partial_{\mathbf{x}})u + G(u) = 0, \tag{3.1}$$

where L is a linear operator with constant coefficients and G is a nonlinear function of u and its derivatives. We imagine that such an equation has been derived from physical principles and is perhaps an "exact" model of phenomena of interest. We also imagine, and this is why we write it in such a general form, that this equation is **hopelessly complicated** to solve exactly so we are forced to infer properties of the phenomena of interest by studying various simplifications of (3.1).

We begin such a study by first considering solutions of very small amplitude. Since G is nonlinear, we suppose that $G(u)$ is small if u is small and so can be ignored for the moment. This results in (3.1) collapsing to the simpler equation $L(\partial_t, \partial_{\mathbf{x}})u = 0$, which is a linear equation and can therefore be solved using the Fourier transform. We suppose this simplified equation admits monochromatic plane wave solutions of the form

$$u = \epsilon e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{3.2}$$

with constant amplitude ϵ. Here $0 < \epsilon \ll 1$ and the *frequency* ω and *wave vector* $\mathbf{k}$ are real quantities linked by the *dispersion relation*

$$L(-i\omega, i\mathbf{k}) = 0. \tag{3.3}$$

(Our assumption that ω is real valued here is consistent with the assumption that dissipation has been ignored in deriving (3.1).) There may be many solutions of the algebraic equation (3.3) and we concentrate on one of them

$$\omega = \omega(\mathbf{k}). \tag{3.4}$$

We have made some progress into understanding (3.1) by finding some solutions of the linear equation obtained by ignoring the nonlinear part, which seems natural provided these solutions have small amplitude (and small u implies small $G(u)$). However, over long time and long spatial scales, the accumulation of small nonlinear effects may become significant and (3.2) may not describe the behavior of solutions of the hard problem (3.1) on such scales.

Perhaps the most reasonable thing to try next is a *regular perturbation expansion.* However, in a manner related to the treatment of the nonlinear oscillator ODE in [22], we encounter secular terms so this approach will fail for long times.

Example 3.1. *(failure of regular perturbation) Let* $u_1 = \epsilon e^{i(\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|^2 t)}$ *be the small amplitude plane wave solution of the (linearization of) cubic nonlinear Schrödinger equation*

$$iu_t + \Delta u = |u|^2 u.$$

Consider a regular perturbation expansion by writing

$$u = u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots$$

Substituting, we find

$$\epsilon^2(i\partial_t u+\Delta)u_2+\epsilon^3(i\partial_t u+\Delta)u_3+\dots = |(u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots)|^2(u_1+\epsilon^2 u_2+\epsilon^3 u_3+\dots)$$

and using the form of u_1 *we obtain*

$$\epsilon^2(i\partial_t + \Delta)u_2 + \epsilon^3(i\partial_t + \Delta)u_3 = \epsilon^3 e^{i(kx-|k|^2 t)} + O(\epsilon^4).$$

Evidently, we should require $(i\partial_t u + \Delta)u_2 = 0$ *so we choose* $u_2 = 0$*. The effect of the (cubic) nonlinearity occurs at* ϵ^3 *in the expansion and suggests that we choose* u_3 *to satisfy*

$$(i\partial_t + \Delta)u_3 = e^{i(kx-|k|^2 t)}.$$

By Duhamel's formula,

$$u_3(t) = i\int_0^t S(t-\tau)e^{i(kx-|k|^2\tau)}d\tau,$$

where

$$S(t)\phi(x) = \int e^{i(kx-|k|^2 t)}\widehat{\phi}(k)dk.$$

We calculate to find

$$u_3(t) = i\int_0^t e^{ikx}e^{-i|k|^2(t-\tau)}e^{-i|k|^2\tau}d\tau.$$

Observe that the τ *dependence cancels in the integrand leaving*

$$u_3(t) = i\int_0^t d\tau e^{i(kx-|k|^2 t)}$$

which is a secular term growing linearly with t*.*

As regular perturbation fails due to the appearance of secular terms, we try a different approach based on a multiple scale expansion. We imagine that the complex amplitude of the "carrying wave" is no longer constant but depends on the "slow" variables $T = \epsilon t, \mathbf{X} = \epsilon \mathbf{x}$, and its evolution is prescribed by solvability conditions that eliminate the resonances which led to secular terms above. A similar set of ideas occurs in applying the Poincaré-Lindstedt method to the nonlinear oscillator in [22].

Let us reexpress the (branch of the) dispersion relation by writing

$$(i\partial_t - \omega(-i\partial_{\mathbf{x}}))Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0 \tag{3.5}$$

where $\partial_{\mathbf{x}}$ is the gradient with respect to $\mathbf{x}$ and $\omega(-i\partial_{\mathbf{x}})$ is the (pseudo)-differential operator obtained by replacing $\mathbf{k}$ by $-i\partial_{\mathbf{x}}$ in $\omega(\mathbf{k})$.

In a weakly nonlinear medium responding instantaneously to a finite wave amplitude, the **nonlinearity is expected to affect the dispersion relation**. This means that the frequency ω will depend upon the intensity of the wave, so we replace $\omega(\mathbf{k})$ by $\Omega(\mathbf{k}, \epsilon^2|A|^2)$ with $\Omega(\mathbf{k}, 0) = \omega(\mathbf{k})$. [It is in this step that we retain some aspects of the nonlinearity and throw much of the complicated structure of the nonlinearity away.] Furthermore, the (complex) amplitude A is no longer constant but is modulated in space and time, so we imagine it depends upon the slow variables $\mathbf{X} = \epsilon\mathbf{x}$ and $T = \epsilon t$, that is $A = A(\mathbf{X}, T)$. In (3.5) we now replace ∂_t by $\partial_t + \epsilon\partial_T$ and $\partial_{\mathbf{x}}$ by $\partial_x + \epsilon\nabla$ where $\nabla = \partial_{\mathbf{X}}$. [So, we will consider a "more flexible" version of the plane wave by considering a plane wave solution with an amplitude which can vary on long spatial scales.] Therefore, a candidate generalization of (3.5) (which might allow us to study next order nonlinear effects with no secular terms) is

$$[i\partial_t + i\epsilon\partial_T - \Omega(-i\partial_{\mathbf{x}} - i\epsilon\nabla, \epsilon^2|A|^2)]Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0. \tag{3.6}$$

This can be rewritten as a nonlinear dispersion relation

$$[\omega + i\epsilon\partial_T - \Omega(\mathbf{k} - i\epsilon\nabla, \epsilon^2|A|^2)]A = 0. \tag{3.7}$$

Now, ϵ is small and we Taylor expand Ω to second order about $\epsilon = 0$

$$\begin{aligned} \Omega(\mathbf{k} - i\epsilon\nabla, \epsilon^2|A|^2) &= \omega(\mathbf{k}) + \Omega_{\mathbf{k}}(\mathbf{k}, 0)\cdot(-i\epsilon\nabla) \\ &+ \Omega_{k^ik^j}(\mathbf{k}, 0)([-i\epsilon\nabla^i][-i\epsilon\nabla^j]) + \Omega_I(\mathbf{k}, 0)\epsilon^2|A|^2 + O(\epsilon^3) \end{aligned}$$

where Ω_I indicates differentiation of Ω with respect to the second slot (intensity) and $\Omega_{\mathbf{k}}$ refers to differentiating with respect to the first slot(s) of Ω. We write $\mathbf{v}$ for $\Omega_{\mathbf{k}}(\mathbf{k}, 0)$.

Substituting this expansion for Ω into (3.7) gives

$$(i\partial_T + \mathbf{v}\cdot\nabla)A + \epsilon[\Omega_{k^ik^j}(\mathbf{k}, 0)]\nabla^i\nabla^j A + \epsilon[\Omega_I(\mathbf{k}, 0)]|A|^2 A = O(\epsilon^2) \tag{3.8}$$

at this order of approximation.

Observe that $\Omega_I(\mathbf{k}, 0)$ is just a constant vector and $\Omega_{k^ik^j}(\mathbf{k}, 0)$ is a constant matrix. In particular, if $\Omega_{k^ik^j}(\mathbf{k}, 0)$ is the identity matrix then the second derivative operator in (3.8) is just the Laplacian. We will observe later that the sign of the Ω_I term relative to the sign of the second derivative terms has a drastic influence on the behavior of solutions.

Remark 3.1. Whenever the physical system under consideration is

(1) strongly dispersive,
(2) weakly nonlinear,
and the solutions of interest are
(3) nearly monochromatic plane waves,

the amplitude of the wave will likely [23] be governed by the nonlinear Schrödinger equation. By *strongly dispersive*, we mean that $D^2\omega(k) \neq 0$. *Weakly nonlinear* means the size of the nonlinearity is small relative to the amplitude of the plane wave. A *nearly monochromatic plane wave* has its Fourier transform supported essentially near one particular wave-number $\mathbf{k}$. We can imagine such a wave as an amplitude modulated plane wave as in (3.2) with the modulation taking place on a much longer spatial scale than the period of the spatial oscillation $\sim \frac{1}{|\mathbf{k}|}$.

NLS from nonlinear Klein-Gordon Equation [23]

We carry out the above general derivation for a particular equation, namely the *nonlinear Klein-Gordon Equation*,

$$\begin{cases} \partial_{tt}u - \partial_{xx}u + u = u\bar{u}u, & \{x \in \mathbb{R}, t > 0\} \\ u(x, 0) = A(\epsilon x)e^{ikx}, & \{x \in \mathbb{R}, t = 0\}, \\ u_t(x, 0) = -i\omega(k)A(\epsilon x)e^{ikx}, & \{x \in \mathbb{R}, t = 0\} \end{cases} \tag{3.9}$$

where

$$\omega^2(k) = 1 + k^2. \tag{3.10}$$

Notice that we are considering initial data which is nearly monochromatic which is of amplitude A, of order 1. So, the solutions we are looking for are not necessarily small in amplitude. However, the size of the nonlinearity is much smaller than the amplitude in light of the ϵ^2 in the equation. Therefore, this system is weakly nonlinear. A calculation shows that $w''(k) \neq 0$ which shows this problem is strongly dispersive.

Remark 3.2. The Klein-Gordon equation is physically relevant. It occurs naturally in relativistic quantum mechanics [11], [27]. Explain de Broglie plus relativity modifys Schrodinger dynamics to KG.

In 1967, McCall and Hahn discovered the phenomenon of "self-induced transparency", an effect whereby the leading edge of an optical pulse is used to invert an atomic population while the trailing edge returns the population to its initial state by standard emission. This process is realizable if it takes place in a time short compared to the phase memory of the medium and the pulse is sufficiently strong. In a certain simplification, this process can be modelled by

$$u_{xt} = \sin u.$$

Here x is distance in the medium, t is (retarded) time and the electric field envelope E is proportional to u_x. These comments are taken from [29].

Finally, the Sine-Gordon equation is the continuum limit of a string of plane pendula hanging from a horizontal torsion wire. This is known as the Scott model [25].

We seek a nearly monochromatic wave $u = u^\epsilon$ in the form

$$u^\epsilon \sim u^\epsilon(x, X; t, T_1, T_2, \dots)$$

where $X = \epsilon x,\ T_j = \epsilon^j t$ and $u^\epsilon = u_0 + \epsilon u_1 + \dots$. Substituting in the PDE and balancing powers of ϵ^j yields

$$\begin{aligned}
(\partial_t^2 - \partial_x^2 + 1)u_0 &= 0 \\
(\partial_t^2 - \partial_x^2 + 1)u_1 &= -2\partial_{T_1} u_{0,t} + 2\partial_X u_{0,x} \\
(\partial_t^2 - \partial_x^2 + 1)u_2 &= \pm u_0 \overline{u_0} u_0 - 2\partial_{T_1} u_{1,tt} + 2\partial_X u_{1,x} - [2\partial_{T_2} u_{0,t} + (\partial_{T_1}^2 - \partial_X^2)]u_0
\end{aligned}$$

with initial data

$$\begin{aligned}
u^\epsilon(x, X; 0, 0, \dots) &= A(x)e^{ikx} \\
u_t^\epsilon(x, X; 0, 0, \dots) &= -i\omega(k)A(x)e^{ikx}.
\end{aligned}$$

Exercise 5. *Justify the preceding paragraph by calculating the coefficients of powers of ϵ in the expansion.*

A particular solution of the u_0 equation is

$$u_0(x, X, t, T_1, T_2, \dots) = A(X; T_1, T_2, \dots)e^{i[kx - \omega(k)t]}.$$

At order ϵ^1 we get

$$(\partial_t^2 - \partial_x^2 + 1)u_1 = 2i[\omega\partial_{T_1} A + k\partial_X A]e^{i[kx - \omega(k)t]}.$$

With respect to the fast scales x, t, the coefficient on the right may be considered to be a constant. Integration of this equation in time will generate a secular term unless we require

$$\omega(k)\partial_{T_1} A + k\partial_X A = 0.$$

The dispersion relation implies $k = \omega\omega'$ so we can write

$$\omega(k)[\partial_{T_1} A + \omega'(k)\partial_X A] = 0$$

which shows the amplitude A should propagate at speed $\omega'(k)$

$$A(X; T_1, T_2, \dots) = A(X - \omega'(k)T_1, T_2, \dots).$$

We introduce the natural change of variable $Y = X - \omega'(k)T_1$ to put ourselves in the frame of the moving amplitude. The preceding restriction on A makes the right-side of the u_1 equation vanish and we choose $u_1 = 0$.

At next order ϵ^2,

$$(\partial_t^2 - \partial_x^2 + 1)u_2 = \{\pm A\overline{A}A + 2i\partial_{T_2} A - (\partial_{T_1}^2 - \partial_X^2)A\}e^{i[kx-\omega(k)t]}.$$

We will again encounter a secular term unless we require A to satisfy

$$2i\partial_{T_2} A - (A_{T_1T_1} - A_{XX}) \pm A\overline{A}A = 0.$$

Since $A(X, T_1, T_2, \dots) = A(X - \omega'(k)T_1, T_2, \dots)$ we have

$$A_{T_1} = -\omega'(k)A_X = -\omega'(k)A_Y$$

which after some calculation shows The dispersion relation may be differentiated to show $(\omega'(k)^2 - 1) = \omega(k)\omega''(k)$ and we finally observe

$$2i\partial_{T_2} - \omega(k)\omega''(k)A_{YY} \pm A\overline{A}A = 0. \tag{3.11}$$

Fourier Mode Perspective on Nonlinear Dispersive Waves [29]

The linear evolution

$$[\partial_t + i\omega(-i\partial_{\mathbf{x}})]u = 0 \tag{3.12}$$

has plane wave solutions of the form

$$e^{i(\mathbf{k_0}\dot{x} - \omega(\mathbf{k_0})\mathbf{t})}, \text{ fixed } \mathbf{k_0} \in \mathbb{R}^n.$$

Moreover, the general solution can be expressed

$$u(\mathbf{x}, t) = \int e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)}\widehat{\phi}(\mathbf{k})d\mathbf{k}.$$

Let $0 < \delta \ll |\mathbf{k_0}|$ and imagine $\widehat{\phi}(\mathbf{k}) = 0$ if $|\mathbf{k} - \mathbf{k_0}| \gtrsim \delta$. So, we have a *wave-packet*

$$u(x, t) = \int_{|\mathbf{k}-\mathbf{k_0}|<\delta} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)}\widehat{\phi}(\mathbf{k})d\mathbf{k} \tag{3.13}$$

consisting of a superposition of wave-numbers extremely close to the center wave-number $\mathbf{k_0}$. Some algebraic manipulations shows the wave-packet may be rewritten

$$u(x, t) = e^{i(\mathbf{k_0}\cdot\mathbf{x} - \omega(\mathbf{k})t)} \int_{|\kappa|<\delta} e^{i(\kappa\cdot\mathbf{x} - [\omega(\mathbf{k_0}+\kappa) - \omega(\mathbf{k_0})]t)}\tilde{c}(\kappa)d\kappa \tag{3.14}$$

where $\mathbf{k} = \mathbf{k_0} + \kappa$ and $\tilde{c} = \widehat{\phi}(\mathbf{k_0} + \kappa)$. Since $|\kappa| < \delta$, to change $e^{ik\cdot x}$ on the order of $0(1)$ requires x to change on the order of $0(\frac{1}{\delta})$. On what scale should t change for $e^{i[\omega(k_0+\kappa 0)-\omega(k_0)]t}$ to change $O(1)$ if $|\kappa| < \delta$?. Expanding via Taylor's theorem,

$$\omega(k_0 + \kappa) = \omega(k_0) + \nabla_k\omega(k_0)\cdot\kappa + \frac{1}{2}\omega_{ij}''(k_0)\kappa^i\kappa^j + O(\delta^3),$$

so

$$[\omega(k_0 + \kappa) - \omega(k_0)] = \mathbf{v_g}\cdot\kappa + \frac{1}{2}\omega_{ij}''\kappa^i\kappa^j + O(\delta^3).$$

Let's assume $\mathbf{v_g} = 0$ since we can always change variables to remove a transport term. Let's also assume $\omega''_{ij}(k_0) \neq 0$ (as a matrix) so $e^{i\frac{1}{2}\omega''_{ij}(k_0)\kappa^i\kappa^k t}$ changes $O(1)$ if t changes $O(\frac{1}{\delta^2})$.

The preceding remarks show that the integral term in (3.13) may be thought of as $A(X, T_2)$ where $X = \epsilon x$ and $T_2 = \epsilon^2 t$. Therefore, the wave-packet is a nearly monochromatic wave.

Summarizing, we have observed that the linear dynamics (3.12) has nearly monochromatic solutions of the form

$$u(x,t) = A(x,t)e^{i(k_0\cdot x - \omega(k)t)} \tag{3.15}$$

where

$$A(x,t) = \int_{|\kappa|<\delta} e^{i(\kappa\cdot x - \tilde{\omega}(\kappa)t)}\tilde{c}(\kappa)d\kappa \tag{3.16}$$

with $\tilde{\omega}(\kappa) = \omega(k_0+\kappa) - \omega(k_0)$ and $A(x,t)$ changes $O(1)$ when x changes $O(\frac{1}{\delta})$ or when t changes $O(\frac{1}{\delta^2})$. Observe also that the linear dynamics sends $\tilde{c}(\kappa) \longmapsto \tilde{c}(\kappa)e^{-i\tilde{\omega}(\kappa)t} = \tilde{c}(\kappa,t)$ which satisfies

$$\partial_t\tilde{c}(\kappa,t) + i\tilde{\omega}(\kappa)\tilde{c}(\kappa,t) = 0. \tag{3.17}$$

Suppose now that we consider the nonlinear evolution

$$\partial_t u + i\omega(-i\partial_x)u = i|u|^2 u$$

for u in the form (3.15). We wish to calculate $|u|^2 u$ and gain some insights into the dynamics of $\tilde{c}(\kappa,t)$ in this nonlinear case. We calculate

$$(Ae^{i[\cdot]})\overline{(Ae^{i[\cdot]})}(Ae^{i[\cdot]}) = A\overline{A}Ae^{i[\cdot]}.$$

Using (3.16), a calculation shows

$$A\overline{A}A(x,t) = \int_{|k_i|<\delta} e^{i(\kappa_1-\kappa_2+\kappa_3)\cdot x}e^{-i[\tilde{\omega}(\kappa_1)-\tilde{\omega}(\kappa_2)+\tilde{\omega}(\kappa_3)]t}\tilde{c}(\kappa_1)\overline{\tilde{c}(\kappa_2)}\tilde{c}(\kappa_3)d\kappa_1 d\kappa_2.d\kappa_3$$

This calculation identifies the result of interacting the wave (3.15) with its complex conjugate and with itself again: a *three wave interaction*. Observe that $|\kappa_1 - \kappa_2 + \kappa_3| < 3\delta$ so the frequency content of A broadens through the nonlinear action but still $3\delta \ll |k_0|$ so we are retaining the nearly monochromatic wave structure. The contribution to the dynamics of $\tilde{c}(\kappa,t)$ due to the nonlinear interaction is

$$i\int_{\kappa_1-\kappa_2+\kappa_3=\kappa,\ |\kappa_i|<\delta} e^{i\kappa\cdot x}e^{-i[\tilde{\omega}(\kappa_1)-\tilde{\omega}(\kappa_2)+\tilde{\omega}(\kappa_3)]t}\tilde{c}(\kappa_1)\overline{\tilde{c}(\kappa_2,t)}\tilde{c}(\kappa_3,t)d\kappa_{123} \tag{3.18}$$

and (3.17) changes with the nonlinearity to

$$\partial_t\tilde{c}(\kappa,t) + i\tilde{\omega}(\kappa)\tilde{c}(\kappa,t) = i\int_{\kappa_1-\kappa_2+\kappa_3=\kappa,\ |\kappa_i|<\delta} e^{-i[\tilde{\omega}(\kappa_1)-\tilde{\omega}(\kappa_2)+\tilde{\omega}(\kappa_3)]t}\tilde{c}(\kappa_1)\overline{\tilde{c}(\kappa_2,t)}\tilde{c}(\kappa_3,t)d\kappa_{123} \tag{3.19}$$

Taylor expanding $\tilde{\omega}(\kappa) = \omega(k_0+\kappa) - \omega(k_0)$ and using $\nabla_k\omega(k_0) = v_g = 0$ gives

$$\partial_t\tilde{c}(\kappa,t) + i\frac{1}{2}\omega''_{ij}(k_0)\kappa^i\kappa^j\tilde{c}(\kappa,t) = i\int_{\kappa_1-\kappa_2+\kappa_3=\kappa,\ |\kappa_i|<\delta} e^{-i\frac{1}{2}\omega''_{ij}(k_0)[\kappa_1^i\kappa_1^j - \kappa_2^i\kappa_2^j + \kappa_3^i\kappa_3^j]}\tilde{c}(\kappa_1)\overline{\tilde{c}(\kappa_2,t)}\tilde{c}(\kappa_3,t)d\kappa_{123}. \tag{3.20}$$

The equation (3.19) gives an explicit evolution of the Fourier mode $\tilde{c}(\kappa, t)$ due to the linear dispersion and teh nonlinear three-wave interaction. It (and therefore the approximation (3.20) reveals some of the complicated structure of the cubic nonlinearity. We will return to these complications later.

Remark 3.3. 3 wave resonances, compare with failure of regular perturbation.

3.2. **Fiber Optics.** [29] [?]

3.3. **Hartree-Fock.**

3.4. **Gross-Pitaevskii; superstuff.** [29], [7]

Describe the Ginzburg-Landau theory of superconductivity and superfluidity. Note the appearance of topological obstructions to solving a minimization problem for S^1 valued maps on $\mathbb{R}^2$. Sketch the votex motion result of [7]. Make connection with vorticity in fluid dynamics via Madelung transformation.

Exercise 6. *Show that for $x \in \mathbb{R}^2$,*

$$\int_{\epsilon<|x|<1} |\nabla \frac{x}{|x|}|^2 dx \to \infty \text{ as } \epsilon \to 0.$$

Find the rate of divergence as $\epsilon \to 0$.

3.5. **Bose-Einstein Condensates?**

3.6. **Zakharov System.** [32], [29]

3.7. **Universal Derivation of KdV.** The following remarks, taken from [26], reveal why the KdV equation is frequently encountered in wave propagation problems. Like the NLS equation, one can identify a class of problems where the next order effects due to dispersion and weak nonlinearity lead to the KdV equation.

Wave processes in various homogeneous media are described by the *wave equation* $\partial_t^2 \psi = c_0^2 \partial_x^2 \psi$. There are 3 assumptions made in deriving this equation.

(1) No dissipation. (Assume the equation is invariant under $t \longmapsto -t$.)
(2) Weak nonlinearity. (Small amplitude solutions $\implies$ small nonlinearity.)
(3) No dispersion. (Propagation velocity does not depend on frequency and wavelength.)

We wish to identify an improved model to describe wave behavior by including terms to account for nonlinear and dispersive effects. Of course, an even better model will include dissipation. The dynamical effect of dissipation is to cause the wave to decay. Dispersion tends to blur wave packets and nonlinearity can steepen fronts.

Solutions of the wave equation $\partial_t^2 \psi = c_0^2 \partial_x^2 \psi$ may be written

$$\psi(x,t) = \psi_1(x - c_0 t) + \psi_2(x + c_0 t), \ c_0 > 0.$$

For small dispersive and nonlinear effects, we can treat waves travelling in different directions independently. A wave moving in the positive x-direction, e.g. ψ_1, satisfies

$$\partial_t \psi_1 + c_0 \partial_x \psi_1 = 0. \tag{3.21}$$

We search for the corrections to this equation. Note that we are considering *real*-valued solutions to an equation with *real*-valued coefficients.

What is the correction due to dispersion? Let the exact law of dispersion for linear waves in the medium be

$$\omega(k) = ku(k).$$

As $k \to 0$, the velocity of propagation should go to c_0. So, when we expand $u(k)$ in a power series in k, the first term is c_0. Recall that the dispersion relation represents the (linearized) PDE which we know to have real coefficients. For ω to be real (no dissipation), it is necessary that u be expandable in even powers of (ik).

For small k, therefore, we have to first interesting correction, the dispersion relation

$$\omega(k) = c_0 k - \beta k^3. \tag{3.22}$$

In order to obtain such a dispersion relation, we must modify (3.21) by adding a term

$$\partial_t + c_0 \partial_x + \beta \partial_x^3 \psi = 0. \tag{3.23}$$

What is the correction due to nonlinearity? Most physically relevant wave equations describe some oscilliatory behavior about an equilibrium. We assume that the wave equation describes some conserved quantity by assuming it to be of the form

$$\partial_t \psi + \partial_x j = 0. \tag{3.24}$$

The derivation of the correction due to nonlinearity rests on obtaining an approximate expression for the flux j in terms of ψ. From (3.23), we have $j = c_0\psi + \beta\partial_x^2\psi$. The next approximation of j, resulting from turning on the simplest possible nonlinear effect, involves adding on a term $\frac{1}{2}\alpha\psi^2$. Addition of such a term to j and rewriting (3.24) reveals

$$\partial_t \psi + c_0 \partial_x \psi + \beta \partial_x^3 \psi + \alpha \psi \partial_x \psi = 0. \tag{3.25}$$

As we have seen several times already, the change of variable $y = x - c_0 t$ will remove the c_0 term collapsing (3.25) to the KdV equation $u_t + \beta u_{xxx} + \alpha u u_x = 0$.

3.8. **Long waves on shallow water.** [31]

The KdV equation arises natural as an approximation to the equations describing long waves on shallow water.

3.9. **Kadomtsev-Petviashvili Equations.** [25]

The Kadomtsev-Petviashvili equations are a two dimensional generalization of the KdV equation. They were introduced to study the stability of the KdV soliton to transverse dispersion.

3.10. **Generalized KdV and NLS Equations, KdV_p and NLS_p.** [13], [4]

A basic issue is to understand the competition between nonlinear and dispersive effects. We have observed that NLS and KdV arise as approximate problems to account for weak nonlinearity and dispersion from rather general principles. Moreover, we have described particular physical settings where NLS is a natural model. To better understand the interplay between nonlinearity and dispersion, we introduce some generalizations of KdV and NLS. The *generalized KdV equation* is

$$\partial_t u + \partial_x^3 u + \partial_x f(u) = 0. \tag{3.26}$$

The special case when $f(z) = \frac{1}{p+1} z^{p+1}$ will be denoted KdV_p. Similarly, we introduce the *generalized nonlinear Schrödinger equation*

$$i\partial_t u + \Delta u + f(|u|^2)u = 0 \tag{3.27}$$

and the special case when $f(z) = z^{\frac{p-1}{2}}$ will be denoted NLS_p.

One can also consider generalizations of the dispersive terms by, for example, replacing ∂_x^3 by ∂_x^5 but we do not formalize such generalizations with any notation at this time. There are also natural generalizations of the KP equations and the Zakharov system.

Exercise 7. *A. Suppose $x \in \mathbb{R}$ and u is a solution of KdV_p. Find $\alpha, \beta \in \mathbb{R}$ such that $u_\sigma = \sigma^\alpha u(\sigma x, \sigma^\beta t)$ also solves KdV_p.*
B. Suppose $x \in \mathbb{R}^n$ and u is a solution of NLS_p. Find $\alpha, \beta \in \mathbb{R}$ such that $u_\sigma = u(\sigma x, \sigma^\beta t)$ also solves NLS_p.

This exercise identifies the scaling invariance of NLS_p and KdV_p.

4. Basic Dynamical Effects

4.1. **Dispersion.** The solution of the initial value problem for Airy's equation

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, & \mathbb{R} \times \{t > 0\} \\ u(x,0) = \phi(x), & \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{4.1}$$

can be explicitly written using the dispersion relation as

$$u(x,t) = \int e^{i(\xi x + \xi^3 t)} \widehat{\phi}(\xi) d\xi = (S(t)\phi)(x). \tag{4.2}$$

We can factor out a common ξ in the exponential to rewrite the exponential above as $e^{i\xi(x+\xi^2 t)}$. This shows that the wave number ξ is "moved to the left" at speed ξ^2.

This point of view on the dynamics leads to a nice *heuristic* picture guiding one's intuition into many of the basic dynamical properties of dispersive waves. Suppose that ϕ is a function compactly supported near the origin. We imagine that ϕ is a sum of different oscillating functions, some of fast spatial oscillation, some oscillating quite slowly. The fast oscillating parts of ϕ should "move to the left" extremely fast while the slower oscillations move left more slowly. At a later time, a fixed interval of x containing the origin should no longer contain any extremely high frequencies in ϕ since they have all moved out of the interval. As a result, the solution of Airy's equation should be quite smooth in the fixed interval at the later time since there are no high frequency parts required to express it there. We can therefore expect that the solution becomes smoother locally in x and it should also decay locally in x.

Note that we can multiply Airy's equation by u and integrate by parts to prove that $\partial_t \int_{\mathbb{R}} u^2(x,t)dx = 0$. This shows there is no decay in the mass as u evolves but the "move to the left at speed ξ^2" picture suggests there should be local smoothing and decay despite the conservation of L^2 mass.

We can turn the dispersive smoothing heuristic around by imagining a function with oscillations growing faster and faster as $x \to \infty$. These oscillations could perhaps be arranged to "move to the left" under the Airy dynamics in such a way that they all overlap at a particular location in x at some time $t > 0$ causing something bad to happen. Such a construction is carried out in [2].

Decay estimate

The Airy equation has solution given by (4.2). Stationary phase suggests there should be decay as $t \to \infty$ like $t^{-\frac{1}{3}}$. This turns out to be the case. For the Schrödinger equation, the corresponding representation of the solution has $|\xi|^2$ in place of ξ^3 and we are in $\mathbb{R}^n$. A quick guess based on stationary phase suggests decay like $t^{-\frac{1}{2}}$ as $t \to \infty$. However, since we are integrating over $\mathbb{R}^n$, we encounter such decay from each separate $1d$ integral leading to $t^{-\frac{n}{2}}$.

The calculations which follow show the solution of the linear Schrödinger equation decay as $t \to \infty$ at the rate $t^{-\frac{n}{2}}$. Moreover, we obtain a new way to express the solution as a convolution and make a nice contact with the heat equation.

Consider the i.v.p for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, & \mathbb{R}^n \times \{t > 0\}, \\ u(x,0) = \phi(x), & \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{4.3}$$

The solution may be expressed using the Fourier transform and the dispersion relation as

$$u(x,t) = S(t)\phi(x) = \int e^{ix\cdot\xi} e^{-i|\xi|^2 t} \widehat{\phi}(\xi) d\xi. \tag{4.4}$$

The reader should verify that the correct choice of sign has been made by formally differentiating under the integral sign. Next, we reexpress the solution of (4.3) as a convolution integral. First, we write out the Fourier transform appearing in (4.4)

$$\int e^{ix\cdot\xi} e^{-i|\xi|^2 t} \int e^{-iy\cdot\xi} \phi(y) dy d\xi.$$

Interchanging the integrations leads to

$$\int\int e^{i(x-y)\cdot\xi - i|\xi|^2 t} d\xi \phi(y) dy. \tag{4.5}$$

Next, we wish to evaluate explicitly the ξ integration. By completing the square above the exponential, the inner integral may be rewritten

$$\int e^{-it|\xi - (\frac{x-y}{2t})|^2} d\xi e^{it|\frac{x-y}{2t}|^2}.$$

The $\frac{x-y}{2t}$ term may be translated away in the above integral. The change of variables $\eta = i^{\frac{1}{2}} t^{\frac{1}{2}} \xi$, $d\eta = i^{\frac{n}{2}} t^{\frac{n}{2}} d\xi$ leads to

$$\frac{1}{i^{\frac{n}{2}} t^{\frac{n}{2}}} \int e^{-|\eta|^2} d\eta e^{it|\frac{x-y}{2t}|^2}. \tag{4.6}$$

The integral of the Gaussian in $\mathbb{R}^n$ gives an (explicit) constant c_n and the inner integral in (4.5) has been calculated. Therefore, we can now reexpress

$$u(x,t) = S(t)\phi(x) = \frac{c_n}{i^{\frac{n}{2}} t^{\frac{n}{2}}} \int e^{i\frac{|x-y|^2}{4t}} \phi(y) dy. \tag{4.7}$$

From (4.7), the following *decay estimate* follows immediately,

$$\|u(t)\|_{L^\infty_x} \leq \frac{C}{t^{\frac{n}{2}}} \|\phi\|_{L^1_x}. \tag{4.8}$$

Remark 4.1. The initial value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & \mathbb{R}^n \times \{t > 0\} \\ u(x,0) = \phi(x), & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has the explicit solution

$$u(x,t) = \frac{c_n}{t^{\frac{n}{2}}} \int e^{-\frac{|x-y|^2}{4t}} \phi(y)dy. \tag{4.9}$$

Observe that the formula (4.9) transforms into (4.7) upon replacing t by it and, similarly, the heat equation transforms into the Schrödinger equation under such a transformation to imaginary time.

Exercise 8. *Prove that the solution of the linear Schrödinger equation decays like $t^{-\frac{n}{2}}$ by using the Fourier multiplier representation of the solution and the method of stationary phase. Assume that the Fourier transform of the data is as nice as you'd like.*

Conformal Transformation

The preceding calculations showed that we have two ways of representing solutions of the initial value problem for the linear Schrödinger equation

$$u(x,t) = \int e^{ix\cdot\xi} e^{-i|\xi|^2 t}\widehat{\phi}(\xi)d\xi = \frac{c_n}{(it)^{\frac{n}{2}}} \int e^{i\frac{|x-y|^2}{4t}} \phi(y)dy. \tag{4.10}$$

If we replace t by $\frac{1}{t}$ in the Fourier multiplier representation, we encounter an expression somewhat similar to the convolution representation. This observation motivates the following calculations which reveal an important symmetry property of the Schödinger equation. By completing the square above the exponential, we can write

$$u(x,t) = e^{it|\frac{x}{2t}|^2} \int e^{-it|\xi - \frac{x}{2t}|^2}\widehat{\phi}(\xi)d\xi.$$

Some manipulations allow us to write

$$\begin{aligned} &= e^{it\frac{|x|^2}{4t}} \frac{(i\frac{1}{4t})^{\frac{n}{2}}}{c_n} \overline{\frac{c_n}{(i\frac{1}{4t})^{\frac{n}{2}}} \int e^{i\frac{|\xi - \frac{x}{2t}|^2}{4(\frac{1}{4t})}} \overline{\widehat{\phi}}(\xi)d\xi} \\ &= e^{it\frac{|x|^2}{4t}} \tilde{c_n} \frac{1}{(it)^{\frac{n}{2}}} \overline{S(\frac{1}{4t})\overline{\widehat{\phi}}(\frac{x}{2t})}. \end{aligned}$$

We summarize. Let $v(x,t) = S(t)\overline{\widehat{\phi}}(x)$. Form the function

$$u(x,t) = \frac{c_n}{(it)^{\frac{n}{2}}} \overline{v}(\frac{x}{2t}, \frac{1}{4t}) e^{i\frac{|x|^2}{4t}}. \tag{4.11}$$

Then $u(x,t) = S(t)\phi(x)$! Not only does the transformation (4.11) manufacture another solution of the linear Schrödinger equation from v, it manufactures one whose initial data is essentially the Fourier transform of the original function. The transformation $v \longmapsto u$ given in (4.11) is called the *conformal transformation.*

We linger a bit and consider the conformal transformation in one spatial dimension. Form the function

$$v(x,t) = t^{-\frac{1}{2}} e^{\frac{x^2}{4it}} \overline{u}(\frac{x}{t} \tag{4.12}$$

A direct calculation shows that

$$\begin{aligned} i\partial_t v = i(-\frac{1}{2}t^{-\frac{3}{2}})e^{-\frac{|x|^2}{4it}}\overline{u} + it^{-\frac{1}{2}}e^{-\frac{|x|^2}{4it}}(\frac{x^2}{4it^2})\overline{u} \\ + it^{-\frac{1}{2}}e^{-\frac{|x|^2}{4it}}\overline{u_x}(\frac{x}{t^2}) + it^{-\frac{1}{2}}e^{-\frac{|x|^2}{4it}}\overline{u_t}(\frac{1}{t^2}). \end{aligned}$$

Similarly,

$$\partial_x^2 v = t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4it}} (\frac{4x^2}{16it^2})\overline{u} + t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4it}} (-\frac{1}{2it})\overline{u}$$
$$+t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4it}} \overline{u_x}(-\frac{1}{t})(-\frac{2x}{4it}) + t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4it}} (-\frac{2x}{4it})\overline{u_x}(-\frac{1}{t}) + t^{-\frac{1}{2}} e^{-\frac{|x|^2}{4it}} \overline{u_{xx}}(\frac{1}{t^2}).$$

Combining these expressions reveals that

$$i\partial_t v + \partial_x^2 v = t^{-\frac{5}{2}} e^{-\frac{|x|^2}{4it}} \left\{ i\partial_t \overline{u} + \partial_x^2 \overline{u} \right\}. \tag{4.13}$$

Now, suppose that $i\partial_t u + \partial_x^2 u - |u|^p u = 0$. Then, we see that

$$i\partial_t v + \partial_x^2 v = t^{\frac{p-4}{2}} |v|^p v. \tag{4.14}$$

In particular, if $p = 4$, the conformal transformation maps solutions of 1d quintic NLS into solutions of the same equation. For other values of p, the conformal transformation maps solutions of NLS_p into an NLS-type equation with a power of t in front of the nonlinearity.

We have seen that for a particular value of p, depending on the spatial dimension, this map sends solutions of the nonlinear problem NLS_p into solutions of NLS_p. This special power of p is called the *conformal power*. On $\mathbb{R}^2$, the conformal NLS_p contains the standard cubic nonlinearity.

Exercise 9. $L^4(\mathbb{R}_x \times \mathbb{R}_t)$ *norm invariance of conformal transformation.*

Local smoothing

Consider the solution of the initial value problem for Airy's equation

$$u(x,t) = \int e^{ix\xi} e^{i\xi^3 t} \widehat{\phi}(\xi) d\xi.$$

The heuristics of dispersion suggest that this solution should be smoother locally in x than the initial data. The following calculation, first observed in the Doctoral Thesis of Luis Vega, shows this smoothing effect in a sharp form.

We first take α derivatives in x by writing

$$\partial_x^\alpha u(x,t) = \int e^{ix\xi} (i\xi)^\alpha e^{i\xi^3 t} \widehat{\phi}(\xi) d\xi.$$

Now, we localize in x by setting $x = x_0$. Write $\tilde{\phi}(\xi) = e^{ix_0\xi}\widehat{\phi}(\xi)$. We change variables in the resulting expression

$$\int e^{i\xi^3 t} (i\xi)^\alpha \tilde{\phi}(\xi) d\xi$$

by writing $\xi^3 = \eta, \ \xi = \eta^{\frac{1}{3}}, \ d\xi = \frac{1}{3}\eta^{-\frac{2}{3}} d\eta$ to obtain

$$\frac{1}{3} \int e^{i\eta t} (i\eta^{\frac{1}{3}})^\alpha \tilde{\phi}(\eta^{\frac{1}{3}}) \eta^{-\frac{2}{3}} d\eta.$$

The point of the change of variables was to rewrite the expression as an inverse Fourier transform. By the Plancherel Theorem, the L_t^2 norm of this expression equals

$$\|\partial_x^\alpha u(x_0,\cdot)\|_{L_t^2} = c_\pi \left(\int |(i\eta^{\frac{1}{3}})^\alpha \tilde{\phi}(\eta^{\frac{1}{3}}) \eta^{-\frac{2}{3}}|^2 d\eta \right)^{\frac{1}{2}}.$$

Write $\xi = \eta^{\frac{1}{3}}$ and change back to get

$$3c_\pi \left(\int |\tilde{\phi}(\xi)|^2 \xi^{2\alpha-2} d\xi \right)^{\frac{1}{2}}.$$

Upon choosing $\alpha = 1$ and observing the lack of dependence upon x_0 we obtain the sharp form of *Kato's local smoothing effect*

$$\|\partial_x u\|_{L^\infty_x L^2_t} = \|\phi\|_{L^2_x}. \tag{4.15}$$

The norm $\| \cdot \|_{L^\infty_x L^2_t} = \| \, \| \cdot \|_{L^2_t} \, \|_{L^\infty_x}$.

Exercise 10. *A. Let u be the solution of the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^5 u = 0, \\ u(0) = \phi. \end{cases}$$

Find β such that

$$\|\partial_x^\beta u\|_{L^\infty_x L^2_t} = \|\phi\|_{L^2_x}.$$

B. If ∂_x^5 is replaced by ∂_x^{2j+1} for $j \in \{1, 2, \dots\}$, can you find $\beta(j)$ so that the smoothing estimate holds?

Notice that the decay property for the Schrödinger equation (4.8) and the smoothing property for Airy's equation (4.15) are properties of **linear** evolutions. We are ultimately interested in understanding the (physically relevant) nonlinear generalizations: NLS and KdV. These linear estimates will play an important role in the rigorous study of the nonlinear equations as will be described later.

4.2. **Modulational Instability.** [29], [25]

The first chapter of these notes showed, through the method of characteristics, that nonlinearities can steepen wave profiles. This process occurs when the characteristics compress as we go forward in time. The method of characteristics is quite suggestive of nonlinear effects but not directly applicable to study the nonlinear terms in the presence of dispersive terms (which are not first order). Ultimately, we will carry out a detailed analysis of the nonlinearity using the representation (3.19). Before doing so, we develop some heuristic understanding of the nonlinearity in the nonlinear Schrödinger equation.

We consider here the cubic nonlinear Schrödinger equation $i\partial_t u + \Delta u \pm |u|^2 u = 0$. Recall that we derived this as the equation which governs the (slowly) varying amplitude of a nearly monochromatic wave acted upon by weak nonlinearity and strong dispersion. The (linear) monochromatic wave at the previous order of approximation had constant amplitude. Note that the NLS does have explicit solutions of constant modulus with no spatial dependence

$$u(x,t) = Ae^{-i\omega(A)t}. \tag{4.16}$$

Substituting into the equation leads to the condition $\omega(A) = \mp |A|^2$, revealing that the nonlinearity changes the linear dispersion relation through the wave's intensity.

We examine the linear stability of this solution. In particular, we wonder if small changes in the amplitude and phase of the constant amplitude spatially independent solution (4.16) lead to the complete destruction or small change in the form of the solution. Therefore, consider a function of the form

$$v(x,t) = Ae^{-i(\omega(A)t + \epsilon\theta(x,t))}(1 + \epsilon\psi(x,t)),$$

where θ and ψ are $\mathbb{R}$-valued. What conditions must θ, ψ satisfy for v to be a solution of NLS? By Taylor expansion of the exponential,

$$v = Ae^{-i\omega(A)t}(1 + i\epsilon\theta + \epsilon\psi) + O(\epsilon^2).$$

We calculate

$$\begin{aligned} i\partial_t v + \Delta v \pm |v|^2 v &= \{\omega(A)(1 + i\epsilon\theta + \epsilon\psi) + \epsilon(i\psi_t - \theta_t) + \epsilon(i\Delta\theta + \Delta\psi) \\ &\pm |A|^2(1 + i\epsilon\theta + 3\epsilon\psi)\}Ae^{-i\omega(A)t}. \end{aligned}$$

Therefore, we should require

$$\begin{aligned} 0 &= \omega(A) + \partial_t\psi + \Delta\theta \pm |A|^2\theta = \psi_t + \Delta\theta, \\ 0 &= \omega(A) - \theta_t + \Delta\psi \pm 3|A|^2\psi = -\theta_t + \Delta\psi \pm 2|A|^2\psi. \end{aligned}$$

We can combine these equations by requiring

$$\psi_{tt} + \Delta^2\psi \pm 2|A|^2\Delta\psi = 0. \tag{4.17}$$

Finally, suppose that $\psi = (constant)e^{ik\cdot x}e^{\sigma t}$. Then we must have

$$\sigma^2 = |k|^2(\pm 2|A|^2 - |k|^2). \tag{4.18}$$

In case $\pm 2|A|^2 - |k|^2 < 0$, the right-side of (4.18) is negative and σ is purely imaginary. For these "modulations" of the spatially boring solution (4.16), the perturbation does not change the solution much. In case $\pm 2|A|^2 - |k|^2 > 0$, which requires we choose $+$ and have $|k|$ be small relative to $|A|$, then σ can be real which leads to exponential growth of the perturbation thereby violently destroying the form of (4.16).

The preceding discussion described the *modulational instability* of the nonlinear Schrödinger equation. This instability mechanism was first observed in 1967 by Benjamin and Feir in the context of water waves so it is sometimes referred to as the *Benjamin-Feir instability.* Our discussion shows the linear instability of long-wave (meaning small spatial frequency) perturbations of the spatially constant solution. Recall that in the physical problem for which NLS is an improved approximation to a linear problem with monochromatic solution, this corresponds to "sideband" modulations of frequency close to the carrier frequency which can grow exponentially in time.

In some cases, the modulational instability can run wild and leads to singularity formation in the solution. In other cases, the instability develops until a wave form which balances the dispersion at high spatial frequencies with the low frequency instability. There is a huge difference in these two behaviors so a characterization of the contexts in which each occurs is an important topic for study.

4.3. **Explicit Solutions and Scaling.**

- Travelling waves

  We begin by recalling some facts from calculus. The function $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$ satisfies $(\operatorname{sech} x)' = \operatorname{sech} x \tanh x$ $(\operatorname{sech} x)'' = (\operatorname{sech} x)^3 + \operatorname{sech} x[1 - (\operatorname{sech} x)^2]$. Since $[1 - (\operatorname{sech} x)^2] = (\tanh x)^2$, $(\operatorname{sech} x)'' = \operatorname{sech} x$. One can then easily verify that $[(\operatorname{sech} x)^{\frac{2}{p}}]'' = \frac{4}{p^2}(\operatorname{sech} x)^{\frac{2}{p}} - \frac{2}{p}\frac{2-p}{p}[(\operatorname{sech} x)^{\frac{2}{p}}]^{p+1}$.

  We look for a *travelling wave solution* $u(x,t) = f(x - ct)$ of the generalized KdV equation $u_t + u_{xxx} + (u^{p+1})_x = 0$. Substituting in the desired form leads to the ODE $-cf' + f''' + (f^{p+1})' = 0$. We integrate once and set the integration constant to 0 since we seek a function f which decays at

$\pm\infty$. The preceding calculus discussion shows that, up to certain constants, the function $f(x) = (\text{ sech } x)^{\frac{2}{p}}$ satisfies this ODE.

These explicit solutions reveal that the dispersive decay can exactly balance with the nonlinear effect resulting in a travelling wave.

- Time-periodic solutions of NLS
- Scaling invariance

  Let $u(x,t)$ be a known solution of the generalized KdV equation $u_t + u_{xxx} + (u^{p+1})_x = 0$. We form a one parameter family of solutions by rescaling,
$$u_\sigma(x,t) = \sigma^{\frac{2}{p}}(\sigma x, \sigma^3 t).$$

5. Hamiltonian Structure and Conserved Quantities

5.1. **Classical Mechanics.** Suppose we are given $L : \mathbb{R}^n_{\dot{y}} \times \mathbb{R}^n_y \times I \longmapsto \mathbb{R}$ where I is an interval in $\mathbb{R}$. Assume that L is smooth and convex with respect to the first slot $\dot{y}$. The function L is called the *Lagrangian*. Form the *Action Functional*
$$I[w] = \int_{t_0}^{t_1} L(\dot{w}(t), w(t), t)dt$$
for $w \in \mathcal{A} = \{\text{competitors subject to some boundary conditions}\}$.

Hamilton's Principle: *Motions of a mechanical system governed by the Lagrangian L coincide with extremals of the Action Functional $I[\cdot]$.*

By the calculus of variations, extremizers of $I[\cdot]$ solve the *Euler-Lagrange equations*
$$-\frac{d}{dt}(L_{\dot{y}}(\dot{y}(t), y(t), t) + L_y(\dot{y}(t), y(t), t) = 0. \tag{5.1}$$
This is a system of n second order equations in $y(t)$.

Define $p(t) = L_{\dot{y}}(\dot{y}(t), y(t), t)$ and $H(p(t), y(t), t) = \sup_{\dot{y}}(p(t) \cdot \dot{y} - L(\dot{y}, y(t), t))$. Notice that H is the Legendre transform of L. Then, the Euler-Lagrange equations (5.1) can be rewritten
$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial y}, \\ \dot{y} = \frac{\partial H}{\partial p} \end{cases} \tag{5.2}$$
This is a system of $2n$ first order equations in $(p(t), y(t))$. The equations (5.2) are called *Hamilton's Equations.*

Remarks

- Mechanical motions are *canonically* described as a consequence of a variational principle.
- Hamilton's equations are special among a general class of equations.
$$\begin{cases} \dot{x} = F(x,y), \\ \dot{y} = G(x,y) \end{cases}$$
  In particular, the form of Hamilton's equations implies phase space volume is conserved under the dynamics. Under certain conditions, this property implies the mechanical motion is recurrent (Poincaré recurrence).

Suppose we write $w = (p, y),\ w \in R^{2n}$. We rewrite Hamilton's equations by writing $H_z = (H_p, H_y)$ and defining the matrix

$$\mathbb{J} = \begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0. \end{bmatrix} \tag{5.3}$$

With these notations, Hamilton's equations (5.2) may be written

$$\dot{w} = \mathbb{J} H_w. \tag{5.4}$$

Note that J is essentially a rotation matrix and in this notation we can interpret the dynamics of Hamilton's equations as "move perpindicular to the gradient of H". Notice also that $\mathbb{J}^2 = -\mathbb{I}$ and Hamilton's equations are defined on an *even* dimensional space. This suggests there may be a nice connection with complex numbers.

Complex Notation: For $(x, y) \in \mathbb{R}^2$, define $z = x + iy,\ \overline{z} = x - iy,\ i^2 = -1$. Notice that $z, \overline{z}$ give x, y and vice-versa. Define $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Observe that $\partial_{\overline{z}z} = (\partial_x + i\partial_y)(x + iy) = 0$. If $f(x, y)$ and $g(x, y)$ are two $\mathbb{R}$-valued functions, we can form $u(x, y) = f(x, y) + ig(x, y)$. Then, u is *complex-analytic* if, upon reexpressing x and y via $z,\ \overline{z}$, we see that $\partial_{\overline{z}} u = 0$. Analytic functions depend only upon z and not $\overline{z}$ (or vice versa).

Now, suppose we write $z_1 = y_1 + ip_1, z_2 = y_2 + ip_2, \ldots$. Reexpress $H(p, y) = H(z, \overline{z})$. Consider the following system of n equations in the complex variables $z_j(t)$ given by

$$\dot{z_j} = iH_{\overline{z_j}}. \tag{5.5}$$

Calculating $\dot{z_j}$ and taking the $\partial_{\overline{z_j}}$ derivative shows this system is equivalent with (5.2).

Example 5.1. *Let*

$$H = H(p, y) = |p|^2 + |y|^2 = \sum_j p_j^2 + y_j^2.$$

We can rewrite this using the notation above as

$$H(z, \overline{z}) = |z|^2 = \sum_j |z_j|^2 = \sum_j z_j \overline{z_j}.$$

Hamilton's Equations then read

$$\begin{cases} \dot{p_j} = -y_j = -\frac{\partial H}{\partial y}, \\ \dot{y_j} = p_j = \frac{\partial H}{\partial p} \end{cases}$$

and can be rewritten

$$\dot{z_j} = iH_{\overline{z_j}} = iz_j.$$

This says the dynamics moves the complex components z_j of the complex vector z "perpindicular to z_j with speed $|z_j|$."

Example 5.2. *(slightly fancier)*
Change the Hamiltonian in the previous example by writing

$$H(z, \overline{z}) = \sum_j j^2 |z_j|^2 = \sum_j j^2 z_j \overline{z_j}.$$

Hamilton's equations are then

$$\dot{z_j} = ij^2 z_j \tag{5.6}$$

which says the complex number z_j moves perpindicular to itself with speed $j^2|z_j|$. We still have the motion of each of the components of z moving indepedently of the other components, which is to say the motion is uncouple. In this example, the speed of motion also depends on the index j.

Example 5.3. *(NLS)*
Let $u : \mathbb{T} \times I \longmapsto \mathbb{C}$ where $\mathbb{T}$ is the 1d torus and I is a time interval. Form the Hamiltonian Functional

$$H = H[u, \overline{u}] = \int_{\mathbb{T}} |\nabla u|^2 = \int_{\mathbb{T}} \nabla u \nabla \overline{u} dx.$$

(Recall that this is just the classical Dirichlet energy.) The usual calculus of variations argument shows

$$\lim_{\tau \to 0} \frac{H[u, \overline{u} + \tau \overline{v}] - H[u, \overline{u}]}{\tau} = \int_{\mathbb{T}} (-\Delta u) \overline{v} dx = \langle -\Delta u, v \rangle.$$

Therefore, $H_{\overline{u}}[u, \overline{u}] = -\Delta u$ and Hamilton's equations in this case read

$$\dot{u} = i H_{\overline{u}} = -i \Delta u$$

which is precisely the linear Schrödinger equation!

The initial value problem

$$\begin{cases} i\partial_t u + \Delta u = 0, & \mathbb{T} \times \{t > 0\} \\ u(x, 0) = \phi(x), & \mathbb{T} \times \{t = 0\}. \end{cases} \tag{5.7}$$

has solution

$$u(x, t) = \sum_k e^{ikx} e^{-i|k|^2 t} \widehat{\phi}(k).$$

We can reexpress the Hamiltonian in terms of the Fourier transform by writing

$$\begin{aligned} H[u, \overline{u}] &= \int_{\mathbb{T}} \nabla u \nabla \overline{u} dx \\ &= \int_{\mathbb{T}} \left(\sum_k e^{ikx} (ik) e^{-i|k|^2 t} \widehat{\phi}(k) \right) \overline{\left(\sum_{k'} e^{ik'x} (ik') e^{-i|k'|^2 t} \widehat{\phi}(k') \right)} dx \\ &= \sum_k |k|^2 |\widehat{\phi}|^2. \end{aligned}$$

So, Schrödinger's equation is an infinite dimensional Hamiltonian system in which each Fourier coefficient behaves like the solution of a Harmonic oscillator. Recall that the dispersion relation calculation showed that

$$\frac{d}{dt} \widehat{u(t)}(k) = i|k|^2 \widehat{u(t)}(k)$$

which is essentially the same as appeared in (5.6).

5.2. **Action symmetries and conservation laws.** KdV and NLS have been derived as approximate models for various physical phenomena. From physical principles, we expect the KdV and NLS evolutions to take place leaving certain integral quantities, associated with mass, energy, momentum, etc. to be time invariant. Time-invariant quantities are said to be *conserved.* For example

$$\partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x u^2 = 0,$$

can be multiplied by u and rewritten

$$\frac{1}{2}\partial_t u^2 + \partial_x(uu_{xx} - \frac{1}{2}u_x^2 + [\frac{1}{3}u^3]) = 0.$$

Upon integrating over all of $x \in \mathbb{R}$ (assuming that u and its derivatives decay as $|x| \to +\infty$) we learn

$$\partial_t \int u^2 dx = 0.$$

Therefore, the KdV evolution satisfies *conservation of the L^2 norm*

$$\|u(t)\|_{L^2_x} = \|u(0)\|_{L^2_x}.$$

This places a basic constraint on the dynamics: all the motion takes place on a sphere in L^2_x. We are interested in finding other conserved quantities to constrain (and therefore better understand) the dynamics. However, the method used above to discover the L^2 conservation law for KdV requires a bit of cleverness that may not be easily available for discovering more complicated conservation laws. A systematic approach to discovering some conservation laws is contained in: **E. Noether's Principle:** *If a variational principle is invariant under a family of transformations, then solutions of its Euler-Lagrange equation satisfy a conservation law.*

Recall that mechanical motions are characterized as extremizers of the action which is a variational principle. The present discussion borrows from [**?**], [29], [**?**].

Let $L : \mathbb{R}^m \times \mathbb{R} \longmapsto \mathbb{R},\ L = L(p, w)$ and $u : \mathbb{R}^m \longmapsto \mathbb{R}$. Let $\mathbb{U} \subset \mathbb{R}^m$. Form $I[u] = \int_{\mathbb{U}} L(Du, u)dx$. The calculus of variations shows that u is a smooth critical point of $I[\cdot]$ if and only if the Euler-Lagrange equation

$$-\sum_{i=1}^{m} \partial_{x_i}(L_{p_i}(Du, u)) + \partial_w L(Du, u) = 0$$

is satisfied.

Noether's principle allows us to look for invariances of the variational principle instead of for the conservation laws. The former turn out to be easier to find than the latter.

Notation The following notation is introduced to flexibly describe a one parameter family of transformations. Let $\mathbf{g} : \mathbb{R}^m \times \mathbb{R} \longmapsto \mathbb{R}^m,\ w : \mathbb{R}^m \times \mathbb{R} \longmapsto \mathbb{R},\ \phi : \mathbb{R}^m \longmapsto \mathbb{R}^m$. We define a *domain variation* $x' = \mathbf{g}(x, \tau) = x + \tau\phi(x) + o(\tau),\ \phi = \frac{\partial g}{\partial \tau}|_{\tau=0}$. We define a *function or target variation* $u'(x') = w(x(x'), \tau) = u(x) + \tau v(x) + o(\tau),\ v = \frac{\partial x}{\partial \tau}|\tau = 0$. Naturally $\mathbb{U}' = \mathbf{g}(\mathbb{U}, \tau)$.

The value of the domain and target varied action at parameter value τ is

$$i(\tau) = \int_{\mathbb{U}'} L(Du', u')dx'.$$

We are interested in characterizeing situations where $i(\cdot)$ is independent of its argument or when $i(\cdot)$ is infinitessimaly invariant: $i'(0) = 0$.

Lemma 5.1.

$$i'(0) = \int_{\mathbb{U}} L_{p_i}[v_{x_i} - u_{x_j}\phi^j_{x_j}] + (\partial_w L)v + L\phi^i_{x_j} dx. \tag{5.8}$$

Proof. follow your nose and change variables. Expand out the jacobian determinant using divergence. □

Theorem 5.1. (Noether) *Given $L : \mathbb{R}^m \times \mathbb{R} \longmapsto \mathbb{R}$, $\mathbb{U} \subset \mathbb{R}^m$, $u : \mathbb{R}^m \longmapsto \mathbb{R}$, we form the action functional $I[u] = \int_U L(Du, u)dx$. Suppose that u is a smooth critical point of the action functional $I[\cdot]$. Assume that $I[\cdot]$ is invariant under a family of transformations $u \longmapsto u'$, $x \longmapsto x'$ (so $\mathbb{U} \longmapsto \mathbb{U}'$), for all regions $\mathbb{U}$,*

$$\int_{\mathbb{U}} L(Du, u)dx = \int_{\mathbb{U}'} L(Du', u')dx'. \tag{5.9}$$

Then,

$$\partial_{x_i} \left\{ \partial_{p_i} L(Du, u)[v - u_{x_j}\phi^j_{x_j}] + L(Du, u)\phi^i \right\} = 0. \tag{5.10}$$

Proof. It suffices to show that the integrand in the lemma may be rewritten as the divergence appearing in (5.10). This may be verified using the fact that u is a solution to the Euler-Lagrange equation. ☐

Remark 5.1. In case $u : \mathbb{R}^m \longmapsto \mathbb{C}$, the conclusion changes to the statement

$$\partial_{x_i} \left\{ \frac{\partial L}{\partial(\partial_{x_i} u)}[v - u_{x_j}\phi^j] + \frac{\partial L}{\partial(\partial_{x_i} \overline{u})}[v - \overline{u}_{x_j}\phi^j] + L\phi^i \right\} = 0 \tag{5.11}$$

5.3. **Lagrangian Structure of NLS.** The nonlinear Schrödinger dynamics may be described using a variational principle. To show this, we introduce a Lagrangian, form the associated action functional and then calculate NLS as the Euler-Lagrange equation characterizing smooth critical points of the action functional.

Consider the nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = 0,$$

posed for $x \in \mathbb{R}^d$. We assume ψ and its derivatives are smooth and vanish as $|x| \to \infty$. The nonlinearity f is a smooth function of its argument and we define

$$F(\lambda) = \int_0^\lambda f(s)ds.$$

Define

$$L = \frac{i}{2}(\overline{\psi}\psi_t - \psi\overline{\psi_t}) - [|D\psi|^2 + F(|\psi|^2)]. \tag{5.12}$$

Evidently, $L = L(\psi, \overline{\psi}, \psi_t, \overline{\psi_t}, D\psi, D\overline{\psi})$. Form the action functional

$$I[\psi] = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} L \ dxdt, \tag{5.13}$$

defined for $\psi \in A$, some appropriate class of admissible functions.

The usual calculus of variations argument shows that if ψ is a smooth critical point of $I[\cdot]$ then ψ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial \psi} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \frac{\partial L}{\partial(\partial_{x_i}\psi)} + \frac{\partial}{\partial t} \frac{\partial L}{\partial(\partial_t \psi)}.$$

We calculate the various terms using L to find (the complex conjugate of) NLS

$$-i\overline{\psi_t} + \Delta\overline{\psi} + f(|\psi|^2)\overline{\psi} = 0.$$

Noether's theorem applied to NLS We write $\mathbb{R}^m_y = \mathbb{R}^1_t \times \mathbb{R}^d_x$, distinguishing the time variable t from the spatial variables x. A one parameter family of transformations is defined when we specify $\phi^0,\ \phi$ and v

$$\begin{cases} t \longmapsto t' = t + \tau\phi^o(x,t,\psi), \\ x \longmapsto x' = x + \tau\phi(x,t,\psi), \\ \psi(x,t) \longmapsto \psi'(x',t') = \psi(x,t) + \tau v(x,t). \end{cases} \tag{5.14}$$

Suppose that the NLS action in (infintessimally) invariant under the family of transformations (5.14). Noether's theorem implies that (5.11) holds. Recalling the distinguished time variable, if we integrate over the spatial domain, we obtain the conservation law

$$\partial_t \int_{\mathbb{R}^d} \left\{ \frac{\partial L}{\partial(\partial_t\psi)}[v - \psi_t\phi^0 - D_x\psi\cdot\phi] + \frac{\partial L}{\partial(\partial_t\overline{\psi})}q[v - \overline{\psi_t}\phi^0 - D_x\overline{\psi}\cdot\phi] + L\phi^0 \right\} dx = 0.$$

We apply this formalism to identify certain invariances of the NLS action functional and thereby infer certain conservation laws for NLS.

Invariance by phase shift: Consider the transformation $\psi \longmapsto \tilde{\psi} = e^{i\tau}\psi$. For tiny τ, this is equivalent to

$$\begin{cases} t \longmapsto \tilde{t} = t, \\ x \longmapsto \tilde{x} = x, \\ \psi \longmapsto \tilde{\psi} = \psi + i\tau\psi. \end{cases} \tag{5.15}$$

This transformation leaves the Lagrangian invariant and therefore the action functional is also invariant so Noether's theorem applies. Comparing with (5.14), we see that $\phi^0 = 0,\ \phi = 0,\ v = i\psi$. Recalling the Lagrangian and calculating

$$\frac{\partial L}{\partial(\partial_t\psi)} = \frac{i}{2}\overline{\psi},\ \frac{\partial L}{\partial(\partial_t\overline{\psi})} = \frac{i}{2}\psi,$$

we find that

$$\int_{\mathbb{R}^d} (\frac{i}{2}\overline{\psi})(i\psi) + (-\frac{i}{2}\psi)(-i\overline{\psi})dx = -\int_{\mathbb{R}^d} |\psi|^2 dx,$$

is conserved. Define

$$N = \int_{\mathbb{R}^d} |\psi|^2 dx. \tag{5.16}$$

The conserved quantity N represents the mass, wave action, plasmon number, or wave power in various applications of NLS as a model equation. Note that the conservation law prior to spatial integration related to this invariance is

$$\partial_t|\psi|^2 + D_x\cdot\{i(\psi D_x\overline{\psi} - \overline{\psi}D_x\psi)\} = 0. \tag{5.17}$$

The quantity $\{i(\psi D_x\overline{\psi} - \overline{\psi}D_x\psi)\}$ may then be interpreted as a current.

The invariance by phase shift is sometimes referred to as *gauge invariance.*

Invariance by time translation: We define a transformation

$$\begin{cases} t \longmapsto t' = t + \tau\phi^0, \\ x \longmapsto x' = x, \\ \psi \longmapsto \psi' = \psi. \end{cases} \tag{5.18}$$

This transformation leaves the Lagrangian and, hence, the action functional invariant. In the notation of (5.14), we have $\phi = 0,\ v = 0, \phi^0 \neq 0$. Noether's theorem implies the time invariance of

$$\int_{\mathbb{R}^d} \left\{ \frac{i}{2}\overline{\psi}\left[-\psi_t \phi^0\right] + (-\frac{i}{2})\left[-\overline{\psi_t}\phi^0\right] + L\phi^0 \right\} dx.$$

Substituting L from (5.12) reveals that

$$H = \int_{\mathbb{R}^d} |D\psi|^2 - F(|\psi|^2) dx \tag{5.19}$$

is conserved. This quantity is the *Hamiltonian* for the NLS equation. It represents the energy in various applications.

Exercise 11. *Find the flux associated to the Hamiltonian. In particular, use the full strength of Noether's theorem to identify G such that $\partial_t H + D_x \cdot G = 0$.*

Invariance by space translation: We define a transformation

$$\begin{cases} t \longmapsto t' = t, \\ x \longmapsto x' = x + \tau b, \\ \psi \longmapsto \psi' = \psi. \end{cases} \tag{5.20}$$

This transformation leaves L invariant so the associated action functional is also invariant. Here $\phi = b,\ \phi^0 = 0,\ v = 0$. Noether's theorem implies

$$\int_{\mathbb{R}^d} \frac{i}{2}\overline{\psi}[-D_x\psi \cdot b] + (-\frac{i}{2}\psi)[-D_x\overline{\psi} \cdot b] dx$$

is conserved. Therefore, $\frac{i}{2}\int_{\mathbb{R}^d}[\psi D_x\overline{\psi} - \overline{\psi} D_x \psi] dx \cdot b$ is conserved. Since b was arbitrary, we have that the *linear momentum* of solutions of NLS

$$\mathbf{P} = i\int_{\mathbb{R}^d} [\psi D_x \overline{\psi} - \overline{\psi} D_x \psi] dx \tag{5.21}$$

is conserved.

Let's also define the *mass center* of the NLS wave,

$$\mathbf{X} = \frac{1}{N}\int_{\mathbb{R}^d} x|\psi|^2 dx. \tag{5.22}$$

Note that

$$N\frac{d\mathbf{X}}{dt} = \int x\partial_t |\psi|^2 dx = -\int x D_x \cdot \{i(\psi D_x \overline{\psi} - \overline{\psi} D_x \psi)\} dx.$$

We integrate by parts to find

$$N\frac{d\mathbf{X}}{dt} = \mathbf{P}. \tag{5.23}$$

The mass center moves with constant momentum.

Exercise 12. (Galilean Invariance) *Consider the transformation*

$$\begin{cases} t \longmapsto t' = t, \\ x \longmapsto x' = x - \mathbf{c}t, \\ \psi(x,t) \longmapsto \psi'(x',t') = e^{-i[\frac{1}{2}\mathbf{c}\cdot x' + \frac{1}{4}|\mathbf{c}|^2 t']}\psi(x' + \mathbf{c}t', t'). \end{cases} \tag{5.24}$$

Verify that this transformation leaves the action associated to NLS invariant. Use this fact to find a conserved quantity.

Scaling invariance for power law nonlinearities Suppose $f(|\psi|^2) = q|\psi|^{p-1}$, $q = \pm 1$. The resulting equation NLS_p is invariant under the scale transformation

$$\begin{cases} t \longmapsto t' = \frac{t}{\lambda^2}, \\ x \longmapsto x' = \frac{x}{\lambda}\psi(x,t) \longmapsto \psi'(x',t') = \lambda^{\frac{2}{p-1}}\psi(\lambda x', \lambda^2 t') \end{cases} \tag{5.25}$$

Is the associated action invariant?

Psuedoconformal invariance

Morawetz identity?

Variance Identity

type out $\partial_t^2 |u|^2 = D_x \cdot D_x \cdot S_{ij}$. Then, the appearance of the double divergence motivates multiplying by $|x|^2$ and integrating, so that IBP yields a nice identity for the variance. Conclude that negative energy initial data (focussing case) evolves into a singularity.

5.4. **Lagrangian Structure of KdV.** See Mirua's survey article [24] Be sure to cover the galilean transformation for KdV to justify introducing the spectral parameter λ into the Schrödinger eigenvalue problem, see (4.6) in [24].

5.5. **Hamiltonian Structure of NLS.** Verify that NLS_p for $p = 1 + \frac{4}{n}$ has the property that the conformal transformation (4.11) maps solutions to solutions. Verify that L^q_{xt} for $q = \frac{2(n+2)}{n}$ is invariant under the conformal transformation.

5.6. **Symplectic Hilbert Space.** [18], [19], [5]

5.7. **Insights from Finite-dimensional Hamiltonian Systems.** [1]

Anticipated Phenomena, Behavior

Integrability vs. Ergodicity

New Behavior occurring in Infinite Dimensions

6. Integrable Model: KdV Equation

6.1. **History.**

- Russell's Observations of waves in a channel
- Korteweg and de Vries
- Fermi-Pasta-Ullam Experiment
- Kruskal and Zabusky discover the Soliton [17]
- Gardener-Greene-Kruskal-Miura discover IST [10]
- Lax Pair[25]

6.2. **Inverse Scattering Transform (IST) overview. Some explicit calculus**

$\text{sech}x = \frac{2}{e^x+e^{-x}}$, $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$, $\text{sech}^2 x + \tan^2 x = 1$.

Form $f(x) = \left(\text{sech}\frac{x}{2}\right)^2$.

$$f'(x) = 2\left(\text{sech}\frac{x}{2}\right)\left(\text{sech}\frac{x}{2}\right)' = \left(\text{sech}\frac{x}{2}\right)^2\left(\tanh\frac{x}{2}\right) .$$

$$\begin{aligned} f''(x) &= \left(\text{sech}\frac{x}{2}\right)^2\left(\tanh\frac{x}{2}\right)^2 + \tfrac{1}{2}\left(\text{sech}\tfrac{x}{2}\right)^4 \\ &= \left(\text{sech}\frac{x}{2}\right)^2\left[\left(\text{sech}\frac{x}{2}\right)^2 + \left(\tanh\frac{x}{2}\right)^2\right] - \tfrac{1}{2}\left(\text{sech}\tfrac{x}{2}\right) \\ &= \left(\text{sech}\frac{x}{2}\right)^2 - \tfrac{1}{2}\left(\text{sech}\tfrac{x}{2}\right)^4 \end{aligned}$$

$$\frac{1}{2}f^2 = \frac{1}{2}(\text{sech}\frac{x}{2})^4$$

$$\Rightarrow \; f'' + \frac{1}{2}f^2 - f = 0 \; .$$

More generally,

$$f_c(x) = 3c\big(\text{sech}\frac{\sqrt{c}}{2}\,x\big)^2 \;\; \text{solves} \;\; f'' + \frac{1}{2}f^2 - cf = 0.$$

Travelling wave solutions of KdV. Can we find solutions u in the form $u(x,t) = f(x-ct)$ (with f decaying as $|x| \to \infty$) of the KdV equation $u_t + u_{xxx} + u_x = 0$? Inserting the ansatz for u leads to the ODE $-cf' + f''' + (\frac{1}{2}f^2)' = 0$ which we rewrite as $f'' + \frac{1}{2}f^2 - cf = 0$. So. one possible solution is $u(x,t) = f_c(x-ct) = 3c\big(\text{sech}\frac{\sqrt{c}}{2}\,(x-ct)\big)^2$. This is a *travelling wave solution* of KdV which is distinguished with the name *soliton.* The dispersion in KdV suggests localized pulses should spread out, resulting in decay, but the nonlinearityi sharpens fronts. The solitons reveal these two effects can balance in KdV.

We will study a method for solving the KdV initial value problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(\frac{1}{2}u^2) = 0 \\ u(x,0) = \phi(x), \end{cases}$$

which has several unexpected twists. Here we will assume that ϕ is a very nice function. A basic test of our understanding of this new method will be to show that

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(\frac{1}{2}u^2) = 0 \\ u(x,0) = f_c(x) \text{ (nice) }, \end{cases}$$

has the solution $u(x,t) = f_c(x-ct)$.

The presentation that follows is largely unmotivated but does provide a direct illustration of the method and it's underlying ideas. Later, we will also explain how the inverse scattering method for solving KdV was found and to what extent the method generalizes to other evolution equations.

Schrödinger Eigenvalue Problem The Schrödinger Eigenvalue problem

$$\frac{d^2}{dx^2}\;\psi + (V(x) + \lambda)\psi = 0, \;\; x \in \mathbb{R}, \tag{6.1}$$

arises naturally in Quantum Mechanics. Indeed, the dynamics of a quantum wave function $\Phi(x,t)$ are governed by the true dependent Schrödinger equation

$$i\hbar\frac{\partial \Phi}{\partial t} \;\; = \;\; E\Phi \;\; = \;\; -\frac{\hbar}{2m}\Delta\Phi + V(x)\Phi$$

Assuming $\Phi(x,t) = e^{-i\lambda t}\psi(x)$ leads to $\lambda\hbar\psi = -\dfrac{\hbar^2}{2m}\Delta\psi + V(x)\psi$ which up to sign and constants is the Schrödinger eigenvalue problem (6.1).

From the theory of Sturm-Liouville problems or basic QM, we know that, for appropriate potentials, there are some real eigenvalues λ_n. Eigenfunctions corresponding to distinct eigenvalues are orthogonal, etc. In case $V(x) = 0$, solutions are of the form

$$\psi(x) = e^{\pm\sqrt{-\lambda}x}$$

which are exponentially decaying/growing if $\lambda < 0$ and oscillatory if $\lambda > 0$.

Remark 6.1. If we replace $V(x)$ in (Schoeigen) by $V(x,\alpha)$ where α is some parameter, the eigenvalues may vary with α. The associated eigenfunctions may also depend upon α. $\rightsquigarrow \lambda = \lambda(\alpha)$, $\psi = \psi(\alpha)$.

Bold Fact. *The eigenvalue problem*

$$\frac{\partial^2\psi}{\partial x} + (\tfrac{1}{6}v(x,t) + \lambda)\psi = 0 \tag{6.2}$$

and the evolution $-\partial_t u + \partial_x^3 u + \partial_x(\frac{1}{2}u^2) = 0$ *(note: – sign is merely a convenience) are spectacularly interrelated!*

The main point of this section is to introduce this relationship and forecast some aspects of its systematic study.

Notice that if ψ vanishes so does $\partial_x^2\psi$. So we solve (6.2) for u,

$$u(x,t) = -6\left(\lambda + \frac{1}{\psi}\,\frac{\partial^2 t}{\partial x^2}\right) \tag{6.3}$$

We can calculate u_t, u_x, u_{xxx} in terms of λ and ψ. For example

$$\begin{aligned} u_x &= -6\left(\frac{\psi_{xxx}}{\psi} - \frac{\psi_{xx}\psi_x}{\psi^2}\right) = -6\frac{1}{\psi^2}\,(\psi\psi_{xxx} - \psi_x\psi_{xx}) \\ &= -\frac{6}{\psi^2}\,\partial_x(\psi\psi_{xx} - \psi_x^2)\ . \end{aligned}$$

A tedious calculation shows $-u_t + u_{xxx} + \partial_x(\frac{1}{2}u^2) = 0$ may be rewritten in terms of ψ as

$$\lambda_t\psi^2 - \frac{\partial}{\partial x}\,\psi^2\,\frac{\partial}{\partial_x}\left(\frac{\psi_{xxx} - \psi_t + (\frac{1}{2}u - 3\lambda)\psi_x}{\psi}\right) \;=\; 0\ . \tag{6.4}$$

Integrate this over $x \in \mathbb{R}$, assuming ψ is a nice decaying eigensolution of (6.2) and that u is also nice and decaying, to conclude $\lambda_t = 0$! If $u(x,t)$ evolves according to KdV then the eigevalue problem (6.2) with *t-dependent* potential $u(x,t)$ has *t-independent* eigenvalues.

Since $\lambda_t = 0$, $\psi^2\,\frac{\partial}{\partial x}(\qquad) = \text{const.}$

Our decay assumptions on u and $\psi \Rightarrow \text{const} = 0$, so

$$\frac{\partial}{\partial x}\left(\frac{\psi_{xxx} - \psi_t + (\frac{1}{2}u - 3\lambda)\psi_x}{\psi}\right) \;=\; 0\ .$$

From (6.2),

$$\begin{aligned} \psi_{xxx} &= \partial_x(-\{\frac{1}{6}u + \lambda\}\psi) = -\frac{1}{6}u_x\psi - \frac{1}{6}u_x\psi_x - \lambda\psi_x\ , \\ &\Rightarrow \frac{\partial}{\partial x}\left(\frac{\psi_t + \frac{1}{6}u_x\psi + 4\lambda\psi_x - \frac{1}{3}u\psi_x}{\psi}\right) \;=\; 0 \end{aligned}$$

and therefore

$$\frac{\partial_t + \frac{1}{6}u_x\psi + 4\lambda\psi_x - \frac{1}{3}u\psi_x}{\psi} \;=\; \text{const.}$$

Since this constant is independent of x, we can determine its value by studying the asymptotic behavior as $x \to \pm\infty$. The appearance of ψ_t suggests we may be able to learn how the eigenfunctions change as the potential $u(x,t)$ evolves.

Bound states. $\lambda = -\kappa^2$.

Suppose $\lambda = -\kappa^2$, $\kappa \in \mathbb{R}$ is an eigenvalue with associated eigenfunction $\psi(x)$. Since u, u_κ vanish as $x \to \pm\infty$, we must have

$$\psi(x) \to A\, e^{-\kappa x} \quad \text{as} \quad x \to +\infty \ .$$

Normalization: $A = 1$, for all time. Note that the normalization guarantees $\psi_t \to 0$ as $x \to +\infty$.

$$\frac{\psi_t + \frac{1}{6} u_x \psi + 4\lambda \psi_x - \frac{1}{3} u \psi_x}{\psi} \underset{x \to +\infty}{\longrightarrow} 4\lambda \ \frac{\psi_x}{\psi} \ = \ -4\lambda\kappa = 4\kappa^3 \ .$$

Therefore

$$\psi_t + \tfrac{1}{6} u_x \psi + 4\lambda \psi_x - \tfrac{1}{3} u \psi_x - 4\kappa^3 \psi = 0 \ .$$

Multiply by ψ and integrate to learn

$$\frac{d}{dt} \frac{1}{2} \int \psi^2 dx - 4\kappa^3 \int \psi^2 dx \ = \ 0 \ .$$

Define $c^{-1}(t) = \displaystyle\int \psi^2(x,t) dx$ and we learn

$$\frac{d}{dt} c^{-1}(t) = 8\kappa^3 c^{-1}(t) \implies c(t) = c(0)\, e^{-8\kappa^3 t} \ .$$

Scattering states. $\lambda = k^2 \geq 0$.

We choose the solution of (6.2) to have the asymptotic forms

$$\psi(x,t) \to e^{ikx} + R(K,t) e^{-ikx} \quad \text{as} \quad x \to -\infty$$
$$\psi(x,t) \to T(k,t) e^{ikx} \quad \text{as} \quad x \to +\infty$$

Interpretation: "scattering", (explain more)

$\overset{e^{ikx}}{\longrightarrow}$

| potential $v(x,\cdot)$ |

$R(k)\, e^{-ikx} \leftarrow$ $\qquad\qquad$ $\rightarrow T(k)\, e^{ikx}$

A basic property of the reflection coefficient $R(k,t)$ and the transmission coefficient is $|R(k,t)|^2 + |T(k,t)|^2 = 1$ (inferred from Wronskian analysis).

$$\frac{\psi_t + \frac{1}{6} u_x \psi + 4\lambda \psi_x - \frac{1}{3} u \psi_x}{\psi} \underset{x \to -\infty}{\longrightarrow} \frac{R_t\, e^{-ikx} + 4\lambda i\kappa e^{ikx} - 4\lambda i\kappa R e^{-ikx}}{e^{ikx} + Re^{-ikx}}$$

$$= \frac{(R_t - 4i\lambda k R) e^{-ikx} + 4i\lambda k\, e^{ikx}}{e^{ikx} + Re^{-ikx}}$$

For this to be constant, we must have

$$R_t - 4i\lambda k R = 4i\lambda k R \Rightarrow R_t = 8i\lambda k R = 8ik^3 R$$

$$\Rightarrow \boxed{\begin{array}{l} R(k,t) = R(k,0)\, e^{8ik^3 t} \\ \\ |T(K,t)| = |T(K,0)| \end{array}} \tag{6.5}$$

Summary. If $u(x,t)$ evolves according to KdV, $\partial_t u + \partial_x^3 + \partial_x(\frac{1}{2}u^2) = 0$, in the Schrödinger eigenvalue problem

$$\frac{\partial^2 \psi}{\partial x^2} + (\tfrac{1}{6} u(x,t) + \lambda)\psi = 0$$

then

(i) The eigenvalues are time independent.
(ii) The L^2 mass of the eigenfunction with eigenvalue $\lambda = -\kappa^2 < 0$ satisfies a simple linear ODE.
(iii) The reflection coefficient of the scattering states satisfies a simple linear ODE.

The initial condition $u(0)$ determines the *scattering data* $\{-\kappa^2, c_\kappa(0), R(K,0)\}$. Assuming the potential u evolves with t according to KdV, we know the scattering data evolves along simple ODEs.

Schematically, $u(0) \underset{\text{(direct scattering)}}{\longrightarrow} \{-\kappa^2, c_\kappa(0), R(K,0)\} \underset{\text{(evolve linearly)}}{\longrightarrow} \{-\kappa^2, c_\kappa(t), R(K,t)\}$

Inverse scattering. The initial data for KdV, namely, $u(x,0)$, determines the scattering data $\{\kappa_n, c_n(0), R(K,0)\}$. The converse is also true.

Gelfand–Levitan–Machenko Equation. Form the following function

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{i} R(\kappa)e^{i\kappa x}dK + \sum_{n=1}^{N} c_n\, e^{-\kappa_n x} \tag{6.6}$$

So, B is determined with the scattering data. Consider the equation

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z)B(y+z)dz = 0\ , \quad y \geq x \tag{6.7}$$

This is an integral equation in K called the *Gelfand–Levitan–Machenko equation.* Suppose we solve this equation for K.
Then

$$\frac{1}{6}u(x) = 2\,\frac{\partial}{\partial x}\,K(x,x)! \tag{6.8}$$

The scattering data determines the potential, so we can reconstruct $u(t)$ from the scattering data at time t.

6.3. **Soliton emerges from IST.** Take initial data for KdV to be $p(x) = 12\,\mathrm{sech}^2 x = f_4(x)$. We know that the solution is a travelling wave $u(x,t) = 12\,\mathrm{sech}^2(x+4t)$. Our goal is to carry out the inverse scattering method for this data to recover the known solution.

The eigenvalue problem becomes $\dfrac{\partial^2\psi}{\partial x^2} + 2\mathrm{sech}^2 x\psi = -\lambda\psi$. For $-\lambda = \kappa^2 = 1$, it can be shown that there is exactly one associated eigenfunction $\psi(x) = \frac{1}{2}\mathrm{sech}x$ after normalization. Note that $\frac{1}{2}$ is chosen so that

$$\psi(x) \to e^{x} \quad \text{as} \quad x \to -\infty$$
$$\psi(x) \to e^{-x} \quad \text{as} \quad x \to +\infty\ .$$

Exercise 13. *Verify that $\psi(x)$ is an eigenfunction for $\lambda = -1$ by calculus.*

We need to find the normalization constant $c(0)$,

$$\begin{aligned} c(0) &= \left(\int_{-\infty}^{\infty} \psi^2(x)dx\right)^{-1} = \left(\tfrac{1}{4}\int_{-\infty}^{\infty} \mathrm{sech}^2 x\,dx\right)^{-1} \\ &= \left(\tfrac{1}{2}\int_{0}^{\infty} \mathrm{sech}^2 x\,dx\right)^{-1} = \left(\tfrac{1}{2}\int_{0}^{1} d(\tanh x)\right)^{-1} = 2 \end{aligned}$$

The time evolution of the normalization constant is

$$c(t) = c(0)\, e^{-8t} \;=\; 2e^{-8t}\ . \tag{6.9}$$

It is a *fact* that sech^2 potentials are reflectionless.
$\Rightarrow R(K,t) = 0 \Rightarrow |T(K,t)|^2 = |T(K,0)|^2 = 1.$

The Gelfand-Levitan-Marchenko equation in this case becomes

$$K(x,y,t) + B(x+y,t) + \int_x^\infty K(x,z,t)B(y+z,t)dz = 0$$

where $B(x,t) = c(t)e^{-x} = 2e^{-8t-x}$. We seek a function $K(x,y,t)$ satisfying

$$K(x,y,t) + 2e^{-8t-x-y} + 2\int_x^\infty K(x,z,t)e^{-8t-y-z} = 0\ .$$

Let's look for K in the form $K(x,y,t) = w(x,t)e^{-y}$,

$$\begin{aligned}
w(x,t) + 2e^{-8t-x} + 2w(x,t)\int_x^\infty e^{-8t-2z}\, dz &= 0 \\
\int_x^\infty e^{-2z} dz &= \tfrac{1}{2}e^{-2z}|_x^\infty \\
&= -\tfrac{1}{2}e^{-2x} \\
w(x,t)(1+e^{-8t-2x}) &= -2e^{-8t-x} \\
w(x,t) &= \frac{-2\, e^{-8t-x}}{1+e^{-8t-x}}
\end{aligned}$$

So

$$\begin{aligned}
K(x,y,t) &= \frac{-2\, e^{-8t-x-y}}{1+e^{-8t-2x}} \\
\Rightarrow\ K(x,x,t) &= -\frac{2e^{-8t-2x}}{1+e^{-8t-2x}} = -2 + \frac{2}{1+e^{-8t-2x}} \\
2\,\frac{\partial}{\partial x}\, K(x,x,t) &= \frac{8e^{-8t-2x}}{(1+e^{-8t-2x})^2} \;=\; \frac{8}{(e^{4t+x}+e^{-4t-x})^2} \\
&= 2\text{sech}^2(x+4t)
\end{aligned}$$

But $\frac{1}{6}u(x,t) = 2\frac{\partial}{\partial x}\, K(x,x,t) \;\Rightarrow\; u(x,t) = 12\text{sech}^2(x+4t)$, as we had hoped.

2-Soliton via the scattering method. [20]

6.4. Miura transform → Ricatti → Eigenvalue problem. Miura transform. [24]

Ricatti equation.

Gardner's extended Miura transform.

Infinitely many conserved quantities.

Suppose v solves KdV_2 (aka modified KdV)

$$\partial_t v + \partial x^3 v - 6v^2 v_x = 0.$$

Form

$$u = v^2 + v_x.$$

Then

$$\partial_t u + \partial_x^3 u - 6uu_x = 0!$$

Remark: Similar in spirit to Hopf–Cole Transformation

$$u_t + uu_x = \epsilon u_{xx}.$$

Form $w = -2\epsilon \frac{u_x}{u}$. Then w solves

$$w_t = \epsilon w_{xx}.$$

We calculate

$$\begin{aligned}
\partial_t u &= 2vv_t + v_{xt} \\
\partial_x u &= 2vv_x + v_{xx} \to 6u\partial_x u = 12(v^2 + v_x)vv_x + 6(v^2 + v_x)v_{xx} \\
\partial_x^2 u &= 2vv_{xx} + 2v_x^2 + v_{xxx} \\
\partial_x^3 u &= 2vv_{xxx} + 2v_x v_{xx} + 4v_x v_{xx} + v_{xxxx} \\
&= 2vv_{xxx} + 6v_x v_{xx} + v_{xxxx}.
\end{aligned}$$

Combining these expressions reveals that

$$\partial_t u + \partial_x^3 u - 6uu_x = 0.$$

Indeed, we have that

$$\begin{aligned}
&2vv_t + v_{xt} + 2vv_{xxx} + 6v_x v_{xx} + v_{xxxx} \\
&-[12v^3 v_x + 12v(v_x)^2 + 6v^2 vv_{xx} + 6v_x v_{xx}] \\
&6\partial_x(6(v^2 v_x)) = 12vv_x^2 + 6v^2 v_{xx} \\
&2v(v_t + v_{xxx} - 6v^2 v_x) + \partial_x(v_t + v_{xxx} - 6v^2 v_x) = 0.
\end{aligned}$$

Summarizing, we have that

$$(\partial_t u + \partial_x^3 u - 6uu_x) = \left(2v + \frac{\partial}{\partial_x}\right)(v^2 + v_{xxx} - 6v^2 v_x)$$

if $u = v^2 + v_x$.

Suppose u is known.

The ODE $\boxed{v_x + v^2 = u}$ is called a Ricatti equation. Solving the Ricatti equation begins with the transformation

$$v = \frac{\psi_x}{\psi}.$$

Calculating

$$v_x = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} \Rightarrow v_x + v^2 = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} + \frac{\psi_x^2}{\psi^2} = \frac{\psi_{xx}}{\psi}.$$

So, the Ricatti equation transforms into

$$\psi_{xx} - u\psi = 0.$$

This is almost, but not quite, the Schrödinger eigenvalue problem. We soup things up a bit using the Galilean invariance of KdV and lack of Galilean invariance of mkdV.

Remark: The Ricatti equation in a more general form should include a lower order term αv.

Galilean Invariance of KdV

Suppose u solves KdV in the form $u_t + u_{xxx} - 6uu_x = 0$. Form the change of variables

$$\begin{cases} t' &= t \\ x' &= x + \alpha t \\ u'(x', t') &= u(x,t) - \beta. \end{cases}$$

Select α, β such that $\partial_{t'} u' + \partial^3_{x'} u' - 6u'\partial_{x'} u' = 0$. A calculation shows that

$$\alpha + 6\beta = 0$$

yields a certain cancellation. This identifies a one parameter family of solutions of *kdV* obtained from any one solution.

The same calculation shows that kdV_2 does not possess this family of solutions.

These observations motivated Gardener to generalize Miura's transformation.

Exercise: Let w solve $w_t + w_{xxx} - 6(w + \epsilon w^2)$(Gardener equation)$w_x = 0$. Form $u = w + \epsilon w_x + \epsilon^2 w^2$. (The expression for u in terms of w is called *Gardener's Transformation*; it is a Ricatti-type ODE in w.) Show

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0.$$

The Ricatti substitution leads to Schrödinger eigenvalue problem

$$\boxed{\epsilon w_x + \epsilon^2 w^2 + w = 0.}$$

Let $\boxed{w = \epsilon \frac{\psi_x}{\psi}}$

$$w_x = \epsilon \frac{\psi_{xx}}{\psi} - \epsilon \frac{\psi_x^2}{\psi^2}$$

$$\epsilon^2 \frac{\psi_{xx}}{\psi} - \epsilon^2 \frac{\psi_x^2}{\psi^2} + \epsilon^2 \frac{\psi_x^2}{\psi^2} + \frac{\psi_x}{\psi} = u.$$

$$\epsilon^2 \psi_{xx} + \psi_x = \psi u.$$

$$\psi_{xx} + \frac{1}{\epsilon^2}\psi_x = \frac{1}{\epsilon^2}\psi u.$$

Multiply by e^g. Consider $(e^g\psi)_{xx} = e^g\psi_{xx} + 2e^g g'\psi_x + e^g(g')^2\psi + e^g g''\psi$.

$$(e^g\psi)_{xx} - 2e^g g'\psi_x - e^g(g')^2\psi - e^g g''\psi + \frac{1}{\epsilon^2}e^g\psi_x = \frac{1}{\epsilon^2}e^g\psi u$$

choose g s.t.

$$-2g' + \frac{1}{\epsilon^2} = 0$$

$$g' = \frac{1}{2\epsilon^2}, \ \boxed{g = \frac{1}{2\epsilon^2}x} \Rightarrow g'' = 0$$

$$(e^g\psi)_{xx} = \left(\frac{1}{\epsilon^2}u + (g')^2\right)(e^g\psi).$$

Let $\boxed{\phi = e^g\psi}$

$$\boxed{\phi_{xx} = \left(\frac{1}{\epsilon^2}u + \frac{1}{4\epsilon^4}\right)\phi}$$

This is a Schrödinger eigenvalue problem.

Let us reexpress the w-equation in conservative form

$$w_t + (-3w^2 - 2\epsilon^2 w^3 + w_{xx})_x = 0.$$

Now w is linked to u though the (generalized) diuna transformation

$$u = w + \epsilon w_x + \epsilon^2 w_x^2.$$

Suppose that w could be solved in the form

$$w = \sum_{j=0}^{\infty} \epsilon^j P_j \text{ Ansatz}$$

where 8_j is a polynomial in u and its derivatives.

Exercise: Formally solve for w in terms of u in our asymptotic series to find P_0, P_1, P_2.

Then

$$\partial_t w + p_x(\text{stuff}) = 0 \Rightarrow \sum_{j=0}^{\infty} \epsilon^j \partial_t P_j + \partial_x(\text{stuff}) = 0.$$

Now integration in x yields (assuming decay to kill ∂_x (stuff))

$$\partial_t \int P_j = 0!$$

Using independence implied by arbitrariness of ϵ. So there exists infinitely many conserved quantities of KdV. (We saw this via Leonard earlier.)

6.5. **Lenard Operator.** An algorithmic approach to generating the collection of equations described above, due to Lenard, is presented in the book of Das [8]. This approach exploits a "bi-hamiltonian" formulation of the KdV equation.

Let $A_0 = 1$ and define $A_j,\ j = 1, 2, \ldots$ recursively by writing

$$(D^3 + \frac{1}{3}(Du + uD))A_j = DA_{j+1}. \tag{6.10}$$

Now set

$$-u_t = (D^3 + \frac{1}{3}(Du + uD))A_n \tag{6.11}$$

to observe the n-th equation in the KdV hierarchy. One can verify that the $n = 0$ case identifies the PDE characterizing translation of the potential. The $n = 1$ case gives the KdV equation.

Exercise 14. *Explicitly write out a few more equations in the KdV hierarchy.*

6.6. **Lax Pair for KdV..** Consider a potential $u(x, \alpha)$ (where $\alpha \in \mathbb{R}$ is a parameter) for Schrödinger's equation

$$\frac{d^2}{dx^2}y + [\lambda - u(x, \alpha)]y = 0, \tag{6.12}$$

with decaying boundary conditions as $|x| \to \infty$. For a fixed function of $x, u(x)$, replacement of the potential $u(x)$ by a translate $u(x+\alpha)$ will not affect the eigenvalues λ_j. Note that functions $u(x + \alpha)$ satisfy the PDE $u_\alpha - u_x = 0$.

We will see shortly that if u satisfies the **nonlinear** PDE $u_\alpha + uu_x + u_{xxx} = 0$, the λ_j is also time independent. In fact, we will see that this equation, known as the KdV equation, is one element of a set of infinitely many nonlinear equations whose solutions are one-parameter families of potentials for (6.12) with λ_j independent of the parameter.

Express the Sturm-Liouville problem (6.12) as

$$Ly = \lambda y,\ L = D^2 - u(x,t),\ D = \frac{d}{dx}. \tag{6.13}$$

Taking the time derivative of both sides of this equation and rearranging reveals

$$Ly_t - u_t y = \lambda_t y + \lambda y_t. \tag{6.14}$$

We now impose a time dependence on y which is quite general. Our goal will to make clever choices about this time dependence to force $\lambda_t = 0$. Suppose that

$$y_t = By \tag{6.15}$$

where B is a **linear** differential operator which depends upon u and the higher spatial derivatives of u. Using this in (6.14) gives

$$LBy - \lambda By - u_t y = \lambda_t y$$

$$(-u_t + [LB - BL])y = \lambda_t y, \tag{6.16}$$

where we used the linearity of B in the second step. Note that we have a wide freedom in the choice of B. We can restrict B to certain classes of operators at our convenience.

Example 6.1. *Let $B_1 = aD$ where a is initially allowed to be any function of u and higher spatial derivatives of u. From the definition of L, we have*

$$\begin{aligned} [L, B_1]y &:= [LB_1 - B_1 L]y \\ &= (D^2 - u)(ay_x) - aD(D^2 y - uy) \\ &= 2a_x y_x x + a_x x y_x + au_x y. \end{aligned}$$

If $2a_x = 0, a_x x = 0$, (so that $a = const$) then $[L, B_1]y = au_x y$ and the PDE on u is

$$(u_t - au_x)y = -\lambda_t y. \tag{6.17}$$

So, if we require u to satisfy $u_t - au_x = 0$, then $\lambda_t = 0$. We observed this before just after (6.12).

Next, we consider a more complicated form of the operator B.

Example 6.2. *Let $B_3 = aD^3 + fD + g$. Here, we will take a to be constant and allow f, g to depend upon u and its higher spatial derivatives. Simple calculations show*

$$[L, B_3] = (2f_x + 3au_x)D^2 y + (f_x x + 2g_x + 3au_x x)Dy + (g_x x + au_{xxx} + fu_x)y. \tag{6.18}$$

Again, we obtain a PDE for u provided the coefficients of $D^2 y$ and Dy vanish. This can be forced by choosing

$$\begin{aligned} f &= -\frac{3}{2}au + c_1 \\ g &= -\frac{3}{4}au_x + c_2 \end{aligned}$$

where c_1, c_2 are arbitrary integration constants. For such a choice of f, g we get

$$[L, B_3]y = [\frac{1}{4}a(u_{xxx} - 6uu_x) + c_1 u_x]y$$

which gives the evolution equation for u through (6.16), $-u_t + \frac{1}{4}a(u_{xxx} - 6uu_x) = 0$. *We can choose $a = -4$ to obtain a simpler expression.*

We have seen that taking perhaps the simplest choice of the operator B (in selecting B_1) above that we were led to a simple PDE for u, namely $u_t - u_x = 0$, to guarantee time independence of the eigenvalues. With the slightly more complicated choice B_3 we were led to the nonlinear evolution $u_t + 6uu_x + u_{xxx} = 0$ which leaves the λ_j time invariant. This approach can be continued by making higher order choices of B, selecting certain functions to cause higher order derivative terms in y to vanish, leading to an infinite family of equations. The solutions of these equations all have the property that the eigenvalues of the problem (6.12), with the evolving solution u the potential, have time-independent eigenvalues.

The preceding discussion is originally due to Lax [21] and was taken from [20].

6.7. **Direct Scattering (long topic).** sech ^{2} potentials are reflectionless. identify decay hypotheses on the potential. (Deift-Trubowitz?) [20]

6.8. **Inverse Scattering (long topic).** [20]
explain the G-L-M equation. comment on potential recovery from scattering data in higher dimensions.

Scattering and Inverse Scattering

The goal of this lecture is to "explain" the Gelfand–Lewiston–Marchenko equation. We will fail. In particular, we wish to see that with appropriate "scattering data" one can reconstruct the potential giving rise to such data.

Consider the operator $\mathcal{L} = -\partial_x^2 + q$ acting on $x \in \mathbb{R}$. q smooth decaying as $|x| \to \infty$. For $q = 0$, the solutions of

$$-\partial_x^2 f = k^2 f,\ \ k \in \mathbb{R}$$

are *superpositions of* e^{ikx}, e^{-ikx}. For $q \neq 0$, the decay assumption $\Rightarrow$ solutions of

$$\mathcal{L} f = k^2 f,\ \ k \in \mathbb{R}$$

should be nearly superpositions of e^{ikx}, e^{-ikx}, for $|x|$ large. Any three solutions of $\mathcal{L} f = k^2 f$ must be linearly dependent. Let's choose a *convenient basis* of solutions. The *Jost Functions* are characterized by

$$\left.\begin{array}{lcl} f_+(x,k) & \simeq & e^{ikx} \text{ as } x \to +\infty \\ f_-(x,k) & \simeq & e^{-ikx} \text{ as } x \to -\infty \end{array}\right. \quad \mathbb{C}\text{-valued.}$$

Clearly,

$$\overline{f_+(k)} = f_+(-k),\ \ \overline{f_-(k)} = f_-(-k).$$

By linear dependence

$$\left[\begin{array}{lcl} f_-(k) & = & a_+(k)\bar{f}_+(k) + b_+(k) f_+(k) \\ f_+(k) & = & a_-(k)\bar{f}_-(k) + b_-(k) f_-(k). \end{array}\right.$$

Recall the *Wronskiga bilinear form*

$$w(f,g) = \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - f'g.\ \ w(f,g) = -w(g,f).\ \ \text{antisymmetric.}$$

Claim. If f, g are solutions of $\mathcal{L} f = k^2 f$, $\bar{w}(f,g)$ is independent of x.

Proof. $\frac{d}{dx}\bar{w}(f,g) = 0$.

We can use this fact and asymptotics implied by decay of q to infer properties linking $a_\pm(k)$, $b_\pm(k)$.

$$\begin{aligned} \bar{w}(f_+(k),\bar{f}_+(k)) &= -2ik & \qquad \tfrac{\bar{w}(f_-(k),f_+(k))}{2ik} &= a_+(k) \\ \bar{w}(f_-(k),\bar{f}_-(k)) &= 2ik & \qquad \tfrac{\bar{w}(f_-(k),\bar{f}_+(k))}{2ik} &= -b_+(k). \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \bar{w}(f_-(k),f_+(k)) &= \bar{w}(f_-(k),a_-(k)\bar{f}_-(k)+b_-(k)f_-(k)\to 0) \\ &= a_-(k)(2ik) \\ \bar{w}(f_-(k),f_+(k)) &= \bar{w}(a_+(k)f_+(k)\to 0+b_+(k)\bar{f}_+(k),f_+(k)) = b_+(k)(2ik) \\ &\Rightarrow \boxed{a_-(k)=a_+(k)} \to a(k) \end{aligned}$$

Exercise. Show

$$1 = |a_-|^2 - |b_-|^2 \text{ and } 1 = |a_+|^2 - |b_+|^2.$$

Show $R_+(-k) = R_+(k)$ and $R_-(-k) = \overline{R_-(k)}$.

Let's divide the connecting relation by $a(k)$. Can we divide by $a(k)$? We'll see more on this later.

$$\begin{aligned} T(k)f_-(k) &= \bar{f}_+(k) + R_+(k)f_+(k) \\ T(k)f_+(k) &= \bar{f}_-(k) + R_-(k)f_-(k) \end{aligned}$$

where

$$T(k)(\text{transmission coefficient})(k) = \frac{1}{a(k)},\ R_\pm(\text{reflection coefficient})(k) = \frac{b_\pm(f)}{a(k)}.$$

In light of the asymptotic behavior of $f_\pm$ near $x = \pm\infty$, we can view the first of these equations as a harmonic wave Cauchy from $x = +\infty$, being partly transmitted and partly reflected by q, the potential.

Dividing the exercise by $|a(k)|^2 \Rightarrow |T(k)|^2 + |R_\pm(k)|^2 = 1$.

For k large, gf is small compared to $k^2 f$ so most of the signal coming in from $+\infty$ will be transmitted $\Rightarrow$

$$\lim_{k\to\infty} R_\pm(k) = 0,\ \lim_{k\to\infty} T(k) = 1.$$

Asymptotic behavior for k large

Ansatz: $f_+(x,k) = e^{ikx+p(x,k)} \simeq e^{ikx}(1+p+\dots) \simeq e^{ikx}\left(1+\frac{p_1}{k_1}+\dots\right)$. We expect $p\to 0$ as $k\to\infty$. Since f_+ satisfies $\mathcal{L}f_+ = k^2 f_+$, we can calculate to find

$$p'' + 2ikp' + (p')^2 = q.$$

Based on our large k expectations, take

$$p \simeq \frac{p_1}{k} + \frac{p_2}{k^2} + \dots (\to 0 \text{ as } k\to\infty).$$

$\Rightarrow$

$$\boxed{p_1' = \frac{q}{2i}} \text{ at order } k^0.$$

$$\boxed{p_2 = \frac{q}{4}} \text{ at order } k^1.$$

So, the large asymptotic behavior as $k \to \infty$ is determined by the potential (not surprising). We will use this in the other direction in a while...

$$\boxed{\begin{array}{rcl} f_+(k,k) & = & e^{ikx}\left(1+\frac{p_1}{k}+\dots\right) \\ p_1' & = & \frac{q}{2i} \end{array}}$$

Complex Values of k

The Jost solutions

$$\begin{array}{rcl} f_+(x) & = & e^{ikx} \text{ as } x \to +\infty \\ f_-(x) & = & e^{-ikx} \text{ as } x \to -\infty \end{array}$$

make sense and we bounded for $k \in \mathbb{C}$, Im $k > 0$.

Lemma 6.1. *For all x,*

$$\begin{array}{rcl} e^{-ikx} f_+(x,k) - 1 & \in & H_+^2 \\ e^{ikx} f_-(x,k) - 1 & \in & H_+^2. \end{array}$$

Definition. $g \in H_+^2$ if g is analytic on $UHP = \{k : \text{Im } k > 0\}$ and

$$\int_{\text{Im } k=c_1} (g(k))^2 < c_2 < \infty \text{ for all } c_1 > 0.$$

Since $\frac{\bar{w}(f_+(k),f_-(k))}{2ik} = a(k)$, we know $a(k)$ is analytic in UHP. The zeros of $a(k)$ in $\bar{U}HP$ are important. Suppose $a(n) = 0$. Then $\bar{w}(f_+(n), f_-(n)) = 0 \Rightarrow df_+(n) = f_-(n)$. When Im $n > 0$, this shows $f_+(x,n)$ decays exponentially as $x \to +\infty$ and $f_-(x,n)$ decays exponentially as $x \to -\infty \Rightarrow f_+(n)$ is square integrable and is therefore a *proper eigenfunction* of $\mathcal{L}$. Since $\mathcal{L}$ is self-adjoint, its eigenvalues are *real* $\Rightarrow$ the zeros of $a(k)$ lie on the imaginary axis $n = in$.

Next, we calculate the derivative of $a(k)$ at n $\left(\dot{a} = \frac{d}{dk}a\right)$

$$\dot{a}(in) = -\frac{1}{2n}\{\bar{w}(\dot{f}_-, f_+) + \bar{w}(f_-, \dot{f}_+)\}.$$

Using $df_+(n) = f_-(n) \to$

$$\boxed{\dot{a}(in) = -\frac{1}{2n}\{d^{-1}\bar{w}(\dot{f}_-, f_-) + d\bar{w}(f_+, \dot{f}_+)\}.}$$

The functions $f_\pm$ satisfy

$$-f_{xx} + qf = k^2 f \times \dot{f}$$

$\Rightarrow$

$$-\dot{f}_{xx} + q\dot{f} = k^2\dot{f} + 2kf \times f \text{ subtract}$$

$$(\dot{f}f_x - f\dot{f}_x)_x = 2kf^2$$

$$(\bar{w}(\dot{f}, f))_x$$

$\Rightarrow$

$$\bar{w}(\dot{f}_+, f_+)(x) = -2k\int_x^\infty f_+^2 dx, \ \bar{w}(\dot{f}_-, f_-)(x) = 2k\int_{-\infty}^x f_-^2 dx.$$

Set $k = in$ and substitute into $\dot{a}$ equation. Use linear dependence relation to eliminate

$$f_- \Rightarrow \boxed{\dot{a}(in) = -id \int_{-\infty}^{\infty} f_+^2(x, in) dx} \Rightarrow \text{zeros are simple.}$$

Let

$$B(x,y) = \int e^{-iky} \underbrace{(f_+(x,k) - e^{ikx})}_{\substack{\text{skew symmetric in } k.\ (\bar{f}_+(k) = f_+(-k)) \\ \Rightarrow B \text{ is Real.}}} dk.$$

Facts:

- f_+ analytic in k in UHP.
- $f_+(x,k)e^{-ikx} - 1 \in H_+^2$

$\Big)$ = (Paler Wienar) $\Rightarrow B(x,y)$ is supported on $[x, \infty)$ and B is nice.

Inverting,

$$\begin{aligned} f_+(x,k) &= e^{ikx} + \int_x^{\infty} B(x,y) e^{iky} dy \; IBP \\ &= e^{ikx} - B(x,x)\frac{e^{ikx}}{ik} + \frac{i}{k} \int B_y(x,y) e^{iky} dy \end{aligned}$$

But recalling the asymptotics as $k \to \infty$,

$$e^{ikx} - B(x,x)\frac{e^{ikx}}{ik} + \frac{i}{k}\int B_y(x,y)e^{iky}dy = e^{ikx} + \frac{e^{ikx}p_1}{k} + \dots$$

$\Rightarrow$

$$iB(x,x) = P_1(x) \text{ so } \boxed{q(x) = -2\frac{d}{dx}B(x,x)}$$

6.9. **KdV Phenomenology via IST.**

- Solitary Wave Resolution [28]
- Almost Periodicity of KdV Flow on $\mathbb{T}$

7. Integrable Structure

7.1. **Lax Pair and other integrable models.** insert a summary of what appeared above. abstraction allowed to see NLS as completely integrable.

7.2. **Ablowitz-Kaup-Newell-Segur System.**

7.3. **NLS phenomenology via IST.**

7.4. **Other integrable models.** KPI, KPII, Davey-Stewartson, sine-Gordon, ??

We have seen that the eigenvalues of the eigenvalue problem

$$\frac{\partial^2 \psi}{\partial_x^2} + \left(\frac{1}{6}u(t) + \lambda\right)\psi = 0$$

with time-dependent potential $u(x,t)$ satisfying the kdV equation

$$-\partial_t u + \partial_x^3 u + \partial_x\left(\frac{1}{2}u^2\right) = 0$$

are time invariant: $\partial_t \lambda = 0$!

- A direct calculation (solve eigenvalue problem for u, apply $kdV \to (\partial_t \lambda)\psi^2 + \partial_x(\ \ ) = 0 \Rightarrow \lambda_t = 0$) revealed this.
- Miura transform suggested a relationship between kdV and Schrödinger eigenvalue problem.

Peter Lax discovered a deeper explanation of $\lambda_t = 0$ which led Zakhorov and Shobat and later Ablowitz, Kemp, Newel and Segure to discover the "kdV miracles" exist in other nonlinear PDE.

Suppose we have a linear Hamiltonian evolution problem

$$\begin{cases} i\partial_t \phi &= H\phi \\ u(0) &= \phi \end{cases} \qquad \phi \mapsto \psi(t) = e^{-iHt}\phi := \text{(know form of unitary operator here)}\mathcal{U}(t)\phi$$

We would like to construct operators whose eigenvalues are time invariant.

$$A(0) \mapsto A(t)?\text{ s.t. if } A(0)\phi = \lambda\phi \text{ then } A(t)\psi(t) = \lambda\psi(t).$$

$$\left.\begin{array}{rcl} A(o)\overline{\mathcal{U}(t)}(\mathcal{U}(t)\phi) &=& A(o)\phi \;=\; \lambda\phi \\ \mathcal{U}(t) & & \qquad\qquad \mathcal{U}(t) \\ (\mathcal{U}(t)A(o)\overline{\mathcal{U}(t)})\phi(\epsilon) &=& \lambda\psi(t) \end{array}\right\} \Rightarrow \begin{array}{c} A(t) = \mathcal{U}(t)A(o)\overline{\mathcal{U}(t)}. \\ A(t) \text{ unitarily equivalent to } A(o). \end{array}$$

Rewriting,

$$\overline{\mathcal{U}(t)}A(t)\mathcal{U}(t) = A(o).$$

Take ∂_t

$$\bar{\mathcal{U}}_t A\mathcal{U} + \bar{\mathcal{U}}A_t\mathcal{U} + \bar{\mathcal{U}}A\mathcal{U}_t = 0.$$

$$\mathcal{U}(t) = e^{-iHt} \Rightarrow \mathcal{U}_t = -iH\mathcal{U} = (H \text{ is time independent}) - i\mathcal{U}H$$

$$\boxed{\frac{\partial \mathcal{U}}{\partial t} = \underbrace{(-iH)}_{\text{antihermitian}} \mathcal{U}}$$

$$\bar{\mathcal{U}}_t = iH\bar{\mathcal{U}} = i\bar{\mathcal{U}}H.$$

$$\bar{\mathcal{U}}(iHA + A_t - iAH)\mathcal{U} = 0$$

$$(A_t - i[A,H]) = 0$$

$$\boxed{\frac{\partial A}{\partial t} = i[A(t),H].} \qquad \text{Heissenberg Picture}$$

We mimic the preceding in the case where we have a nonlinear *Hamiltonian* evolution.

Let

$$L(u(x,t)) = L(t)$$

denote a linear operator we hope to find. We assume it is Hermitian and its eigenvalues are time independent. For this to be true, there must exist(?) a unitary operator $\mathcal{U}(t)$ such that

$$\overline{\mathcal{U}(t)}L(t)\mathcal{U}(t) = L(o).$$

Differentiate ?? t

$$* \qquad \frac{\overline{\partial \mathcal{U}}}{\partial t}L\mathcal{U} + \bar{\mathcal{U}}\frac{\partial L}{\partial t}\mathcal{U} + \bar{\mathcal{U}}L\frac{\partial \mathcal{U}}{\partial t} = 0.$$

We do not know the form of $\mathcal{U}(t)$, but since $\mathcal{U}$ is unitary

$$\bar{\mathcal{U}}(t)\mathcal{U}(t) = 1 \Rightarrow \frac{\partial \bar{\mathcal{U}}}{\partial t}\mathcal{U} + \bar{\mathcal{U}}\frac{\partial \mathcal{U}}{\partial t} = 0 \Leftrightarrow \frac{\partial \mathcal{U}(t)}{\partial t} = B(t)\mathcal{U}(t) \text{ with } B(t) \text{ antihermitian. Why?}$$

Substituting this into $*$ gives

$$\# \qquad \frac{\partial L(t)}{\partial t} = [B(t), L(t)] \qquad \text{(operator equation)}$$

as a condition for $L(t)$ to be isospectral.

Given $L(o)$, self-adjoint. We wish to flow $L(o)$ into $L(t)$ in such a way that the eigenvalues of $L(o)$

$$L(o)v = \lambda v$$

are t-independent. "Isospectral deformation".

A *unitary operator* $\mathcal{U}$ satisfies $\bar{\mathcal{U}}\mathcal{U} = I$. (More precisely, Range$(\bar{\mathcal{U}}) = \mathcal{H}$ and $\mathcal{U}$ is isometric

$$\langle \mathcal{U}v, \mathcal{U}w\rangle = \langle v, w\rangle \text{ for all } v, w \in \mathcal{H}.$$

Fact: $\hat{L} = \mathcal{U}^{-1}L\mathcal{U}$ has the same spectrum as L. Form $w = \mathcal{U}^{-1}v$ where v is an eigenfunction with eigenvalue λ.

$$\begin{aligned} \tilde{L}w &= \mathcal{U}^{-1}L\mathcal{U}(\mathcal{U}^{-1}v) = \mathcal{U}^{-1}Lv = \mathcal{U}^{-1}\lambda v = \lambda(\mathcal{U}^{-1}v) \\ &= \lambda w. \end{aligned}$$

One must also show that $\lambda \notin \sigma(L) \to \lambda \notin \sigma(\bar{L})$ and this may be done.

So conjugation by unitary operators is a way to generate an isospectral operator $\tilde{L}$ from a given operator L. To make a flow, we need a device to make a flow of unitary operators.

Let B be an antiselfadjoint operator $\langle u, Bv\rangle = -\langle Bu, v\rangle$.

Examples:

$$\begin{aligned} \int u\overline{(iv)} &= -\int (iu)v \\ \int u\partial_x v &= -\int (\partial_x u)x. \end{aligned}$$

Form the family of unitary operators $\mathcal{U}(t)$ by solving

$$\begin{cases} \mathcal{U}_t &= B\mathcal{U} \\ \mathcal{U}(o) &= I \end{cases} \qquad B = B(t).$$

Verify that such a $\mathcal{U}(t)$ will consist of unitary operators. Now, look at

$$\mathcal{U}^{-1}(t)L(t)\mathcal{U}(t) = \overline{\mathcal{U}(t)}L(t)\mathcal{U}(t) = L(o).$$

Take a t-derivative

$$\underset{\substack{\| \\ -\bar{\mathcal{U}}B}}{\bar{\mathcal{U}}_t} \quad L\mathcal{U} + \bar{\mathcal{U}}L_t\mathcal{U} + \bar{\mathcal{U}}L\underset{B\mathcal{U}}{\mathcal{U}_t} = 0.$$

$$\bar{\mathcal{U}}(L_t + [L, B])\mathcal{U} = 0.$$

Isospectral deformation conditions:

$$\underset{\text{“Lax Pair”}}{L_t} = [B(t), L(t)].$$

Assume L is linear in $u(x, t)$. Then the left side of # is a multiplication operator. Consequently, the right side must be as well.

Therefore, *if* we can find a linear operator $L(t)$ and a second operator $B(t)$, which is *not* necessarily linear, such that the commutator $[B(t), L(t)]$ is a multiplication operator and is proportional to the evolution of $u(x,t)$ according to the nonlinear equation, *then* the eigenvalues of $L(t)$ would be time invariant.

Specialization to kdV

Set $L(t) = \partial_x^2 + \frac{1}{6}u(x,t) \Rightarrow \frac{\partial L(t)}{\partial t} = \frac{1}{6}\frac{\partial u}{\partial t}$. $B(t)$ must be antihermitian so must be odd in the number of derivatives. Choose simplest form

$$B(t) = a\partial_x.$$

$$[B(t), L(t)] = \frac{a}{6}\frac{\partial u}{\partial x}.$$

Then

$$\frac{\partial L}{\partial t} = [B, L] \Rightarrow \frac{1}{6}\frac{\partial u}{\partial t} = \frac{a}{6}\frac{\partial u}{\partial x}$$

$$\boxed{\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x}}$$

Next natural choice

$$\begin{aligned}
B_3 &= a_3\partial_x^3 + a_1(Du + uD) \\
[B_3, L] &= \left(\frac{a_3}{6} - a_1\right)(\partial_x^3 u) + \frac{a_1}{3}u\partial_x u \\
&\quad + \underbrace{\left(\frac{a_3}{2} - 4a_1\right)}_{\substack{\text{make zero} \\ \frac{a_3}{2} = 4a_1 \\ a_3 = 8a_1}}((\partial_x^2 u)\partial_x + (\partial_x u)\partial_x^2) \\
\frac{\partial L}{\partial t} &= [B, L] \Rightarrow \frac{\partial u}{\partial t} = \partial_x^3 u + u\partial_x u. \;\; kdV.
\end{aligned}$$

The Lax Pair for kdV is

$$\begin{cases} L(t) = \partial_x^2 + \frac{1}{6}u \\ B(t) = 4\partial_x^3 + \frac{1}{2}(\partial_x u + u\partial_x). \end{cases}$$

Summary of Lax method

Suppose $L(t)\psi(t) = -\lambda\psi(t)$ and $\frac{\partial L(t)}{\partial t} = [B(t), L(t)]$, where $\frac{\partial \psi(t)}{\partial t} = B(t)\psi(t)$. Then $\frac{\partial \lambda}{\partial t} = 0$.

Check this: Take true derivation of eigenvalue problem

$$\begin{aligned}
\frac{\partial L}{\partial t}\psi + L\frac{\partial \psi}{\partial t} &= -\frac{\partial \lambda}{\partial t}\psi - \lambda\frac{\partial \psi}{\partial t} \\
\frac{\partial L}{\partial t}\psi + LB\psi + \lambda B\psi &= -\frac{\partial \lambda}{\partial t}\psi \\
\frac{\partial L}{\partial t}\psi + [LB - BL]\psi &= -\frac{\partial \lambda}{\partial t}\psi \\
&\quad + B(t)(L(t) + \lambda)\psi \\
\left(\frac{\partial L}{\partial t} + [L, B]\right)\psi &= -\frac{\partial \lambda}{\partial t}\psi.
\end{aligned}$$

Assume

$$L(t)\psi(t) = -\lambda\psi(t)$$

and

$$\frac{\partial\psi(t)}{\partial t} = B(t)\psi(t),$$

with

$$\frac{\partial\lambda}{\partial t} = 0.$$

This requires that L and B satisfy the *compatibility condition*

$$\frac{\partial L(t)}{\partial t} = [B(t), L(t)].$$

Let $\phi = \binom{\phi_1}{\phi_2}$

Generalize "$L(t)\psi(t) = -\lambda\psi(t)$" by writing

$$\left(\sigma_3\tfrac{\partial}{\partial x} - q\sigma_+ + r\sigma_-\right)\phi = -i\zeta\phi$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Generalize "$\frac{\partial\psi(t)}{\partial t} = B(t)\psi(t)$" by writing

Note:
$$\begin{aligned} \sigma_3^2 &= I \\ \sigma_\pm\sigma_3 &= \mp\sigma_\pm = -\sigma_3\sigma_\pm \\ \sigma_\pm\sigma_\mp &= \tfrac{1}{2}(I \pm \sigma_3). \end{aligned}$$

$$\frac{\partial\phi}{\partial t} = (P\sigma_+ + Q\sigma_- + R\sigma_3)\phi.$$

Take $\frac{\partial}{\partial x}$.

Here $q = q(x,t)$, $r = r(x,t)$ are the dynamical variables, independent of spectral ??? ζ. The coefficient functions P, Q, R do depend on ζ and are functionals of $q(x,t)$, $r(x,t)$.

Compatibility

$$\begin{aligned} \sigma_3\frac{\partial}{\partial x}\phi &= (q\sigma_+ - r\sigma_- - i\zeta\phi) \\ &\Rightarrow \\ \frac{\partial}{\partial x}\phi &= (q\sigma_+ + r\sigma_- - i\zeta\sigma_3)\phi \end{aligned}$$

Take $\frac{\partial}{\partial t}$.

A tedious calculation writes

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial x}\phi &= (A_1\sigma_+ + A_2\sigma_- + A_3\sigma_3 + A_4 I)\phi \\ \frac{\partial}{\partial x}\frac{\partial}{\partial t}\phi &= (B_1\sigma_+ + B_2\sigma_- + B_3\sigma_3 + B_4 I)\phi \\ A_4 &= B_4 \text{ comes out in the wash.} \end{aligned}$$

Compatibility requires

$$\begin{aligned} A_1 &= B_1 \to q_t = P_x + 2qR + 2i\zeta P \\ A_2 &= B_2 \to r_t = Q_x - 2rR - 2i\zeta Q \\ A_3 &= B_3 \to R_x = qQ - sP. \end{aligned}$$

We can write the compatibility condition:

$$\begin{aligned} L &= \frac{\partial}{\partial x} - A(x,3) = \frac{\partial}{\partial x} - q\sigma_+ - r\sigma_- + i\zeta\sigma_3 \\ B &= P\sigma_+ + Q\sigma_- + R\sigma_3. \end{aligned}$$

Compatibility requires

$$\frac{\partial L}{\partial t} = [B, L].$$

Example: Choose $r = 6$. The r_t equation gives

$$R = \frac{1}{12}Q_x - \frac{i\zeta}{6}Q.$$

The R_x equation gives

$$P = \frac{1}{6}8Q - \frac{1}{72}Q_{xx} + \frac{i\zeta}{36}Q_x.$$

Using these in the q_t equation becomes

$$q_t = -\frac{1}{72}Q_{xxx} + \frac{1}{3}qQ_x + \frac{1}{6}q_xQ - \frac{\zeta^2}{18}Q_x.$$

If we choose to write $q = -\frac{1}{36}u(x,t)$, this becomes

$$u_t = \frac{1}{2}\left(D^3 + \frac{1}{3}(Du + uD)\right)Q + 2\zeta^2 DQ$$

which encodes the *kdV* hierarchy with appropriate choices of Q.

Example: Choose $q(x,t) = \sqrt{\kappa}\bar{\psi}$, $r(x,t) = \sqrt{\kappa}\psi$, $\kappa \in \mathbb{R}$. Since $q = \bar{r}$, consistency of the q_t, r_t equations requires that

$$Q = (\text{sgn } \kappa)\bar{P}.$$

If we choose

$$R = 2i\zeta^2 + i\kappa\bar{\psi}\psi,$$

we obtain that

$$Q = i\sqrt{\kappa}\psi_x - 2\zeta\sqrt{\kappa}\psi \text{ (using } R_x \text{ eq. and } Q = (\text{sgn } \kappa)\bar{P}).$$

What do the r_t, q_t equations give? r_t equation $\rightarrow$

$$\begin{aligned} \sqrt{\kappa}\psi_t &= i\sqrt{\kappa}\psi_{xx} - 2\sqrt{\kappa}\psi_x - 2(\sqrt{\kappa}\psi)(2i\zeta^2 + i\kappa\bar{\psi}\psi) - 2i\zeta(i\sqrt{\kappa}\psi_x - 2\zeta\sqrt{\kappa}\psi) \\ \sqrt{\kappa}\psi_t &= i\sqrt{\kappa}\psi_{xx} - 2i\kappa^{3/2}|\psi|^2\psi. \end{aligned}$$

$$\boxed{i\psi_t + \psi_{xx} - 2\sqrt{\kappa}|\psi|^2\psi = 0}$$

Exercise: Let $r(x,t) = -q(x,t) = \frac{1}{2}w_x(x,t)$. Let $P = Q = \frac{i}{4\zeta}\sin w$. Verify, using q_t, r_t, R_x equations that

$$R = \frac{i}{4\zeta}\cos w$$

and

$$w_{x,t} = \sin w \qquad \text{(sine-gordon equation).}$$

Note:

$$\begin{aligned} r &= q = \frac{1}{2} w_x \\ Q &= -P = \frac{i}{4\zeta} \sinh w \\ \Rightarrow R &= \frac{i}{4\zeta} \cosh w \\ w_{xt} &= \sinh w \qquad \text{(sinh-gordon equation).} \end{aligned}$$

Various nonlinear PDE can be formulated as *compatibility conditions* between

- a time dependent problem
- a time dependence of the eigenfunctions

$\Rightarrow$ time invariance of eigenvalues.

This perspective unifies various integrable equations as isospectral flows.

8. **Wellposedness**

Remark 8.1. Physical considerations motivate our interest in a wider class on nonlinear wave equations than those to which IST applies. To what extent is the phenomena we observe via IST in integrable cases an expression of the underlying integrability vs. the general behavior of nonlinear wave equations? To begin to answer this, we first need to construct the solutions somehow using a different technique than IST.

8.1. **Local Wellposedness.**

ODEs, typical properties of proofs, initial value problem, initial-boundary value problem periodic initial value problem. Scaling heuristic, sharp LWP problem

The integrable cases have "formulae" giving the solution of an initial value problem in terms of the initial data. This appears to be more than can be hoped for in the nonintegrable case. We will be interested in wellposedness properties of

$$\text{kdV}_p \quad \begin{cases} \partial_t u + \partial_x^3 u + \partial_x \left(\frac{1}{p+1} u^{p+1} \right) = 0 \\ u(x,0) = \phi(x) \end{cases} \qquad u : \mathbb{R} \times [0,T] \to \mathcal{R}$$

$$\text{NLG}_p \quad \begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1} u = 0 \\ u(x,0) = \phi(x) \end{cases} \qquad u : \mathbb{R}^d \times [0,T] \to \mathbb{C} \ .$$

ODE wellposedness. The initial value problem for an ODE is

$$\dot{y} = f(t,y) \qquad y(t) = y_0 \tag{$*$}$$

Basic issues: Does the solution exist? If so, for how long? Is it unique? Does it depend continuously on the initial data y_0?

Examples.

$$\begin{cases} \dot{y} = y^2 \\ y(0) = a \end{cases} \quad \text{has solution} \quad y(t) = \frac{1}{a-t} \ .$$

This shows solutions can exist and not be global in time.

$$\begin{cases} \dot{y} = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases} \quad \text{has solution} \quad y(0) = 0 \ .$$

But $y(t) = \alpha t^{3/2}$ is also a solution. This shows solutions are not necessarily unique.

Theorem. *If $f(t, y)$ is continuous and satisfies a Lipschitz condition for y in a domain D, then for $(t_0, y_0) \in D$, there exists a unique exact solution at $(*)$ for $|t - t_0| \leq h$, with $h > 0$ appropriately small.*

Exercise. Write out a proof of this theorem using the Picard Contraction Mapping Theorem. References: Arnold ODE, Hurewicz, Coddington-Levinson.

Recall that a Lipschitz function satisfies
$|f(t, y_1) - f(t, y_2)| \leq M|y_1 - y_2|$.

Idea.

$$y(t) = y_0 + \int_1^t f(\tau, y(\tau))dt$$

Build a sequence of approximate solutions $\{y_j\}$,

$$\begin{aligned} y_0(t) &= y_0 \\ y_{j+1}(t) &= y_0 + \int_0^t f(\tau, y_j(t))d\tau \ . \end{aligned}$$

Show y_j is Cauchy,

$$\begin{aligned} \|y_{j+1} - y_j\| &\leq \| \int_0^t f(\tau, y_j(\tau))d\tau - \int_0^t f(\tau, y_{j-1}(\tau))d\tau \\ &\leq T \sup_\tau \|f(\tau, y_j(t)) - f(\tau, y_{j-1}(\tau))\| \\ &\leq \underbrace{TM}_{\text{make small}} \quad \|y_j - y_{j-1}\| \ . \end{aligned}$$

Definition. The initial value problem (KdVp) is *wellposed on the time interval* $[0, T]$ *for data in* H if there exists a uniquely defined continuous map

$$H \ni \phi \mapsto u \in X$$

with $X \subset C([0, T]; H)$ and u "solves" KdVp. If $T < \infty$, the problem is *locally wellposed* (LWP). If T may be taken arbitrarily large, the problem is said to be *globally wellposed* (GWP). A similar definition applies to (NLSp).

Remarks.

(1) Rough spaces H require a careful notion of "solves".
(2) Two notions of continuity in definition.
(3) Typical proofs of LWP have $T \sim \|\phi\|_H^{-\alpha}$, $\alpha > 0$. In particular, big data has a short lifetime.
(4) GWP in H follows from LWP if $\|u(t)\|_H$ is bounded. How can we prove a globalizing estimate $\|u(t)\|_H < c(\phi)$ $\forall t$? Often this is done by exploiting the available conserved quantities. Available conserved quantities usually involve L^2 or H^1 norms and therefore we encounter difficult issues in LWP proof since such fractions are so rough.
(5) There are various strengthenings or relaxations of the definition. For example $\phi \mapsto u$ mapping $H \mapsto X$ may be better than continuous. Perhaps $u(t)$ is better than the function space H, etc.

(6) There are other natural problems to consider other than the initial value problem (KdVP) or the line above. For example, the periodic initial value problem and various initial boundary value problems may naturally occur in applications or due to curiosity.

8.2. **Energy/compactness method.** Construction of approximate solutions
Bounds $\Longrightarrow$ compactness
extract subsequence, Gronwall implies uniqueness

Energy method
The general theory of parabolic PDE shows (find ref: Temam?, Bona-Smith?)

$$\begin{cases} \partial_t u^\epsilon + \partial_x^3 u^\epsilon + \partial_x \left(\frac{1}{p+1} u\epsilon^{p+1}\right) = -\epsilon \partial_x^4 u^\epsilon \\ u(x,0) = \phi(x) \end{cases}$$

has solutions. There is nothing terribly special about the 4th order parabolic term. In fact, we could proceed with a standard second order regularization or boost the power even higher. Moreover

$$\partial_t \int_R (u^\epsilon)^2 dx = -\epsilon \int_{\mathbb{R}} (\partial_x^2 u)^2 dx < 0$$

$\Rightarrow \{u^\epsilon\}$ is uniformly bounded in L^2. A similar calculation shows $\{u^\epsilon\}$ is uniformly bounded in H^1. Therefore, $\{u^\epsilon\}$ is compact in L^2 and has a convergence subsequence $\{u^{\epsilon'}\}$ but $\{u^{\epsilon'}\}$ may not be uniquely defined.

This procedure of using compactness gives a rather quick and cheap proof of existence of solutions obtained as limits of a natural sequence of approximations. This leave uniqueness as an unresolved issue.

Uniqueness (via Gronwall inequality). Let $u, \tilde{u}$ be two solutiosn of KdV. Form $w = u - \tilde{u}$. Then w satisfies

$$\begin{aligned} \partial_t w + \partial_x^3 w &= \tilde{u}\tilde{u}_x - u u_x \\ &= -w w_x + w\tilde{u}_x - \tilde{u} w_x \ . \end{aligned}$$

Multiply by w and integrate,

$$\partial_t \int_{\mathbb{R}} w^2 dx + \int_{\mathbb{R}} w \partial_x^3 w \, dx = -\int_{\mathbb{R}} w^2 w_x \, dx + \int_{\mathbb{R}} w^2 \tilde{u}_x - \tilde{u}\, w w_x \, dx.$$

Integration by parts then reveals that

$$\begin{aligned} \partial_t \int_{\mathbb{R}} w^2 dx &= \int_{\mathbb{R}} w^2 (\tilde{u}_x + \tilde{u}_x) \, dx \\ &\le C \int_{\mathbb{R}} w^2 \ . \end{aligned}$$

(provided we assume enough regularity).

Gronwall inequality. If $F' \le cF$, $F \ge 0$ and $F(0) = 0$ then $F \equiv 0$. *Prove this as an exercise.*

Therefore if $\phi \to u$ and $\phi \mapsto \tilde{u}$ then $w(0) = 0 \Rightarrow \int w^2(0) dx = 0$, so, provided $\|\tilde{u}_x\|_{L^\infty} \le C$, we know that $w \equiv 0 \Rightarrow u = \tilde{u}$ which proves uniqueness.

Intuition behind the selection of the spaces involved in fixed point approach.

Initial data space. *What space(s) are scaling invariant?* Not all problems have a scaling structure.
• $s_p = \frac{1}{2} - \frac{2}{p}$: heuristic for ill posedness of $s < s_p$.
• *Sharp LWP problem.* Find largest space H in which KdV p is LWP. Restrict to a class of spaces H, e.g. H^s, $s \in \mathbb{R}$. $\rightsquigarrow$ extra information due to Sharpnesss, dynamical insights. mathematically natural problem. counterexamples to multilinear estimates are related to weak turbulence wave resonances.

What spaces enjoy a globalizing estimate?
• Interpolation argument for energy conservation $\Rightarrow$ globalizing estimate
• This question distinguishes problems with likely GWP from those with possible blow-up.

What spaces have the algebra property?
• Stabilize the mapping properties for the nonlinearity.

$$\begin{aligned} u &\mapsto u^P \\ u &\mapsto \partial_x u^P, \quad \text{etc.} \end{aligned}$$

for all fixed t.
• Perhaps we can succeed at lower regularity by taking advantage of space-time norms?

Space-time Space

8.3. **Modular estimates for fixed point argument. Duhamel's principle**
Homogeneous problem estimates
Inhomogeneous problem estimates
Nonlinear bridge estimate
Contraction estimate
Example: KdV_p **following** [13] smoothing effect & flow property $\rightsquigarrow$ maximal function

Modular decomposition and associated estimates

Homogeneous problem

$$\begin{cases} \partial_t w + \partial_x^3 w = 0 \\ w(0) = \phi \end{cases} \tag{8.1}$$

$$w(0) = \int e^{i(x\xi + t\xi^3)} \hat{\phi}(\xi) d\xi$$

$$\|w\|_X \leq \|\phi\|_H \quad \text{homogeneous estimate}$$

Inhomogeneous problem

$$\begin{cases} \partial_t v + \partial_x^3 v = h \\ w(0) = 0 \end{cases} \tag{8.2}$$

$$v(x,t) = \int_0^t S(t-t')h(x,t')dt'$$

$$\|v\|_X \leq \|h\|_B \quad \text{inhomogeneous estimate}$$

Nonlinear problem

$$\partial_t u + \partial_x^3 u + N(u, \partial_x) = 0 \;\Leftrightarrow\; u(t) = S(t)\phi - \int_0^t S(t-t')N(u,\partial_x)(t')dt'$$

where the part after the $\Leftrightarrow$ is an integral equation and $v(0) = \phi$.

Fixed point estimate requires *Nonlinear bridge estimate*:

$$\|N(u,\partial_x)\|_B \leq \tilde{N}(\|u\|_X).$$

Contraction mapping approach to LWP. The initial value problem KdVp is equivalent to the integral equation (Duhamel's formula)

$$v(t) = S(t)\phi - \int_0^t S(t-\tau)\left(\partial_x\left[\frac{1}{p+1}u^{p+1}\right](\tau)\right)d\tau \; .$$

We wish to find a function $u : \mathbb{R}\times[0,T] \to \mathbb{R}$ which satisfies this integral equation. This equivalence lets us define our notion of solution as one which satisfies the integral equation.

Denote the right-hand side by $\Phi_\phi[u]$. We will have found a solution if we find a fixed-point of Φ_ϕ: $\Phi_\phi[u] = 0$.

One way to find that a fixed point exists is to show the mapping is a contraction on a Banach space. That is, we'd like to show

$$\|\Phi_\phi[u] - \Phi_\phi[v]\|_{\text{Ł}} \;\leq\; \theta\|u-v\|_{\text{Ł}} \; , \qquad 0 < \theta < 1 \; .$$

Then, the Picard fixed-point theorem $\Rightarrow \exists! u \in \text{Ł}$ such that $\Phi_\phi[u] = u$.

A necessary condition for Φ_ϕ to be a contraction on Ł is that it map bounded subsets of Ł into bounded subsets of Ł. Let's first try to find an approach to verify this simpler property.

Let $\phi \in H$. Let $u \in \text{Ł}$. Form

$$\Phi_\phi[u] = S(t)\phi - \int_0^t S(t-\tau)\left(\partial_t\frac{1}{p+1}u^{p+1}(\tau)\right)d\tau \; .$$

Prove $\|\Phi_\phi[u]\|_{\text{Ł}} < \infty$.

This probably requires us to show that $\|S(t)\phi\|_L \leq c\|\phi\|_H$ (linear homogeneous estimate).

$$\|\int_0^t S(t-\tau)\left(\partial_x\frac{1}{p+1}u^{p+1}(\tau)\right)d\tau\|_{\text{Ł}} \;\leq\; c\|u\|_{\text{Ł}}^{p+1}$$

(nonlinear estimate). (L^p, H^s as Ł look hopeless.)

The nonlinear estimate can be attacked in two stages: $\partial_t u + \partial_x^3 u = f$ and $u(0) = 0$ has solution $u(t) = \int_0^t S(t-\tau)f(\tau)d\tau$. Suppose we prove

$$\|\int_0^t S(t-\tau)f(\tau)d\tau\|_{\text{Ł}} \;\leq\; \|f\|_B$$

(linear inhomogeneous estimate). Then we'd like to show there is a nonlinear bridge estimate

$$\|\partial_x\, u^{p+1}\|_B \;\leq\; c\|u\|_{\text{Ł}}^{p+1}$$

(nonlinear estimate).

Strategy: Prove all the linear estimates we can. Pray these estimates "match up" to build a bridge (Kenig-Ponce-Vega).

How should we choose L and B? We want LWP in H^1 and then we plan to iterate to get GWP. We want

$$u \in C([0,T]; H'_x) \subset L^\infty([0,T]; H'_x) \qquad \text{"flow property"}$$

$$\left\| \left\| \partial_x \int_0^t S(t-\tau)(\partial_x u^2)(\tau) d\tau \right\|_{L^2_x} \right\|_{L^\infty_T} \qquad \text{Cauchy-Schwarz}$$

$$\left\| \left\| \left(\int_0^t 1^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t |S(t-\tau)(u\partial_x^2 u(\tau))|^2 d\tau \right)^{\frac{1}{2}} \right\|_{L^2_x} \right\|_{L^\infty_T}$$

We have chosen to concentrate on the term with all derivatives landing on one function since it is probably the worst.

$$\left\| \left(\iint_0^t |u\partial_x^2 u|^2 d\tau\, dx \right)^{\frac{1}{2}} \right\|_{L^\infty_T} = \left(\int_0^T \int |u\partial_x^2 u|^2 dx\, d\tau \right)^{\frac{1}{2}}$$

$\Rightarrow$ flow through $H' \rightsquigarrow B = \| \ \|_{L^2_T H^1_x}$. In particular, the flow property naturally leads us to the space $L^2_T H^1_x$ as a candidate for the norm on the inhomogeneity.

We have to swallow one extra derivative. Recall the Kato smoothing estimate:

$$\|\partial_x \partial_x S(t)\phi\|_{L^\infty_x L^2_t} \ \leq \ \|\partial_x \phi\|_{L^2_x}$$

$$\begin{aligned}
\left(\int_0^T \int |u\partial_x^2 u|^2 dx\, d\tau \right)^{\frac{1}{2}} &= \left(\int_x \int_0^T |u|^2\, |\partial_x^2 u|^2\, d\tau\, dx \right)^{\frac{1}{2}} \\
&\leq \left(\int_x \|u\|^2_{L^\infty_T} \int_0^T |\partial_x^2 u|^2 d\tau\, dx \right)^{\frac{1}{2}} \\
&\leq \|\partial_x^2 u\|_{L^\infty_x L^2_t} \left(\int_x \|u\|^2_{L^\infty_T} dx \right)^{\frac{1}{2}} \\
&= \|\partial_x^2 u\|_{L^\infty_x L^2_t}\ \|u\|_{L^2_x L^\infty_t \in [0,T]}
\end{aligned}$$

The flow property and the Kato smoothing estimate have led us to another norm, the maximal function norm, and together these norms define a space in which we may possibly be able to prove a contraction estimate.

Summarizing, suppose $y : \mathbb{R} \times [0,T] \to \mathbb{R}$ satisfying

$$\|y\|_{L^\infty_T H'_x} + \|y\|_{L^2_x L^\infty_T} + \|D_x \partial_x y\|_{L^\infty_x L^2_T} < c$$

Then $y\partial_x y \in L^2_T H^1_x$.

We need to verify that the maximal function estimate $\|S(t)\phi\|_{L^2_x L^\infty_T} \leq c_T \|\phi\|_{H'_x}$ is valid to continue with this approach. This can be done. **do it!**

$$\left(\int_{-\infty}^{\infty} \sup_{t \in [-T,T]} |S(t)\phi(x)|^2 dx \right) \leq^? c_T \|\phi\|_{H'_x}$$

One can show in fact a stronger estimate

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t| \leq T} \sup_{j \leq x < j+1} |S(t)\phi(x)|^2 \right) \leq c_T \|\phi\|_{H^S}$$

provided $s > \frac{3}{4}$. Combining all this stuff gives

Theorem (Kenig-Ponce-Vega, 1991). KdV_p is LWP in H^s, $s > \frac{3}{4}$. And, there are refinements of s based on p. In particular

$$\begin{array}{ll} p=1 & s > \frac{3}{4} \\ p=2 & s \geq \frac{1}{4} \\ p=3 & s \geq \frac{1}{12} \\ p=4 & s \geq 0 \\ p \geq 4 & s \geq S_p \quad \text{(the scaling exponent)} \end{array}.$$

Consider the initial value problem

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 \\ u(0) = t \in H^s(\mathbb{R}) \end{cases}$$

$\Leftrightarrow u(t) = S(t)\phi - \int_0^t S(t-\tau)(u\partial_x u(\tau))d\tau$ ($:= \Phi_\phi[u]$, seek fixed point in some space X) where $S(t)\phi(x) = \int e^{i\cdot(x\cdot\xi + t\xi^3)}\hat{\phi}(\xi)d\xi$.

For Φ_ϕ to be a contraction it must be bounded. Therefore we'll need the space Ł to satisfy

$$\|S(t)\phi\|_{Ł} \leq c\|\phi\|_{H^s} \qquad \text{linear homogeneous} \begin{cases} \partial_t u + \partial_x^3 u = 0 \\ u(0) = \phi \end{cases}$$

$$\|\int_0^t S(t-\tau)(u\partial_x u(\tau))d\tau\|_{Ł} \leq c\|u\|_{Ł}^2$$

We break the second estimate into two pieces

$$\|\int_0^t S(t-\tau)(f(\tau))d\tau\|_{Ł} \leq c\|f\|_B \qquad \text{linear inhomogeneous} \begin{cases} \partial_t u + \partial_x^3 u = f \\ u(0) = \phi \end{cases}$$

$$\|u\partial_x u\|_B \leq \|u\|_\Lambda^2 \quad \text{nonlinear bridge estimate}$$

Last time we discussed the Kenig-Ponce-Vega approach to selecting $Ł, B$. We want $u \in C([0,T]; H^S(\mathbb{R}_x)) \subset L^\infty([0,T]; H^S(\mathbb{R}_x))$ so we verified

$$\|S(t)\phi\|_{L_T^\infty H_x^s} \leq c\|\phi\|_{H_x^s}$$

$$\|\int_0^t S(t-\tau)f(\tau)d\tau\|_{L_T^\infty H_x^s} \leq cT\|f\|_{L_T^2 H_x^s}$$

Our attention focussed on the bridge

$$\|D_x^s(u\partial_x u)\|_{L_T^2 L_x^2} \leq \|uD_x^{s+1}u\|_{L_T^2 L_x^2} + \text{other} \leq^{\text{Hölder}} \|u\|_{L_x^2 L_T^\infty}\|D_x^{s+1}u\|_{L_x^\infty L_T^2} + \text{other}$$

This identifies $Ł = L_x^2 L_T^\infty \cap L_x^\infty L_T^2(D_x^s) \cap L_T^\infty H_x^s \cap$ (other).

To close the circle for iteration requires proving the linear homogeneous estimates/properties

$$\|s(t)\phi\|_{\substack{L_x^2 L_T^\infty \\ L_x^\infty L_T^2(D_x^s) \\ L_T^\infty H_x^s \\ \text{(other)}}} \leq \|\phi\|_{H_x^s} \quad \rightsquigarrow \text{KdV LWP in } H^s(\mathbb{R}),\ S > \tfrac{3}{4}$$

Not surprisingly, the proofs of each of these estimates follow from the explicit form of

$$S(t)\phi(x) = \int e^{i(x\xi+\xi^3 t)}\hat{\phi}(\xi)d\xi$$

Remark. A basic part of the strategy of [KPV] is to show the nonlinearity is a "small" effect compared to the linear part or "small" time intervals. The choice of the space Ł is made to allow us to view the nonlinearity as a "small perturbation" of the linear flow. Note that the basic property of the linear solution is that its space-time Fourier transform is suported on the set $\{(\xi,\lambda) : \lambda = \xi^3\}$.

We'd like to introduce a new space which

- captures the linear homogeneous properties which follow from "lives on cubic"
- flexibly allows us to treat nonlinearities or perturbations.

These motivations led J.Bourgain to introduce the two-parameter spaces $X_{s,b}$ with norm

$$\|w\|_{X_{s,b}} = \left(\iint \left|(1+|\xi|)^s(1+|\lambda-\xi^3|)^b\,|\hat{u}(\xi,\lambda)|\right|^2\,d\xi\,d\lambda\right)$$

Sobolev estimates $\Rightarrow X_{s,b} \subset C([0,T];H^s(\mathbb{R}))$ provided $b >$. We will take Ł $= X_{s,b}$; $b = \frac{1}{2}+$. We need to verify

$$\|S(t)\phi\|_{X_{s,b}} \le c\|\phi\|_{H^s_x} \tag{8.3}$$

$$\|\int_0^t S(t-\tau)(u\partial_x u(\tau))d\tau\|_{X_{s,b}} \le c\|u\|^2_{X_{s,b}} \tag{8.4}$$

$$\begin{aligned}
S(t)\phi(x) &= \int e^{i(x\cdot\xi+t\xi^3)}\hat{\phi}(\xi) = \iint e^{i(x\xi+t\lambda)}\hat{\phi}(\xi)\delta_{\{\lambda=\xi^3\}}(\lambda)d\xi\,d\lambda \\
\Rightarrow (S(t)\phi)^{\text{Ł}}(\xi,\lambda) &= \hat{\phi}(\xi)\delta_{\{\lambda=\xi^3\}}(\lambda) \\
& \quad \left(\left|\iint (1+|\xi|)^s(1+|\lambda-\xi^3|)^b\,\hat{\phi}(\xi)\delta_{\{\xi=\xi^3\}}(\lambda)\right|^2 d\xi\,d\lambda\right) \\
&= \left(\int |(1+|\xi|)^s\hat{\phi}(\xi)|^2 d\xi\right) \\
\|S(t)\phi\|_{X_{s,b}} &\le c\|\phi\|_{H^s_x} \qquad \text{so (1)}
\end{aligned}$$

To check (2), let's split it up as before by first considering

$$\|\int_0^t S(t-\tau)f(x,\tau)d\tau\|_{X_{s,b}} \;\le\; c\|f\|_B$$

and then

$$\|u\partial_x u\|_B \le c\|u\|^2_{X_{s,b}}$$

What is B?

Informally, $\int_0^t S(t-\tau)f(x,\tau)d\tau$ is the solution of $(\partial_t+\partial_x^3)v = f$. $v = (\partial_t+\partial_x^3)^{-1}f$?

Taking Fourier transforms

$$\begin{aligned}
i(\lambda-K^3)\hat{v}(K,\lambda) &= \hat{f}(K,\lambda) \\
\Rightarrow \hat{f}(K,\lambda) &= \frac{1}{i(\lambda-K^3)}\hat{f}(K,\lambda)
\end{aligned}$$

so $\|v\|_{X_{s,b}} \leq \|f\|_{X_{s,b-1}}$. More formally,

$$\begin{aligned}\int_0^t S(t-\tau)f(x,\tau)d\tau &= \int_0^t \left(\iint e^{i(Kx+\lambda\tau)}\, e^{iK^3(t-\tau)} \hat{f}(K,\lambda)\, dK\, d\lambda\right) d\tau \\ &= \iint e^{i(Kx+K^3t)} \underbrace{\int_0^t e^{i(\lambda-K^3)\tau}\, d\tau}_{\frac{e^{i(\lambda-K^3)t}-1}{i(\lambda-K^3)}} \\ &= \iint e^{i(Kx+\lambda t)}\, \frac{\hat{f}(K,\lambda)}{i(\lambda-K^3)}\, dK\, d\lambda + \iint e^{i(Kx+K^3t)}\, \frac{\hat{f}(K,\lambda)}{i(\lambda-K^3)}\, dK\, d\lambda\end{aligned}$$

Matters are now focussed on proving the nonlinear bridge estimate.

$$\boxed{\|u\partial_x u\|_{X_{s,b-1}} \leq \|u\|^2_{X_{s,b}}\ , \qquad b = +\ , \quad \text{which S?}}$$

Bourgain proved this estimate holds for $s \geq 0 \Rightarrow$ LWP of KdV in $L^2(\mathbb{R})$. Kenig-Ponce-Vega proved this estimate holds for $s > -\frac{3}{4} \Rightarrow$ LWP in $H^s(\mathbb{R})$, but fails for $s < -\frac{3}{4}$.

Remark. In Chapter 3 of the typed lecture notes, there is a small section called "Fourier mode perspective on nonlinear dispersive waves". The definitionn of the space $X_{s,b}$ and th eproof of the bilinear estimate

$$\|\partial_x uv\|_{X_{s,b-1}} \leq c\|u\|_{X_{s,b}}\, \|v\|_{X_{s,b}}$$

rigorizes some of that discussion.

8.4. **Dispersive Estimates. Fourier Restriction Phenomena**
Stein-Tomas Theorem
Strichartz Estimates
Smoothing Effect
Oscilliatory integrals and estimates
Periodic Strichartz estimates and conjectures
Exponential sums
Bourgain's relaxed Strichartz estimates
$\mathcal{F}(f) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$ Fourier transform

$$\begin{array}{rll} & \mathcal{F}: L^1 \longmapsto L^\infty & \mathcal{F}(L^1) \longmapsto C_0 \cap L^\infty \\ HY: & \mathcal{F}: L^p \longmapsto L^{p'} & \mathcal{F}(L^p)\ ?\ L^{p'} \\ & \mathcal{F}: L^2 \longmapsto L^2 & \mathcal{F}(L^2) = L^2 \end{array}$$

$\mathcal{F}(L^1)$ distinguishes itself from L^∞ by *pointwise restriction property*:

$$\begin{array}{rcll} \forall x_0 \in \mathbb{R}^d,\ \ L^1 & \ni & f \mapsto \mathcal{F}(f)|_{x_0} \in \mathbb{R} & \text{is bounded} \\ L^\infty & \ni & g \mapsto g|_{x_0} & \text{is not even defined} \end{array}$$

Q: Does $\mathcal{F}(L^p)$ distinguish itself from $L^{p'}$ through its restrictability to certain measure zero sets?

Given $S \subset \mathbb{R}^d$, $|S| = 0$, suppose $A = \{f : S \to \mathbb{C} \mid \|f\|_{A(s)} < \infty\}$ is a class of functions in S. We say that L^p *has A-Fourier Restriction property* to S if

$$(*) \qquad \|\mathcal{F}(f)|_S\|_A \lesssim \|f\|_{L^p}\ , \quad \forall\, f \in L^p\ .$$

Denote by $F(S,P,A)$ the condition $(*)$.

$F(x_0, 1, \text{“}C^\infty\text{”})$ — ok

$F(?, 2, ?)$ — fails

Collapse to $S =$ hypersurface $\subset \mathbb{R}^d$.

Lebesgue measure on $\mathbb{R}^d$ induces *natural surface measure* on S: $d\sigma$.

Various function spaces may be defined using $d\sigma$: $L^1(s; d\sigma)$.

$$F(S, P, L^q_{\text{loc}}(d\sigma)) := \mathcal{F}(S, P, q) \ .$$

Stein observed: L^p has $L^2(d\sigma)$-Fourier Restriction property for hypersurfaces with appropriate curvature for certain $p > 1$.

$$F(S_{\text{curved}}, P, 2)$$

A “hypersurface” in $\mathbb{R}^1$ is a point but $F(x_0, \forall\ p > 1, ?)$ fails.

$\Rightarrow F(\text{affine} \subset \mathbb{R}^d, \forall\ p > 1, ?)$ fails. This shows “flatness” is the enemy.

Knapp homogeneity argument.

$S = \{x_d = |x'|^2 = x_1^2 + x_2^2 + \cdots + x_{d-1}^2\}$ paraboloid. Suppose dn is drawn,

$$\begin{aligned} \hat{f}_\delta &= \chi_{Q_s} \\ \mu(Q_s) &\sim \delta^{d-1} \\ \mu(\operatorname{supp} Q_s) &\sim \delta^{d+1} \end{aligned}$$

(drawing)

$$1 \le p \le 2$$

$$\begin{aligned} \|f_s\|_p &\sim \|\hat{f}_s\|_{p'} \sim \delta^{\frac{d+1}{p'}} \\ \|\hat{f}_s|_s\|_q &\sim \delta^{\frac{d-1}{q}} \end{aligned}$$

$$F(s,p,q) \text{ requires } \delta^{\frac{d-1}{q}} \lesssim \delta^{\frac{d+1}{p'}} \forall\ \delta > 0, \Leftrightarrow 1 \le p \le \frac{q(d+1)}{q(d+1) - (d-1)} \ .$$

“Higher flatness” Knapp example (drawing:)

$$F(s_k, p, q) \text{ requires } 1 \le p \le \frac{q(d-1+k)}{q(d-1+k) - (d-1)} \ .$$

In the hypersurface setting and with $q = 2$, decay of $\hat{\mu}$ implies $F(S, P, Z)$ for certain $1 < P$.

S has *nonvanishing Gaussian curvature* if $\det(\partial^2_{ij}\phi) \neq 0$ where $S = \{x \in \mathbb{R}^d : x_d = \phi(x_1, \ldots, x_d)\}$.

Theorem. *S has nonvanishing Gaussian curvature. σ is natural induced surface measure in S. $d\mu = \psi d\sigma$, $\psi \in C_0^\infty$.*

$$|\hat{\mu}|(n) \le C|n|^{-\frac{(d-1)}{2}} \ .$$

PROOF. Localization, stationary phase, Mouse Lemma.

Stein-Tomas. $S \subset \mathbb{R}^d$ is a hypersurface w.n.vv.g.c.

$\Rightarrow F(S, P, 2)$ for $1 \le p \le \frac{2(n+1)}{n+3}$, i.e.

$$\left(\int |\hat{f}(\xi)|^2 d\mu(\xi) \right)^{\frac{1}{2}} \le C_{p,q} \|f\|_{L^p(\mathbb{R}^n)} \ .$$

PROOF. Note based entirely on the decay proeprty, does use hypersurface. P.Tomas argument:

$$\begin{aligned} \int |\hat{f}|^2 d\mu &= \int \hat{f}\bar{\hat{f}}\, d\mu = \int \hat{f}\hat{\tilde{\bar{f}}}\, d\mu = \int (f * \tilde{\bar{f}})^\wedge\, d\mu = \int f * \tilde{\bar{f}}\;\; d\hat{\mu} \\ &= \int\int \tilde{\bar{f}}\, (x-y) f(y) dy \widehat{d\mu}(x) dx \\ &= \int f(y) \int \bar{f}(y-x)\widehat{d\mu}(x) dy = \int f\, \bar{f} * \hat{\mu} \\ &\leq \|f\|_p\, \|\bar{f} * \hat{\mu}\|_{p'}\ . \end{aligned}$$

reduced to $\|f * \hat{\mu}\|_{p'} \leq \|f\|_p$.

$Rf(T) = \int_{\mathbb{R}^\wedge} e^{-ix\cdot\xi} f(x) dx$ for $\xi \in S$.

Our question is: $R : L^p(\mathbb{R}^n) \to L^2(d\mu)$?

Formal adjoint of R is $R^* f(x) = \int_S e^{ix\cdot\xi} f(\xi) d\mu(\xi)$ for $x \in \mathbb{R}$?

$$\langle Rf, Rf\rangle_{L^2(S,d\mu)} = \langle R^* Rf, f\rangle_{L^2(\mathbb{R}^n)} \overset{\text{Holder}}{\leq} \|R^* Rf\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

But

$$\begin{aligned} R^* Rf(x) &= \int_{\mathbb{R}^n} \int_S e^{i\xi\cdot(x-y)} d\mu(\xi) f(y) dy \\ &= \int_{\mathbb{R}^n} \hat{\mu}(x-y) f(y) dy = \hat{\mu} * f(x) \end{aligned}$$

Let $Tf = \hat{\mu} * f(x)$. Show $T : L^p \to L^{p'}$.

A decomposition of $\mathbb{R}^d$ (Littlewood-Paley).

$B_1 = \{x \in \mathbb{R}^d \mid |x| < 1\}$

$A_0 = \{x \in \mathbb{R}^d \mid 1 \leq |x| < 2\}$

$A_j = \{x \in \mathbb{R}^d \mid \frac{x}{2^j} \in A_0\}$

$$Tf = \hat{\mu} * f = (\chi_{B_1} + \sum_j \chi_{A_j})\hat{\mu} * f = K_{-1} * f + \sum_{j=0}^{\infty} K_j * f$$

$$K_{-1} = (\chi_{B_1}\hat{\mu} \in L^1 \cap L^\infty \Rightarrow \|K^{-1} * f\|_{p'} \leq C\|f\|_{L^p}\ .$$

(Young's inequality: $\|g * h\|_{L^r} \leq \|q\|_{L^q}\, \|h\|_{L^p}$ $\quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.)

We make two estimates

$$\begin{aligned} \|K_j * r\|_{L^\infty} &\leq \|K_j\|_\infty\ |f\|_1 \\ \|K_j * f\|_{L^2} &\leq \|\hat{K}_j\|_{L^\infty}\|f\|_{L^2} \qquad \leftarrow \text{ interpolate} \\ \|K_j\|_\infty &= \|\chi_{A_j}\hat{\mu}\|_\infty \leq (2^j)^{-\frac{(d-1)}{2}} = 2^{-\frac{j(d-1)}{2}} \qquad \text{using decay} \\ \|\hat{K}_j\|_\infty &= \|\hat{\chi}_{A_j} * \mu\|_i = 2^{id}\|\hat{\chi}_{A_0}(2^j\,\cdot) * \mu\|_{L^i} \leq 2^{jd}\, 2^{-j(d-1)} \end{aligned}$$

$\int \chi_{B_{2^{-j}}}(x-y) d\mu(y)$ — hypersurface measure charges $B_{2^{-j}}$ by at most $(2^{-j})^{d-1}$

$T_j f = K_j * f$

$$\begin{aligned} &T_j : L^1 \to L^\infty \quad \|T_j\|_{1,\infty} \leq 2^{-j\frac{d-1}{2}} \\ &T_j : L^p \to L^{p'} \quad \|T_j\|_{p,p'} \leq 2^{-j\frac{d-1}{2}\tanh} 2^{j(1-\tanh)} \\ &T_j : L^2 \to L^2 \quad \|T_j\|_{2,2} \leq 2^j \end{aligned}$$

Pick tanh such that $\tanh\left(\frac{1-d}{2}\right) + (1-\tanh) < \tanh \Rightarrow \tanh > \frac{2}{d+1} \Rightarrow p < \frac{2(d+1)}{d+3}$

Q: Does $\mathcal{F}(L^p)$ distinguish itself from $L^{p'}$ by a restriction property to certain measure zero sets? M.Christ pointed out to me that the answer is yes.

$\exists$ measures μ on $\mathbb{R}^d$ such that
supp$\mu \subset E$ with $|E| = 0$ (in fact, $\dim_H E < d$), μ decays.

Hausdorff dimension property replaces "hypersurface property" in controlling
$\|\hat{K}_j\|_{L^\infty}$
for bounding $\|K_j * f\|_{L^2}$. Decay property is as before and controls $\|K_j\|_{L^\infty}$ for bounding
$\|K_j * f\|_{L^\infty}$.

$\rightarrow$ Flexibility in contructing μ permits finding a distinction between $\mathcal{F}(L^p)$ and $L^{p'}$
$\forall\ 1 \leq p < 2$.

Gerd Mockenhaupt

Stern-Tomas. For a hypersurface $S \subset \mathbb{R}^d$ with nonvanishing Gaussian curvature and $\psi \in C_0^\infty(\mathbb{R}^d)$, we have for $1 \leq p \leq \frac{2(d+1)}{d+3}$

$$\left(\int_S |\hat{f}(\xi)|^2 \psi(\xi) d\sigma(\xi)\right)^{\frac{1}{2}} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

The duality inequality reads

$$\begin{aligned} \left\|\int_S e^{ix\cdot\xi}\hat{\phi}(\xi)\tilde{\psi}(\xi)d\sigma(\xi)\right\|_{L^{p'}(\mathbb{R}^d)} &\leq c\|\hat{\phi}\|_{L^2(\psi d\sigma)} \\ \frac{2(d+1)}{d-1} &\leq p' \leq i \end{aligned}$$

For Schrödinger applications, we desire the dual inequality without the cutoff function. We will find that such an inequality is only possible for $p = \frac{2(d+1)}{d-1}$.

Rescaling argument. Let $\hat{\phi}$ be supported on the vast ball B_1. We know that

$$(*) \qquad \left\|\int_{\mathbb{R}^d-1} e^{i(x'\cdot\xi' + x_d|\xi'|^2}\hat{\phi}(\xi')d\xi'\right\|_{L^p(\mathbb{R}^d)} \leq C\|\hat{\phi}\|_{L^2(\mathbb{R}^{d-1})}$$

Replace $\hat{\phi}(\xi')$ by $\hat{\phi}_\sigma = \hat{\phi}\left(\frac{\xi'}{\sigma}\right)$. Note that supp$\hat{\phi}\left(\frac{\cdot}{\sigma}\right) \subset B_\sigma$. We change variables to see if $(*)$ is invariant w.r.t. σ.

Q: $\|\int_{\mathbb{R}^d-1} e^{i(x'\cdot\xi'+x_d|\xi'|^2}\hat{\phi}_\sigma(\xi')d\xi'\|_{L^p(\mathbb{R}^d)} \leq C\|\hat{\phi}_\sigma\|_{L^2(\mathbb{R}^{d-1})}$?

Re-express right-hand side:

$$\|\hat{\phi}_r\|_{L^2(\mathbb{R}^{d-1})} = \underset{\substack{y' = \frac{\xi'}{\sigma} \\ \sigma^{d-1}dy' = d\xi'}}{\left(\int |\hat{\phi}\left(\tfrac{\xi'}{\sigma}\right)|^2 d\xi'\right)^{\frac{1}{2}}} = \sigma^{\frac{d-1}{2}}\|\hat{\phi}\|_{L^2(\mathbb{R}^{d-1})}$$

Re-express left-hand side:

$$\| \int e^{i(x'\cdot\xi' + x_d|\xi|^2)} \hat{\phi}(\tfrac{\xi'}{\sigma}) d\xi' \quad d\xi' \|_{L^2(\mathbb{R}^d_x)}$$

$$y' = \tfrac{\xi'}{\sigma}$$

$$\sigma^{d-1} dy' = d\xi'$$

$$\sigma^{d-1} \| \int e^{i(\sigma x'\cdot y' + \sigma^2 x_d |y'|^2)} \hat{\phi}(y') dy' \|_{L^{p'}(\mathbb{R}^d_x)}$$

$$z' = \sigma x' \qquad z_d = \sigma^2 x_d$$

$$dz' = \sigma^{d-1} dx' \qquad dz_d = \sigma^2 dx_d$$

$$\sigma^{-(d+1)} dz = dx$$

$$\sigma^{d-1}\, \sigma^{-(\frac{(d+1)}{p})} \| \int e^{i(z'\cdot y' + z_d |y'|^2)} \hat{\phi}(y') dy' \|_{L^{p'}(\mathbb{R}^d_z)}$$

We want

$$\sigma^{d-1}\, \sigma^{-(\frac{(d+1)}{p})} \| \quad \|_{L^{p'}(\mathbb{R}^d_z)} \leq^{?} \sigma^{\frac{d-1}{2}} \|\hat{\phi}\|_{L^2(\mathbb{R}^{d-1})} \quad \forall \text{ large } \sigma$$

This requires that

$$\sigma^{d-1-\frac{d+1}{p'}} = \sigma^{\frac{d-1}{2}} \quad \text{for large } \sigma$$

$$\frac{d-1}{2} - \frac{d+1}{p'} = 0 \Rightarrow \boxed{p' = \frac{2(d+1)}{d-1}}$$

Proposition (Strichartz Inequality for Paraboloid.)

$$\| \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)} \hat{\phi}(\xi) d\xi \|_{L^{2+\frac{4}{d}}(\mathbb{R}^d_x \times \mathbb{R}^1_t)} \;\leq\; C\|\phi\|_{L^2(\mathbb{R}^d)} \;.$$

Recall that $\begin{cases} i\partial_t u + \Delta u & = 0 \\ u(0) = \phi & x \in \mathbb{R}^d \end{cases}$ is solved by

$$S(t)\phi(x) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)} \hat{\phi}(\xi) d\xi$$

so

$$\boxed{\|S(t)\phi\|_{L^{2+\frac{4}{d}}(\mathbb{R}^d_x \times \mathbb{R}^1_t)} \;\leq\; C\|\phi\|_{L^2(\mathbb{R}^d)} \;.}$$

It is also clear that we have the unitarity property of the flow in L^2

$$\|S(t)\phi\|_{L^\infty_t L^2_x} \leq c\|\phi\|_{L^2 x} \;.$$

Interpolation implies then that (genralized stochastic inequalities)

$$\|S(t)\phi\|_{L^q_t L^r_x} \leq c\|\phi\|_{L^2 x} \quad \forall q, r$$

satisfying $0 \leq frac2q = \frac{2}{d} - \frac{d}{r} < 1$. Exponents q, r satisfying these conditions are said to be *admissible.*

Some functional analysis implies **inhomogeneous Strichartz estimates.** Let (q_1, r_1) and (q_2, r_2) each be admissible pairs. Then

$$\| \int_0^t S(t-t') f(x,t') dt' \|_{L^{q_1}_{t\in I} L^{r_1}_x} \;\leq\; c_I \|f\|_{L^{q_2'}_t L^{r_3'}_x} \;.$$

In particular,

$$\| \int_0^t S(t-t') f(x,t') dt' \|_{L^{z+\frac{4}{d}}_{x,t_\epsilon}} \;\leq\; \|f\|_{x,t}^{\frac{2d+4}{d+4}} \;.$$

Application. Cubic NLS on $\mathbb{R}^2$ is LWP in L^2.

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^2 u &= 0 \\ u(0) = \phi & x \in \mathbb{R}^2 \end{cases}$$

$\Leftrightarrow 0(t) = S(t)\phi \mp \int_0^t S(t-t')(|u|^2)(t')dt'$.

RHS := $\Phi_\phi[u]$. Φ_ϕ : (bounded subsets of $L^4_{x,t\in I}$) $\to$ (bold subsets of $L^4_{x,t\in I}$).

$$\begin{aligned} \|\Phi_\phi[u]\|_{L^4_{x,t}} &\le \|S(t)\phi\|_{L^4_{x,t}} + \|\int_0^t S(t-t')(|u|^2 u)(t')dt'\|_{L^4_{x,t}} \\ &\le \|\phi\|_{L^2_x} + \|\,|u|^2 v\|_{L^{4/3}_{xt\in I}} \\ &\le \|\phi\|_{L^2_x} + \|u\|^3_{L^4_{x,t\in I}} \end{aligned}$$

Periodic Strichartz Inequalities

Problem. Let $d \ge 1$, let $S \subset \mathbb{Z}^d$ and $q > 2$. Find the optimal constant $K_q(S)$ (or at least a good estimate) such that

$$\|\sum_{s\in S} \hat{u}(S)\, e^{ix\cdot\xi}\|_{L^q(\mathbb{T}^d)} \le K_q(S)\|\hat{u}\|_{\ell^2(S)} \ .$$

Remark. For $q = 2$ this is valid for all subsets $S \subset \mathbb{Z}^d$ by Plancherel. For $q > 2$, we are asking for a function on $\mathbb{T}^d$ to satisfy the unlikely estimate $\|u\|_{L^q(\mathbb{T}^d)}\| \le \|u\|_{L^2(\mathbb{T})}$ for $q > 2$, by the knowledge that supp $\hat{u} \subset S \subset \mathbb{Z}^d$.

Example. Suppose $S = \{2^j : j \in \mathbb{N}\}$. $K_q(S)$ is bounded for all $q > 2$. Recall the Littlewood-Paley square function $Sf(x) = (\sum_j |S_j f(x)|^2)^{\frac{1}{2}}$ where $S_j f(x) = \sum_{|k|\in\sum 2^j, 2^{j+1}} \hat{f}(k)e^{ikx}$. Since $\|Sf\|_{L^q} \sim \|f\|_{L^q}$ and $|S_j f(x)| = |\hat{f}(2^j)|$, we have that

$$\|f\|_{L^q} \sim \left(\sum_j |\hat{f}(2^j)|^2\right)^{\frac{1}{2}} \quad \forall\ q\ .$$

The relationship of this problem to the Schrödinger equation is as follows.

Let us consider Schrödinger's equation posed on $\mathbb{T}$,

$$\begin{cases} i\partial_t u + \Delta u &= 0 \\ u(0) = \phi & x \in \mathbb{T} \end{cases}$$

The solution operator is

$$S(t)\phi(x) = f(x,t) = \sum_{m\in\mathbb{Z}^d} \hat{\phi}(\mu)\, e^{i(\mu x + \mu^2 t)} \ .$$

The set $S_d = \{(\mu, \mu^2) : \mu \in \mathbb{Z}^d\} \subset \mathbb{Z}^{d+1}$ and we wonder if

$$\|f\|_{L^q(\mathbb{T}^{d+1})} \ \stackrel{?}{\le}\ k_q(S_d)\|\phi\|_{L^2(\mathbb{T})} \ ?$$

Notice this is a Strichartz type inequality.

Sometimes no finite $k_q(S_d)$ may exist. For such a situation, we may inquire about a truncated version of the inequality. Define $S_n = S \cap \{\text{supp } \chi_N \times \mathbb{Z}\}$ where $\chi_N = \chi_{\{m\in\mathbb{Z}^d : |m|<N\}}$. Then we ask for $\|f\|_{L^q(\mathbb{T}^{d+1})} \le K_q(S_N)\|\phi\|_{L^2(\mathbb{T})}$ when $\hat{\phi} = \chi_N\hat{\phi}$. In this case we may end up with estimates of the form

$$\|f\|_{L^q(\mathbb{T}^{d+1})} \ \le\ N^\beta\ \|\hat{\phi}\|_{L^2(\mathbb{Z}^d)} \sim \|\phi\|_{H^\beta(\mathbb{T}^d_x)}$$

where N^β is a loss and the final term $H^\beta(\mathbb{T}^d_x)$ becomes derivative. Motivated by a comparison with the standard Strichartz setting and certain explicit cases:

Periodic Strichartz Conjecture (paraboloid).

$$K_q(S_{d,N})\begin{cases} << N^{\epsilon} & \text{for } q = 2+\frac{4}{d} \\ C_q & q < 2+\frac{4}{d}\ . \end{cases}$$

Two methods have led to partial progress on this conjecture:

- $q = 2S$ is an even integer, exploit L^2 properties
- Tomas argument $\begin{matrix} L^1 \\ L^\infty \end{matrix}$ and major-minor arc decomposition (technical) exponential sums techniques.

Suppose that

$$f(x,t) = \sum_{m\in\mathbb{Z}^d} e^{i(mx+m^2t)}\hat{\phi}(m)\ .$$

Suppose $\hat{\phi} = \chi_N\hat{\phi}$ and $q = 25$.

$$\begin{aligned} \|f\|_{L^B(\pi^{d+1})} &= \left(\int_{\pi^{d+1}} |f^s|^2 dx\right)^{\frac{1}{25}} \\ &= \|f\|^{\frac{1}{s}}_{L^B(\pi^{d+1})} = \|\mathcal{F}(f^s)\|^{\frac{1}{s}}_{L^2(\mathbb{Z}^{d+1})} \end{aligned}$$

$$\mathcal{F}(f^s)(m,P) = \sum{}^{*} \hat{\phi}(m_1)\dots\hat{\phi}(m_s)$$

$\sum^*$ over $\{(m_1,\dots,m_s), |m_i| < N : \begin{matrix} m = m_1+\cdots+m_s \\ p = m_1^2+\cdots+m_s^2\ \} \end{matrix}$.

Set $r_{m,P} = \#\{\ \ \}$ for fixed N.

$$\begin{aligned} \|\mathcal{F}(f^s)\|^2_{L^2(\mathbb{Z}^{d+1})} &= \sum_{m,P} |\sum{}^{*} \hat{u}(m_1)\dots\hat{u}(m_s)|^2 \\ &\le \sum_{m,P} r_{m,P}\sum{}^{*} |\hat{u}(m_1)\dots\hat{u}(m_s)|^2 \\ &\le (\sup_{m,P} r_{m,P})\|u\|^{25}_{L^2} \end{aligned}$$

The problem is therefore reduced to estimating $r_{m,P}$.

Special case. $d = 1$, $d+1 = 2$, $q = 4 = 2.2$. Fix and satisfy m, P. Suppose $m_1, m_2 \in \mathbb{Z}$. How many (m_1, m_2)? $m = m_1 + m_2$, $P = m_1^2 + m_2^2$.

Answer: A bounded number independent of N.

$$\begin{matrix} \Rightarrow & s(\ \cdot\ )\phi & \\ & \| & \\ \|f\|_{L^4(\mathbb{Z}^2)} & & \le \|\phi\|_{L^2(\mathbb{Z})} \end{matrix}$$

Hence the application to cubic NLS above also works for periodic case.

$$\begin{cases} i\partial_t u + \Delta 0 & x \in \mathbb{R}^d \\ u(0) = \phi \end{cases}$$

$$u(x,t) = s(t)\phi(x) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\hat{\phi}(\xi)d\xi = \iint e^{i(x\cdot\xi+\tau t)}\hat{\phi}(\xi)\,\delta_{\{??=|?|^2\}}d??$$

so $\hat{u}(\xi,\tau) = \hat{\phi}(\xi)\,\delta_{\{t=|\xi|^2\}}(\tau)$.

Strichartz inequality

$$\| \iint e^{i(x\cdot\xi+\tau t)}\hat{\phi}(\xi)\,\delta_{\{\tau=|\xi|^2\}}(\tau)d\tau d\xi\|_{L^{p(d)}_{xt}} \leq c\|\phi\|_{L^2_x}$$

Let ψ be a nice cutoff function to, say, $[-1,1]$.

$$(\psi u)^\wedge = \hat{\psi} * \hat{u} = \int \hat{\psi}(\tau-\tau_1)\hat{u}(\xi,\tau_1)d\tau_1$$

so $(\psi u)^\wedge$ looks like $\hat{u}$ only smeared out.

$$\Rightarrow \| \iint e^{i(x\cdot\xi+\tau t)}\hat{\phi}(\xi)\hat{\psi}(\tau-|\xi|^2)d\tau d\xi\|_{L^{p(d)}_{xt}} \leq c\|\phi\|_{L^2_x}$$

where $\psi \sim \chi_{[-1,1]}$ only smooth.

How far can we relax the Fourier-support-near-paraboloid property and retain the same estimate?

We are motivated to consider this because we want as general as possible a criterion to recognize L^p functions when F.T. properties. In particular, nonlinear pole have solutions which may not live on the dispersive surface.

Strichartz inequality

$$\| \iint e^{i(x\cdot\xi+\tau t)}\frac{a(\xi,\tau)}{(1+|\tau-|\xi|^2|)^b}d\xi\, d\tau\|_{L^{P(d)}_{x,t}} \leq^? c\|a\|_{L^2_{\xi,\tau}}\ .$$

Claim: Strichartz $\Rightarrow$ relaxed Strichartz if $b>\frac{1}{2}$.

Stack up parabolic level sets.

Wave-to-the-general.

$$\sum_\ell \frac{1}{(1+|\ell|)^b}\iint_{\{\|tau-|\xi|^2|=\ell+O(1)\}} e^{i(x\cdot\xi+\tau t)}a(\xi,\tau)d\xi\, d\tau$$

$$= \left\| \sum_\ell \frac{1}{(1+|\ell|)^b}\iint_{\substack{|\theta|<1 \\ \tau=|\xi|^2+\ell+O(2)}} \overbrace{e^{i(x\cdot\xi+|\xi|^2t)}e^{i\ell t+\tanh t}a(\xi,|\xi|^2+\ell+\theta)}^{\hat{\phi}_{\ell,\tanh}(\xi)}d\xi\, d\tanh \right\|_P$$

$$\leq \sum_\ell \frac{1}{(1+|\ell|)^b}\int_{|\theta|<1} \underbrace{\|\int e^{ix\cdot 2+|\xi|^2t}+\hat{\phi}_{\ell,\theta}(\xi)d\xi\|_P}_{\|\hat{\phi}_{\ell,\tanh}(\xi)\|_{L^2_\xi}}\, d\theta$$

$$\leq \sum_\ell \frac{1}{(1+|\ell|)^b}\|\ \|\hat{\phi}_{\ell,\theta}(\xi)\|_{L^2_\xi}\ \|_{L^2_\theta}$$

$$a(\xi,|\xi|^2+\ell+?)$$

$$\leq \left(\sum_\ell \frac{1}{(1+|\xi|)^{2h}}\right)^{\frac{1}{2}}\left(\sum_\ell \|\hat{\phi}_{\ell,\tanh}\|^2_{L^2_\xi L^2_\theta}\right)^{\frac{1}{2}}$$

Certainly we also have

$$\left\| \iint e^{i(x\cdot\xi+\tau t)} \frac{a(\xi,\tau)}{(1+|\tau-|\xi|^2|^0} d\xi\, dt \right\|_{L^2_{x,\tau}} \leq \|a\|_{L^2_{\xi,\tau}}$$

So, by interpolation

$$\left\| \iint e^{i(x\cdot\xi+\tau t)} \frac{a(\xi,\tau)}{(1+|\tau-|\xi|^2|^0} d\xi\, dt \right\|_{L^{p(\beta)}_{xt}} \leq \|a\|_{L^2_{\xi,\tau}}$$

for $0 \leq \beta \leq \frac{1}{2}t = b$ and $p = p(\beta)$ satisfying $2 \leq p(\beta) \leq p(d)$.

Suppose $(1+|\tau-|\xi|^2|)^\beta c(\xi,\tau) \in L^2_{\xi,\tau}$. Then the preceding may be rewritten

$$\left\| \iint e^{i(x\cdot\xi+\tau t)} c(\xi,\lambda) d\xi\, d\lambda \right\|_{L^p_{xt}} \leq \|(1+|\tau-|\xi|^2|)c(\xi,\tau)\|_{L^2_{\xi,\tau}}$$

Definition:

$$\|u\|_{X_{S,b}} = \left(\iint (1+|\xi|)^{25}(1+|\tau-|\xi|^2|)^{2b} |\hat{u}(\xi,\tau)|^2 d\xi\, d\tau \right)^{\frac{1}{2}}$$

Fact: $\|u\|_{L^{P(\beta)}_{x,t}} \leq \|u\|_{x_0,\beta}$.

In particular, for the parabolid $\{(\xi,|\xi|^2) : \xi \in \mathbb{R}^d\}$, we have

$$\begin{aligned} \|u\|_{L^{P(\beta)}_{x,t}} &\leq \|u\|_{x_0,b}\ ; \qquad b > \tfrac{1}{2}\ . \\ p(d) &= \frac{2(d+2)}{d}\ . \end{aligned}$$

Cubic NLS on $\mathbb{R}^1$ LWP in $X_{0,b}$. L^6 vs L^4 KdV on $\mathbb{R}$ in $X_{s,b}$, $s \geq 0$.

$X \in \mathbb{R}^2$ Strichartz inequality: $\|S(t)\phi\|_{L^4_{xt}} \leq c\|\phi\|_{L^2_x}$.

Rewrite it as

$$\|(S(t)\phi_1)(S(t)\phi_2)\|^2_{L^2_{xt}} \leq c\|\phi_1\|_{L^2_x}\|\phi_2\|_{L^2_x}\ .$$

Bourgain observed that this bilinear estimate may be refined in the following sense:

Bourgain's Strichartz refinement. Suppose $\hat{p}_i \subset \{\xi : |\xi| \sim N_i\}$. Assume $N_2 \leq N_1$. Then

$$\begin{aligned} \|(S(t)\phi_{N_1})(S(t)\phi_{N_2})\|^2_{L^2_{xt}} &\leq \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}} \|\phi_{N_1}\|_{L^2}\|\phi_{N_2}\|_{L^2} \\ &\sim \|\partial_x^{\frac{1}{2}}\phi_{N_1}\|_{L^2}\|\partial_x^{-\frac{1}{2}}\phi_{N_2}\|_{L^2}\ . \end{aligned}$$

(gaining regularity)

8.5. **Oscilliatory integrals approach to LWP. The initial value problem for KdV_p on the line**
Schrödinger with derivative?
KdV on $\mathbb{R}^+$? KP?

Remark 8.2. Nonlinear term is treated via Hölder with this approach.

Remark 8.3. The use of the smoothing estimate restricts this approach to noncompact domains. But does it really given the likely progress on KdV posed on an interval?

8.6. **Bourgain's approach to LWP.** [3], [?]
Remarks on Duhamel terms
Relaxed Strichartz estimates
L^4_{xt} **treatment of cubic** NLS
$X_{s,b}$ **treatment of cubic** NLS **on** $\mathbb{R}$, $\mathbb{T}$
$X_{s,b}$ **treatment of KdV on** $\mathbb{R}$, $\mathbb{T}$

Remark 8.4. Method applies to polynomial nonlinearities. More general nonlinearites may be approximated by polynomials, [3] ?, [?].

Remark 8.5. Detailed Fourier analysis of nonlinear term goes beyond the Hölder idea in the Oscilliatory integrals approach in special cases where it can be done.

8.7. **Multilinear estimates in** $X_{s,b}$**. Bilinear estimate for KdV** [14]
Counterexamples
Bourgain's refined Strichartz estimate
Bilinear estimates for NLS on $\mathbb{R}$ [15]
Bilinear estimates for NLS on $\mathbb{R}^2$ [6]
Induction on Scales [30]
KPI [?], [?], [?]
Multilinear notation

8.8. **Applications of multilinear estimates. Wellposedness**
Growth of high Sobolev norms
Correction terms and almost conservation laws

8.9. **Global wellposedness. Conservation Law**
Bourgain's high/low frequency technique [4]
I-Method [?], [?]
Correction Terms [?]

9. Long-time behavior of solutions

Remark 9.1. The range of possible behaviors of solutions of nonlinear dispersive equations is vast. Recurrence, dispersive decay, blow-up, turbulence may occur and merit prediction and quantification.

9.1. **Special solutions and their stability.** orbital and asymptotic stability results. Pego-Weinstein, Martel-Merel, Cuccagna, Tsai-Yau, CKSTT

9.2. **Scattering.**

9.3. **Weak turbulence.**

9.4. **Open questions?**

10. Blow-up behavior

10.1. **Physical relevance of blow-up analysis.**

10.2. Existence of singular solutions of NLS.

- Glassey's argument
- Explicit singular solutions via conformal transformation

10.3. NLS blow-up.

- L^2 mass concentration
- upper/lower bounds on rate of blow-up?
- stability of blow-up?

10.4. Zakharov system blow-up.

10.5. KdV blow-up.

11. Infinite Dimensional Hamiltonian Systems

11.1. Invariant Measures. [4]

11.2. Symplectic Capacities. [4]

11.3. Kolmogorv-Arnold-Moser Theory. [4]

11.4. Aubry-Mather Theory?

11.5. Arnold Diffusion? Nekorochev Estimates?

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